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# QUANTUM LOWER LIMIT ON SCATTERING ANGLE IN THE CALCULATION OF MULTIPLE TOUSCHEK-EFFECT

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# Quantum lower limit on scattering angle in the calculation of multipole Tousckek-effect

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**Abstract**

We take into account the quantum lower limit on a scattering angle in calculation of the Intra-Beam Scattering (IBS) affecting the beam sizes and energy spread in the storage rings. The aim is to compare the classical and quantum approaches. Numerical examples for the CLIC and VEPP-4M machines are presented.

# 1 Introduction

As is known [1], there are two definitions in plasma physics for minimal scattering angle ($\theta_{min}$) at interaction between two charged particles. One of them is of a classic origin and is used in Rutherford formulas; particularly, in scattering of the particles with equal charges ($e$) and masses ($m$):

$$\theta_{class} = \frac{2e^2}{\Lambda m V^2}.$$

The other one is connected with a quantum uncertainty relation of particle momentum and coordinate:

$$\theta_{quant} \approx \frac{\hbar}{\Lambda m V}.$$

In these formulas $\Lambda$ is maximal impact parameter scale (applied to plasma – Debye radius); $m$ is particle mass reduced to a center-of-mass system; $V$ is relative velocity; $\hbar$ is Planck constant. The condition of violation of the classical definition is written as

$$\frac{\theta_{class}}{\theta_{quant}} = \frac{2e^2}{\hbar V} < 1. \tag{1}$$

At electron-electron or electron-positron scattering, and also in $e^-p^+$ and $e^+p^-$ processes, condition (1) is fulfilled at relative movement energy $mV^2/2 > 50$ eV (for $p^-p^+$ over 40 keV).

It is interesting to consider ratio (1) with respect to the calculation of electron and positron beam sizes in the storage rings of colliders and SR sources. Under certain conditions the beam sizes in storage rings depend on the processes of intra-beam scattering (multiple Touschek effect) in the presence of radiation friction and quantum excitation of betatron and phase oscillations. In typical machines, at a GeV-order energy, the particle transverse momentum is up to $10^{-4}$ and over with respect to the value of longitudinal one; and, consequently, the characteristic energy of relative movement in collisions exceeds by several orders of magnitude the critical level of 50 eV.

Nevertheless, the quantum limit on a minimal scattering angle is not met in the widespread approach providing the calculation of beam size increase due to Touschek effect; the classical formula (see, e.g. [2, 3]) is used for $\theta_{min}$.

When the intra-beam scattering effect on the sizes is insignificant t his impropriety is unnoted due to the weak dependence of the calculation results on the parameter $\theta_{min}$ introduced under the logarithm sign. The question of classical approach validity may arise in the design of modern storage rings with ultra-small beam emittance of a nanometer-order, in which the role of Touschek effect and, therefore, the required calculation accuracy increase significantly. One must show in what conditions a quantum lower limit on scattering angle is important. If a minimal scattering angle determined by a quantum limit is much larger than the classical one, can this fact lead to a significant increase of the IBS diffusion (in beam sizes and energy spread) in comparison with a classical consideration?

The aim of the work is to try to fill the mentioned gap and to compare the both approaches using particular numerical examples [1].

## 2 Intra-beam scattering parameters

In intra-beam scattering theory the maximal impact parameter scale $\Lambda$ is denoted by $b_{max}$ and is defined as

$$b_{max} = min\left\{\sigma_y, \left(\frac{\gamma V_b}{N}\right)^{1/3}\right\}. \tag{2}$$

Here $\sigma_y$ is a beam vertical size; $\gamma$ is relativistic factor; $N$ is a number of particles in a bunch; $V_b = 8\pi^{3/2}\sigma_x\sigma_y\sigma_z$ is the volume of a bunch with the Gaussian density distribution in laboratory coordinate system. The second argument in braces characterizes the average distance between particles in a co-moving inertial reference system.

An average square of momemtum increment across the original direction of the movement in a scattering plane is found using the Moeller cross section $\sigma$ [2, 3]:

$$\sigma < p_\perp^2 >= \int p_\perp^2 d\sigma = 2\pi \left(\frac{r_0 p_0^2}{p}\right)^2 \ln\left(\frac{p_{\perp max}}{p_{\perp min}}\right).$$

[1]The results of this work were presented at the IBS Mini Workshop (Daresbury, The Cockcroft Institute of Accelerator Sciencies and Technologies, 28-29 August 2007)

The Coulomb logarithm

$$\ln\left(\frac{p_{\perp max}}{p_{\perp min}}\right) = \ln\left(\frac{\theta_{max}}{\theta_{min}}\right)$$

is determined by the ratio of maximal and minimal values of the transverse momenta of a scattering particle (i.e. the ratio of corresponding values of a scattering angle) in center-of-mass system (CMS). In non-relativistic approximation, $p_{\perp max} = p = p_0\gamma\xi$ where $p_0 = mc$ and $\xi = V/(2c)$ is an initial particle velocity in CMS in the light speed units. It is convenient to introduce the characteristic of a transverse momentum spread in a beam (corresponding to $p$ value), which is, for a flat beam case, equal to $\sigma_p = p_0\gamma\sigma_{X'}$, $\sigma_{X'}$ is spread of trajectory angles in radial plane. In the classical limit

$$\ln\left(\frac{p_{\perp max}}{p_{\perp min}}\right) = \ln\left(\frac{p}{p_m}\right)^2 = \ln\frac{\chi}{\chi_m^c}, \qquad (3)$$

where the following notations are used:

$$p_{\perp min} = \frac{r_0 p_0^2}{b_{max} p}, \quad p_m = p_0\sqrt{\frac{r_0}{b_{max}}},$$

$$\chi = \frac{p^2}{\sigma_p^2}, \quad \chi_m^c = \frac{r_0 p_0^2}{b_{max}\sigma_p^2}.$$

In the quantum limit on a minimal scattering angle

$$\ln\left(\frac{p_{\perp max}}{p_{\perp min}}\right) = \ln\left(\frac{p b_{max}}{\hbar}\right) = \frac{1}{2}\ln\frac{\chi}{\chi_m^q}, \qquad (4)$$

$$p_{\perp min} = \frac{\hbar}{b_{max}}.$$

In (4) the classic parameter $\chi_m^c$ is substituted by a quantum analog

$$\chi_m^q = \left(\frac{\hbar}{b_{max}\sigma_p}\right)^2.$$

and, besides, in comparison with (3), there appeared coefficient 1/2 before the logarithm. In addition,

$$p_{\perp min} = \frac{\hbar}{b_{max}}, \quad \frac{p_{\perp max}}{p_{\perp min}} = \frac{\chi}{\chi_m^q}.$$

It can be shown that in practically interesting cases $\chi_m^c/\chi_m^q >> 1$. In fact,

$$\frac{\chi_m^c}{\chi_m^q} = \frac{m^2c^2r_0b_{max}}{\hbar^2} \approx 2\cdot 10^8 b_{max},$$

where $b_{max}$ -in centimeters. For example, in case $b_{max} \sim \sigma_y \sim 10^{-4}$ cm parameter $\chi_m^c$ exceeds parameter $\chi_m^q$ by four orders of magnitude. Actually, the values of parameters $\chi_m^c$ and $\chi_m^q$ approach each other under the beam compression. Nevertheless, it is almost impossible for the ratio $\chi_m^c/\chi_m^q$ to reach the value close to unity as it requires non-realistic ultra-thin/superdense beams ($b_{max} \sim 10^{-8}$ cm!).

# 3 Touschek effect calculation in 2D collision approach

Below we describe the method that was developed and is used to calculate the sizes and life-time of a beam [4, 5, 6].

Based on the theory given in [2], the method expands a "flat beam" approximation into a theory, which takes into account the two-dimensional character of particle relative motion in CMS.

With the aim to describe the two-dimensional character of motion the parameter of transverse oscillation coupling in velocity space $k = \sigma_{X'}/\sigma_{Y'}$ is introduced, where $\sigma_{Y'}$ is the spread of trajectory angles in a vertical plane. In the so-called "round" beam $k \to 1$ and in the flat one $k \to \infty$. Momentum($p$) distribution function in the center-of-mass system (CMS) has the following form [4]:

$$f(k,p)dp = \frac{2kp}{\sigma_p^2}\cdot S(w,k)dp, \quad (p>0),$$

$$S(w,k) = \exp\left[-\frac{w}{2}(1+k^2)\right] I_0\left[\frac{w}{2}(1-k^2)\right]. \tag{1}$$

Here $p = m\nu/2$; the relative velocity and the momentum spread depend on the contribution of the velocity vertical projection $V^2 = V_X^2 + V_Y^2$; $\sigma_p = mc\gamma\sqrt{\sigma_{X'}^2 + \sigma_{Y'}^2}$; $I_0(x)$ is the modified zero-order Bessel function.

At $k \to \infty$ the distribution function approaches the form corresponding to the one-dimensional collision case (see Fig.1) [4]:

$$f(p)dp = \frac{2}{\sqrt{\pi}\sigma_p}\exp\left(-\frac{p^2}{\sigma_p^2}\right)dp, \quad (p>0).$$

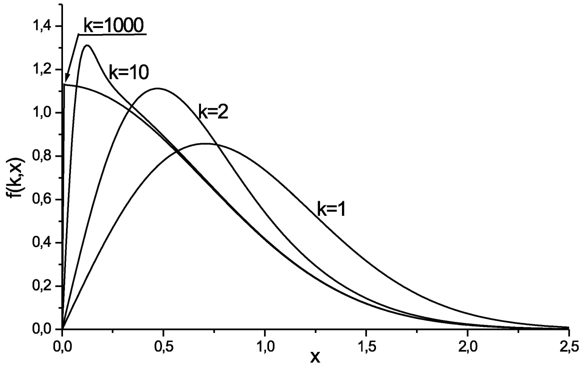


Figure 1: Distribution function $f(k, x)$, $x = p/\sigma_p$.

At $k \to 1$ (the strong betatron coupling case) the distribution becomes the two-dimensional Maxwell one with a characteristic dip near zero:

$$f(p) \propto p \cdot \exp\left(-p^2/\sigma_p\right).$$

Since the Møller cross section is proportional to $1/V^4$, the shift of the distribution function maximum to the area of larger $p$'s at smaller $k$'s should affect the intensity of the IBS processes.

The determinative process in the integrated Touschek effect is the multiple scattering provided that the latter contributes significantly to the energy diffusion in comparison with the synchrotron radiation (SR). Losses of particles (beam lifetime) due to a single intra-beam scattering depend on the steady beam dimensions determined by the total (SR + IBS) diffusion rate, radiative damping and betatron coupling. The betatron coupling in a traditional storage ring is so weak that the beam cross section tilt in relation to ideal axes can be neglected. In this case the coupling is characterized by the ratio of a vertical emittance to a radial one: $æ = \mathcal{E}_Y/\mathcal{E}_X$.

Let denote

$$u = (\sigma_\gamma/\gamma)^2 = u_Q + u_T$$

– the square of relative energy dispersion;

$$v = \mathcal{E}_X = v_Q + v_T$$

– the radial phase volume;

$$k = \sqrt{(1+\alpha_X^2)\beta_Y/(æ(1+\alpha_Y^2)\beta_X)};$$

$$\mathcal{H} = [\eta_X^2 + (\beta_X \eta_X' + \alpha_X \eta_X)^2]/\beta_X$$

– the function describing the excitation of radial betatron oscillations due to an instant change in a particle energy; $\sigma_S = R\alpha\sqrt{u}/Q_S$ – the longitudinal beam size, $Q_S$ – the synchrotron tune, $\alpha$ – the momentum compaction, $R$ – the machine radius; $\beta_Y$, $\alpha_Y$, $\beta_X$, $\alpha_X$, $\eta_X$, $\eta_X'$ are the amplitude and dispersion functions; the prime denotes the derivative with respect to the azimuthal coordinate. Here the indexes $Q$ and $T$ mark the contribution of synchrotron radiation (quantum diffusion) and Touschek effect, respectively. The diffusion coefficients of energy and radial emittance are found through the following sums:

$$D_u = D_u^Q + D_u^T,$$
$$D_v = D_v^Q + D_v^T,$$

where $D_u^Q$ and $D_v^Q$ are determined, for example, in [2]. The Touschek diffusion coefficients may be written as [6]

$$D_u^T = \frac{N r_0^2 c Q_S}{8\pi\gamma^3 R\alpha\sqrt{uv}} \left\langle \frac{\beta_X B(k, \chi_m)}{(\beta_X v + \eta_X^2 u)\sqrt{æ\beta_Y(1+\alpha_X^2)}} \right\rangle,$$

$$D_v^T = \frac{N r_0^2 c Q_S}{8\pi\gamma^3 R\alpha\sqrt{uv}} \left\langle \frac{\beta_X B(k, \chi_m)\mathcal{H}}{(\beta_X v + \eta_X^2 u)\sqrt{æ\beta_Y(1+\alpha_X^2)}} \right\rangle.$$

The angle brackets mean the averaging over the machine azimuth ($\vartheta$); $N$ is the number of particles in a bunch. The factor $B(k, \chi_m)$ is a modified diffusion rate function [4], which in contrast to the analogous one of the one-dimensional collision theory [2] depends on the coupling parameter $k$. In approximation of the classical lower limit on a scattering angle ($\chi_m = \chi_m^c$) is expressed as follows:

$$B(k, \chi_m^c) = \sqrt{\pi}k \int\limits_{\chi_m^c}^{\infty} \sqrt{\frac{1}{\chi}} \cdot \ln\left(\frac{\chi}{\chi_m^c}\right) \cdot S(\chi, k) d\chi; \tag{6}$$

The steady values of $u$ and $v$ are determined from the system of equations

$$u = u_Q + \frac{\tau_E}{2} D_u^T, \tag{3.1}$$

$$v = v_Q + \frac{\tau_X}{2} D_v^T, \tag{3.2}$$

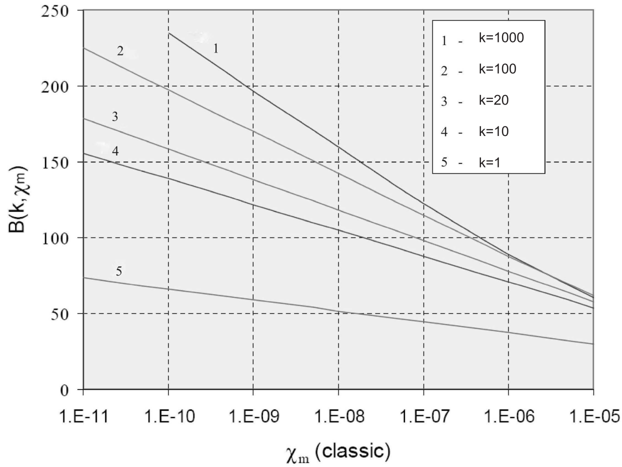


Figure 2: Diffusion factor vs $\chi_m$ in the approximation of classic lower limit on the scattering angle at some values of the coupling parameter $k$.

$\tau_E$ and $\tau_X$ are the damping times for synchrotron and radial betatron oscillations, respectively.

The loss rate (the inverse beam lifetime) due to Touschek processes may be found from [4]:

$$\frac{1}{\tau} = 2\sqrt{\pi} r_0^2 m^3 c^4 N \left\langle \frac{C(k,\varepsilon)}{\sigma_p A_p^2 V} \right\rangle . \tag{8}$$

Here

$$C(k,\varepsilon) = \sqrt{\pi} k \varepsilon \int_{\varepsilon}^{\infty} \chi^{-\frac{3}{2}} \left[ \frac{\chi}{\varepsilon} - \frac{1}{2} \ln \frac{\chi}{\varepsilon} - 1 \right] \cdot S(\chi, k) d\chi,$$

the modified "loss function", which depends on the parameters $k$ and $\varepsilon = A_p^2/\gamma\sigma_p^2$. $A_p$ is the "energy aperture" limiting the deviation of the longitudinal momentum value from the equilibrium one.

At $k \to \infty$ the functions $B(k, \chi_m)$ and $C(k, \varepsilon)$ take the form of the corresponding functions of the one-dimensional approximation [4].

# 4 Account of the quantum lower limit on scattering angle

Diffusion rate of particle energy in the process of multiple Touschek scattering is proportional to a quantity

$$\{\sigma V < p_\perp^2 >\}_{cm} = \int_{p^*}^{\infty} \sigma V < p_\perp^2 > f(k,p)dp,$$

where $p^* = p_m$, $\chi_m = \chi_m^c$ or $p^* = \sigma_p \sqrt{\chi_m^q} = \hbar/b_{max}$, $\chi_m = \chi_m^q$ depending on the applied approach for a minimal scattering angle (classical and quantum respectively). In this case the form of the expression for factor $B$ is slightly changed: under fulfilled condition (1), $\chi_m = \chi_m^q$ should be put in (6) to account the quantum limit on the momentum and the Coulomb logarithm in integrand should be written with the coefficient 1/2 (see (4)):

$$B(k,\chi_m^q) = \frac{1}{2}\sqrt{\pi}k \int_{\chi_m^q}^{\infty} \sqrt{\frac{1}{\chi}} \cdot \ln\left(\frac{\chi}{\chi_m^q}\right) \cdot S(\chi,k)d\chi. \tag{9}$$

# 5 Examples for CLIC and VEPP-4M

Results of the CLIC Damping Ring beam energy spread and emittance calculation in the classical approximation versus the coupling parameter are presented in Fig.3. The quantum approximation results are practically coincide with those shown in figure. The same situation takes a place also in the case of VEPP-4M[2].

The reason is that for the conventional storage ring as well as for the projected "nano-emittance" machine the integrals (6) and (9) have rather close values, in spite of a several order difference between $\chi_m^c$ and $\chi_m^q$. The dependence of diffusion factor $B$ on the parameter $\chi_m$ in classic and quantum approximations is plotted in Fig.4 and 5 for two ranges of $\chi_m$ variation. There are the points determined by the CLIC ring parameters in one of rhe ranges and the points for VEPP-4M are in another.

[2]It was examined [7], all the Touschek effect calculations performed in the paper are in a wholly satisfactory agreement with the results obtained by means of some other methods, particularly, by Piwinsky's one [8, 9]

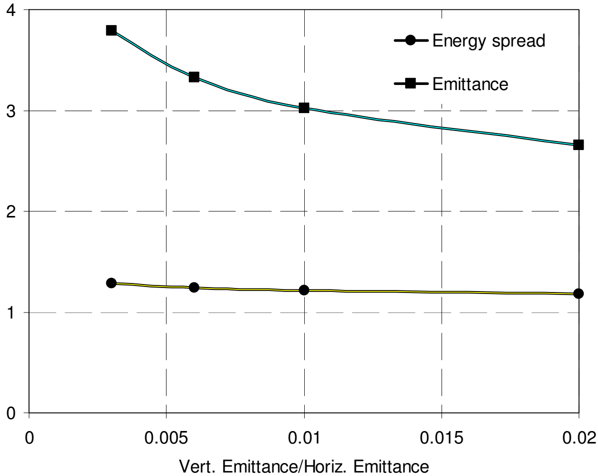


Figure 3: Relative increase of the beam emittance and energy spread due to IBS versus the ratio of vertical and horizontal beam emittances for the CLIC Damping Ring at $E = 2.24$ GeV and $I = 0.34$ mA/bunch

Suppose that the parameter $\chi_m^c$ for CLIC (Fig.4) is increased by a factor of $3 \cdot 10^4$ to a level of $\sim 10^{-2}$ owing to decreasing the vertical beam size (see Fig.6). At the same time, the point for CLIC on the curve of quantum approximation (Fig.4) moves through nine orders to a level of $\sim 10^{-2}$ in view of the relation $\chi_m^c/\chi_m^q \propto b_{max}$. In this case, it follows from an analysis of both $B(\chi_m)$ curves in Fig.6 that the diffusion factor of classic approximation is about 2 times larger than that of quantum approximation. A region where the mentioned curves becomes non linear in a plot with logarithmic absciss scale and where two values of $B$ can be distinguished is severely limited. In practice, this region is not available (see a remark in a first section of the paper).

# 6 Conclusion

It has been shown, that, formally, a quantum lower limit on scattering angle must be included in consideration of the IBS processes. Nevertheless, the CLIC and VEPP-4M Touschek calculation examples demonstrate that an account of the quantum limit of minimal scattering angle instead of the

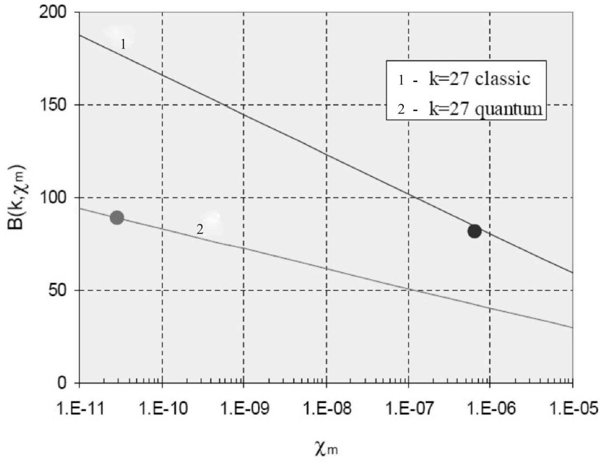


Figure 4: Diffusion factor $B$ as a function of the $\chi_m$ parameter in the classic and quantum approximations. The bold points on curves correspond to the CLIC parameters: 2.242 GeV beam energy, 0.34 mA beam current per bunch and 0.003 ratio of the vertical and radial emittances.

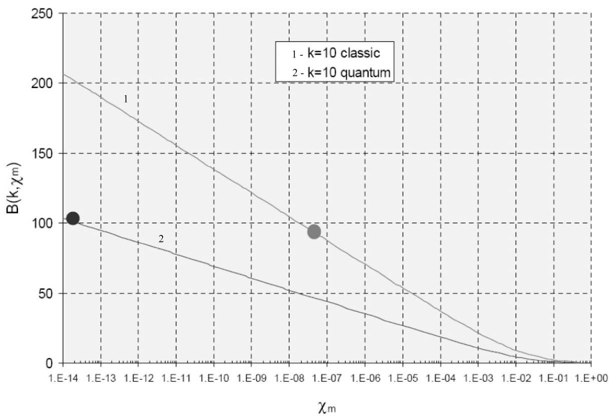


Figure 5: Diffusion factor for the VEPP-4M vs the $\chi_m$ parameter in the classic and quantum approximations at 0.9 GeV, 0.1 mA per bunch and 0.01 ratio of the emittances (the bold points on curves).

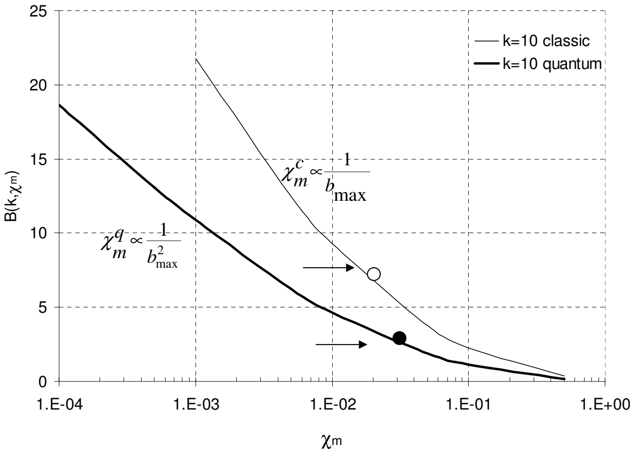


Figure 6: The transformed CLIC example: $b_{max}$ is $3 \cdot 10^4$ times decreased.

classical one does not change notably the numerical results. This conclusion seems to be true for all existing and designed storage rings. An apparent difference between results of classical and quantum approximation can be only in the non-realistic case of super-dense/super-thin beams.

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