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THE BOOTSTRAP CONDITIONS
FOR THE GLUON REGGEIZATION

The bootstrap conditions
for the gluon reggeization[1]

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Abstract

Compatibility of gluon Reggeization with s-channel unitarity requires the vertices of the Reggeon interactions to satisfy a series of bootstrap conditions. In order to derive, in the next-to-leading order (NLO), conditions related to the gluon production amplitudes, we calculate the s-channel discontinuities of these amplitudes and compare them with those required by the Reggeization. It turns out that these conditions include the so called strong bootstrap conditions for the kernel and for the impact factors of scattering particles, which were proposed earlier without derivation, and recently were proved to be satisfied. Besides this, there is a new bootstrap condition, which relates a number of Reggeon vertices and the gluon trajectory.

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1   Introduction

The gluon Reggeization [1, 2] is one of the remarkable properties of QCD, very important at high energy. In particular, the BFKL approach [3] to the description of QCD processes at large center of mass energy $\sqrt{s}$ and fixed (not increasing with s) momentum transfer $\sqrt{-t}$, $s \gg |t|$, is based on this property. Let us specify that, when using the notion of 'gluon Reggeization', we mean the Reggeized form which in the BFKL approach was assumed to be valid for amplitudes with colour octet quantum number and negative signature in exchange channels with fixed momentum transfer. In this paper only such amplitudes will be considered, even if it is not indicated directly. The assumed form for them will be presented explicitly in the next Section.

In the leading logarithmic approximation (LLA), where only the leading terms ($\alpha_S \ln s$)^n are resummed, the assumption concerns the amplitudes in the multi-Regge kinematics (MRK), i.e. at large (growing with s) invariant masses of any pair of produced particles and fixed transverse momenta. The Reggeized form of these amplitudes was proved [4], so that in the LLA the BFKL approach has a firm ground.

Now the approach is intensively developed in the next-to-leading approximation (NLA), when also the terms $\alpha_S(\alpha_S \ln s)^n$ are resummed (for references see, e.g. [5, 6]). In these calculations it is assumed that the Reggeized form of the MRK amplitudes remains valid in the NLA (with the gluon Regge trajectory and the Reggeon vertices taken in the NLO). Besides this, the assumption is extended to production amplitudes in the quasi-multi-Regge kinematics (QMRK), where a pair of produced particles has a fixed (not growing with s) invariant mass.

The hypothesis of the gluon Reggeization is extremely strong, even in the LLA, since amplitudes with any number of produced particles are expressed in terms of the gluon Regge trajectory and a small number of Reggeon vertices. It seems very hard to combine the gluon Reggeization with s-channel unitarity. Indeed, comparison of the amplitudes themselves with their discontinuities in invariant masses of subsets of produced particles, calculated with the help of s-channel unitarity, gives an infinite set of "bootstrap" relations. Nevertheless, it turns out that all of them can be fulfilled if the vertices and trajectory satisfy several bootstrap conditions. This fact is highly non

trivial. Fulfillment of all these conditions generates the basis on which the proof of Reggeization in the LLA was constructed [4]. An analogous proof can be constructed in the NLA as well [7]. The first step in the proof is to derive all bootstrap conditions; in the next step is has to be shown that these conditions are indeed satisfied.

The NLO bootstrap conditions imposed by the bootstrap relations for elastic amplitudes in the NLA were derived several yeas ago [5], and they were shown to be satisfied (see [6] for references). Recently the conditions following from the bootstrap requirement for the QMRK amplitudes were obtained, and their fulfillment was shown in [8]. Note that, since the QMRK in the unitarity relations leads to loss of large logarithms, the QMRK amplitudes used in the BFKL approach are expressed through the gluon trajectory and the Reggeon vertices taken in the leading order (LO).

In this paper we derive the NLO bootstrap conditions imposed by the requirement of the Reggeized form of the gluon production amplitudes in the MRK. We show that these conditions include the so called strong bootstrap conditions for the kernel and the impact factors of scattering particles, which were proposed earlier [9, 10] without derivation, and recently were proved to be satisfied [6]. Besides this, there is a new bootstrap condition, which entangles a number of the Reggeon vertices and the gluon trajectory.

The derivation is based on the calculation of s-channel discontinuities of the production amplitudes. It is shown that certain combinations of discontinuities in the NLA can be expressed through partial derivatives of the real parts of the amplitudes with respect to subenergies of produced particles. Starting from Reggeization these derivatives are found to have a well-defined form. By deriving these combinations of energy discontinuities from the calculated unitarity integrals, and by then comparing them with the form required by the Reggeization we obtain the bootstrap conditions.

The paper is organized in the following way. In the next Section the necessary definitions and notations are introduced, and the multi-Regge form of QCD amplitudes is presented. In Section 3 we find the connection between the s-channel discontinuities of production amplitudes and the partial derivatives of these amplitudes. Assuming the gluon Reggeization, this connection gives us the bootstrap relations which express the discontinuities through the amplitudes themselves. Section 4 is devoted to the calculation of the discontinuities. The bootstrap conditions on the Reggeon vertices and trajectory imposed by the bootstrap relations are derived in Section 5. Section 6 summarizes the obtained results.

Throughout the paper we work in the NLA, and all equations below should not be understood with higher accuracy.

2 Multi-Regge form of QCD amplitudes

To specify the hypothesis of the gluon Reggeization in the form, which was used in the BFKL approach, we briefly review the general structure of the MRK amplitudes. Multi-particle amplitudes depend upon more than one energy variables. Since the general bootstrap relations, which we will derive and discuss in this paper, involve discontinuities not only in the total energy but also in subenergies, we have to discuss the full analytic structure. At first sight, this structure appears somewhat complicated, but we will show that, in the color octet exchange channel in both LA and NLA, there are substantial simplifications. In particular, there are combinations of single discontinuities for which the expressions become simple.

We introduce the light cone momenta p_1 and p_2 related to momenta p_A and p_B of colliding particles A and B:

$$p_A = p_1 + \left(m_A^2/s\right) p_2 \ , \ \ p_B = p_2 + \left(m_B^2/s\right) p_1 \ , \ \ s = 2p_1 p_2 \simeq (p_A + p_B)^2 \ , \tag{2.1}$$

where s is supposed tending to infinity, and use the Sudakov decomposition of momenta in the form

$$p = \beta p_1 + \alpha p_2 + p_\perp \ , \ \ s\alpha\beta = p^2 - p_\perp^2 = p^2 + \vec{p}^{\,2} \ , \tag{2.2}$$

so that the vector sign is used for components of momenta transverse to the (p_A, p_B) plane. The transverse components are supposed to be limited (not growing with s).

Let us consider production of n particles in the process $A + B \rightarrow \tilde{A} + P_1 + ... + \tilde{B}$ in the MRK (see Fig.1). We admit all particles to have non zero masses, reserving the possibility to consider each of them as a compound state or as a group of particles. Of course, since we work here in QCD perturbation theory, our particles are actually partons, i.e. quarks and gluons.

Denoting momenta of the final particles k_i, $i = 0 \div n + 1$,

$$k_i = \beta_i p_1 + \alpha_i p_2 + k_{i\perp} \ , \ \ s\alpha_i\beta_i = k_i^2 - k_{i\perp}^2 = k_i^2 + \vec{k}_i^{\,2} \ , \tag{2.3}$$

we have in the MRK

$$\alpha_0 \ll \alpha_1 \cdots \ll \alpha_n \ll \alpha_{n+1} \ , \ \ \beta_{n+1} \ll \beta_n \cdots \ll \beta_1 \ll \beta_0 \ . \tag{2.4}$$

Eqs. (2.3) and (2.4) ensure that the squared invariant masses

$$s_i = (k_{i-1} + k_i)^2 \approx s\beta_{i-1}\alpha_i = \frac{\beta_{i-1}}{\beta_i}(k_i^2 + \vec{k}_i^{\,2}) \tag{2.5}$$
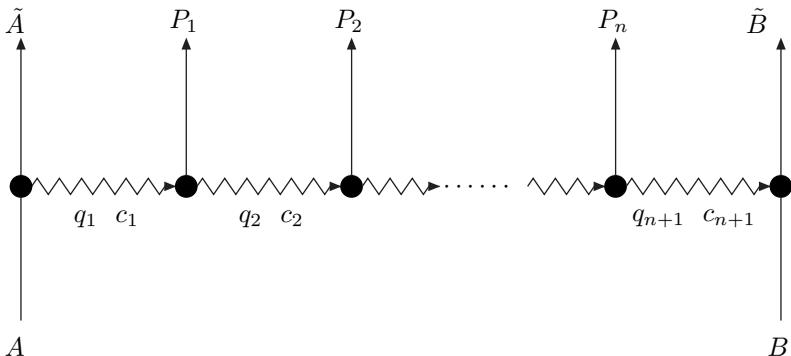
Figure 1: Schematic representation of the process $A + B \rightarrow \tilde{A} + P_1 + \cdots + \tilde{B}$ in the MRK. The zig-zag lines represent Reggeized gluon exchange; the black circles denote the Regeon vertices; q_i are the Reggeon momenta and c_i are the colour indices.

are large compared with the squared transverse momenta, which are supposed to be limited (not growing with s):

$$s_i \gg \vec{k}_i^2 \sim \mid t_i \mid = \mid q_i^2 \mid , \tag{2.6}$$

where

$$t_i = q_i^2 \approx q_{i\perp}^2 = -\vec{q}_i^{\,2} , \tag{2.7}$$

and the product of s_i is proportional to s:

$$\prod_{i=1}^{n+1} s_i = s \prod_{i=1}^{n} (\vec{k}_i^{\,2} + \vec{k}_i^{\,2}) . \tag{2.8}$$

The real part of the production amplitude can be written as (see [5] and references therein)

$$\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} = 2s\Gamma_{\tilde{A}A}^{c_1} \left[\prod_{i=1}^{n} \frac{1}{t_i} \gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1}) \left(\frac{s_i}{\sqrt{\vec{k}_{i-1}^2 \vec{k}_i^2}} \right)^{\omega(t_i)} \right] \tag{2.9}$$

$$\times \frac{1}{t_{n+1}} \left(\frac{s_{n+1}}{\sqrt{\vec{k}_n^2 \vec{k}_{n+1}^2}} \right)^{\omega(t_{n+1})} \Gamma_{\tilde{B}B}^{c_{n+1}} ,$$

where $\omega(t)$ is called gluon Regge trajectory (although actually the trajectory is $j(t) = 1 + \omega(t)$); $\Gamma_{\tilde{A}A}^c$ are the particle-particle-Reggeon (PPR) vertices (they are called also scattering vertices), i.e. the effective vertices for $A \to \tilde{A}$ transition due to interaction with Reggeized gluon; c is the colour index of this gluon; $\gamma_{c_i c_{i+1}}^{P_i}(q_i, q_{i+1})$ are the Reggeon-Reggeon-particle (RRP) vertices (we'll call them also production vertices), i.e. the effective vertices for production of particles P_i with momenta $k_i = q_i - q_{i+1}$ in collisions of Reggeons with momenta q_i and $-q_{i+1}$ and colour indices c_i and c_{i+1}. It is clear that in the MRK only gluons can be produced, so that all P_i must be gluons. All scattering vertices $\Gamma_{P'P}^c$, , the gluon production vertex γ_{ij}^G and the gluon Regge trajectory are known now in the NLO (see, e.g. [5],[6] for references), as it is required in the NLA (in this approximation, in (2.9) either one of the vertices or one of the trajectory functions must be taken in the NLO).

Note that in the amplitude $\mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n}$ there are contributions of various colour states and signatures in the t_i-channels, so that, strictly speaking, on the L.H.S. of (2.9) we should indicate that only contributions of a colour octet with negative signature are retained. But since in this paper we are interested only in such contributions, here and below we have omitted this indication, in order not to introduce unnecessary complications. Therefore, in the following it will be understood that the t-channel exchanges belong to colour octet and negative signature (i.e. the gluon quantum numbers). We remind the reader that in each order of perturbation theory amplitudes with the negative signature do dominate, owing to the cancellation of the leading logarithmic terms in amplitudes with the positive signature, which become pure imaginary in the LLA due to this cancellation.

In the NLA the multi-Regge form for production amplitudes is assumed to be valid also in the QMRK. Of course, in order to escape uncertainties, it is necessary to unambiguously separate two kinds of kinematics. We adopt the separation used in [5]: by definition, in the MRK all squared invariant masses of produced particles are larger than some subsidiary parameter s_Λ, which is taken to be sufficiently large, $s_\Lambda \gg |q_\perp^2|$, where $|q_\perp|$ is a typical value of transverse momenta. Actually the QMRK can be incorporated into the MRK if we introduce the notion of a 'jet'. By definition, we call a 'jet' either a single particle, or a system of two particles with its invariant mass being less than s_Λ. With this definition, instead of speaking of 'production of particles in the QMRK', we will speak about 'production of jets in the

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MRK', where one jet contains two particles. The QMRK amplitudes then have the same form (2.9) as in the MRK, with one of the vertices $\gamma^{P_i}_{c_i c_{i+1}}$ or $\Gamma^c_{\tilde{P} P}$ being substituted by a vertex for the production of a pair of particles. Note that because any two-particle jet in the unitarity relations leads to loss of a large logarithm, scales of energies in (2.9) are unimportant in the NLA for the QMRK amplitudes; moreover, the trajectory and the vertices are needed there only in LO accuracy. All vertices for the production of two-particle jets are known in this order (for references see, e.g. [8]).

In the following sections we will discuss energy discontinuities of multiparticle amplitudes. The multi-Regge form (2.9) does not show the full analytic structure, i.e. the dependence on the energy s and the different subenergies s_i. In order to explain the analytic content of (2.9), it may be useful to remind the reader of the general structure of inelastic production amplitudes in the multi-Regge limit [11, 12, 13]; for simplicity we will restrict ourselves to the $2 \to 3$ scattering process. In this case we have three energy variables s, s_1, and s_2, which, in the double Regge limit, are constrained by the condition $s_1 s_2 = s \vec{k}_1^2$. According to [11], the $2 \to 3$ amplitude in the double Regge limit can be written in the factorized form (2.9). However, the production vertex, in general, has a nontrivial phase structure and an analytic dependence upon the ratio $s_1 s_2 / s = \vec{k}_1^2$:

$$\mathcal{A} = \Gamma(t_1) \ |z_1|^{j_1} \ V^{\tau_1 \tau_2}(t_1, t_2, \eta) \ |z_2|^{j_2} \ \Gamma(t_2) \ , \qquad (2.10)$$

where, for simplicity, we have suppressed the labels A, A' etc.; $z_1 = \cos\theta_1 \approx 2s_1/(t_1 - t_2)$ and $z_2 = \cos\theta_2 \approx 2s_2/(t_2 - t_1)$ denote the cosinus of the cross channel scattering angles, $\eta \approx s_1 s_2 / s = \vec{k}_1^2$ has its origin in the Toller angle, τ_1, τ_2 are the signatures in the t_1 and t_2 channels, respectively, and $j_i = 1 + \omega_i$. We are interested in gluon quantum numbers, i.e. $\tau_1 = \tau_2 = -1$. The production vertex V has the general structure

$$
\begin{aligned}
V^{\tau_1 \tau_2}(t_1, t_2, \eta) &= |\eta|^{j_1} \left(e^{-i\pi j_1} + \tau_1 \right) \left(e^{-i\pi(j_2 - j_1)} + \tau_1 \tau_2 \right) \\
&\times \frac{1}{\omega_1 - \omega_2} \frac{1}{\omega_1} \ V_1^{\tau_1 \tau_2}(t_1, t_2, \eta) \\
&+ |\eta|^{j_2} \left(e^{-i\pi j_2} + \tau_2 \right) \left(e^{-i\pi(j_1 - j_2)} + \tau_1 \tau_2 \right) \\
&\times \frac{1}{\omega_2 - \omega_1} \frac{1}{\omega_2} \ V_2^{\tau_1 \tau_2}(t_1, t_2, \eta) \ .
\end{aligned}
\qquad (2.11)
$$

It is easy to see, by expansion in powers of g, that for $\tau_1 = \tau_2 = -1$ the amplitude is real-valued, i.e. the phase factors can be approximated by ± 2,

and by a suitable change of the energy scales and a redefinition of the vertex factors Γ and V_1, V_2, one arrives at the form (2.9). However, at this stage the information on the analytic structure has been lost. Instead, if we rewrite Eq. (2.10), with a redefinition of the vertex factors, as

$$
\begin{aligned}
\mathcal{A} &= s^{j_1} s_2^{j_2-j_1} \left(e^{-i\pi j_1} + \tau_1\right) \left(e^{-i\pi(j_2-j_1)} + \tau_1\tau_2\right) \ \Gamma(t_1) \\
&\times \ \frac{1}{\omega_1 - \omega_2} \frac{1}{\omega_1} \ V_1^{\tau_1\tau_2}(t_1, t_2, \eta) \ \Gamma(t_2) \\
&+ \ s^{j_2} s_1^{j_1-j_2} \left(e^{-i\pi j_2} + \tau_2\right) \left(e^{-i\pi(j_1-j_2)} + \tau_1\tau_2\right) \ \Gamma(t_1) \\
&\times \ \frac{1}{\omega_2 - \omega_1} \frac{1}{\omega_2} \ V_2^{\tau_1\tau_2}(t_1, t_2, \eta) \ \Gamma(t_2) \ , \quad\quad (2.12)
\end{aligned}
$$

we can associate the phase factors with the energy factors, and the dependence on the energy variables becomes transparent: the first term on the R.H.S. is a function of s and s_2 and has the usual right and left hand cut structure, the second one depends upon s and s_1. Consequently, the single discontinuity in s_1, $disc_{s_1}\mathcal{A}$, is obtained from the second term only and within the NLO can be written as

$$
disc_{s_1}\mathcal{A} = 2\pi i s^{j_2} s_1^{j_1-j_2} \left(e^{-i\pi\text{æ}_2} + \tau_2\right) \ \Gamma(t_1) \ \frac{1}{\omega_2} \ V_2^{\tau_1\tau_2}(t_1, t_2, \eta) \ \Gamma(t_2) \ ,
$$
$$(2.13)$$

whereas the discontinuity in s, $disc_s\mathcal{A}$, has contributions from both terms (this structure has been used to calculate separately the two vertex functions V_1 and V_2 in the LO in [12] and in the NLO in [13]). Moreover, the form (2.12) also allows to compute double discontinuities, e.g. $disc_s \ disc_{s_1}\mathcal{A}$. The most essential property of this representation is the absence of discontinuities in overlapping channels (i.e. $disc_{s_1} \ disc_{s_2}\mathcal{A} = 0$) and in the decoupling of singularities in s and s_1 or s and s_2. Now it is easy to show that at $\tau_1 = \tau_2 = -1$ the sum of two single discontinuities, $disc_{s_1} + disc_s$, in the approximation where the phase factors are taken to be equal to ± 2, is proportional to the full amplitude again:

$$
(disc_s + disc_{s_1}) \mathcal{A} = -\omega_1\pi i \ \mathcal{A}, \quad\quad (2.14)
$$

and we do not need to discuss the two pieces of the production vertex, V_1 and V_2, separately. An analogous discussion holds for the $2 \to 4$ production amplitude [12]: the $2 \to 4$ amplitude can be written as a sum of 5 terms, each of which has a simple analytic structure in a certain subset of energy variables. Single energy discontinuities pick out only a few of these terms,

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whereas certain sums of single discontinuities can be shown to be proportional to the full amplitude (see below).

In the following we will argue that the simple relation (2.14) (and its generalizations to the two gluon production) can be obtained directly from the inspection of QCD perturbation theory, and we then will use these equations for the derivation of bootstrap conditions.

3 Bootstrap relations

Let us first recall the derivation of the bootstrap relation for the elastic amplitude $\mathcal{A}_{AB}^{A'B'}$. In the limit of large s the radiative corrections of order α_S^k to this amplitude divided by s can depend on s only in the form $(\ln^n(-s) + \ln^n s)$ with $n \leq k$ (remember that we consider negative signature). With the NLO accuracy we can put

$$\frac{1}{-2\pi i} disc_s \left(\ln^n(-s) + \ln^n s\right) = \frac{1}{2}\frac{\partial}{\partial \ln s}\Re\left[\ln^n(-s) + \ln^n s\right] , \qquad (3.15)$$

where $disc_s$ denotes the s-channel discontinuity, and $\Re$ indicates the real part. Therefore in the NLA we have

$$\frac{1}{-2\pi i s} disc_s \mathcal{A}_{AB}^{A'B'} = \frac{1}{2}\frac{\partial}{\partial \ln s}\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'B'}\right] . \qquad (3.16)$$

Substituting the Reggeized form (2.9) into $\Re\,\mathcal{A}_{AB}^{A'B'}$ we obtain the bootstrap relation:

$$\frac{1}{-2\pi i s} disc_s\mathcal{A}_{AB}^{A'B'} = \frac{1}{2}\omega(t)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'B'}\right] . \qquad (3.17)$$

The important point is that the s-channel discontinuity on the L.H.S. of the relation (3.17) can be calculated by inserting the amplitude of the form (2.9) into the unitarity condition. Since the amplitudes are expressed through the vertices of the Reggeon interactions and the gluon Regge trajectory, the relation (3.17) imposes a series of restrictions on the vertices and trajectory, which have been formulated as bootstrap conditions for the color octet impact factors and for the BFKL kernel [5]. Note that in order to obtain these conditions in the NLO it is sufficient to retain, on both sides of the relation (3.17), only terms linear in $\ln s$.

The bootstrap relations for the one-gluon production amplitude $A + B \rightarrow A' + G + B'$ in the MRK are derived in a similar way, although this case is slightly more complicated. As discussed before, in this amplitude there

are three energy variables, $s_1 = (p_{A'} + k)^2$, $s_2 = (k + p_{B'})^2$, (k is the momentum of the produced gluon), and $s = (p_A + p_B)^2 \simeq (p_{A'} + p_{B'})^2$, and two momentum transfers, $t_1 = (p_A - p_{A'})^2$ and $t_2 = (p_B - p_{B'})^2$. Recall that we are considering only negative signatures in both t- channels, i.e. the part of the amplitude which is antisymmetric with respect to any of the substitutions $s_1 \to -s_1$ and $s_2 \to -s_2$. Due to the relation $s_1 s_2 = s\vec{k}^{\,2}$, which is fulfilled in all physical channels, this part is also antisymmetric with respect to $s \to -s$. In the MRK logarithms of all energy variables are considered to be large (i.e. on the same footing as $\ln s$ in the elastic amplitude). Let us consider the discontinuities of the amplitude in these variables (for brevity we sometimes call all these discontinuities s-channel ones). In general the determination of discontinuities of inelastic amplitudes is not simple. In particular, when calculating the discontinuity in one of the energy variables one needs to care on which edges of their cuts are the other variables. It is related to the existence of double discontinuities: e.g., the discontinuity in s can have, in turn, discontinuities in s_1 or s_2, so that the single discontinuities are not pure imaginary. Fortunately, for our purposes it is sufficient to consider only imaginary parts of the discontinuities, so that this complication is irrelevant. The second complication is that, because of the relation $s_1 s_2 = s\vec{k}^{\,2}$, the large logarithms compensate each other if they enter in the combination $\ln s_1 + \ln s_2 - \ln s = \ln \vec{k}^{\,2}$, so that radiative corrections of some fixed order in α_S can contain each of the large logarithms in any power through the dependence on $\vec{k}^{\,2}$. Therefore equalities like (3.16) are evidently absent. This difficulty can be overcome noticing that there are combinations of the discontinuities in which contributions related to dependence on $\vec{k}^{\,2}$ cancel. Indeed, the sums $(disc_{s_{1,2}} + disc_s)F(s_1 s_2/s)$ are zero (as well as any superposition of these sums). Note that just these sums, as already indicated in (2.14), are proportion within the NLA to the full amplitude. The most general form for the dependence of radiative corrections (to the amplitude divided by s) on the energy variables is a superposition of functions of $\vec{k}^{\,2}$ multiplied by powers of large logarithms. Then, by noticing that a discontinuity of a product of two functions is expressed through the discontinuities of these functions as

$$f_+ g_+ - f_- g_- = \frac{1}{2}(f_+ - f_-)(g_+ + g_-) + \frac{1}{2}(f_+ + f_-)(g_+ - g_-) , \quad (3.18)$$

we conclude that calculating $(disc_{s_{1,2}} + disc_s)\mathcal{A}_{AB}^{A'GB'}$ we can ignore the analytical properties of the functions of $\vec{k}^{\,2}$ and take their real parts.

Now, if the variables s_1, s_2 and s do not enter into the combination $s_1 s_2/s = \vec{k}^{\,2}$, they can appear in the radiative corrections of order α_S^k to

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the amplitude divided by s only as $\hat{\mathcal{S}} \ln^{n_1}(-s_1) \ln^{n_2}(-s_2) \ln^{n_3}(\pm s)$, with $n_1 + n_2 + n_3 = n \leq k$, where $\hat{\mathcal{S}}$ is the operator of symmetrization with respect to exchanges $s_1 \leftrightarrow -s_1$, $s \leftrightarrow -s$ and $s_2 \leftrightarrow -s_2$, $s \leftrightarrow -s$. Note that the terms containing products of $\ln(-s_i) \ln(s_i)$, where s_i can be s_1, s_2 or s, are forbidden, on the same ground as the terms containing $\ln(-s) \ln(s)$ are forbidden in the elastic amplitudes. In analogy to our treatment of the elastic case, in our actual calculation of energy discontinuities in the next section we will restrict ourselves to terms which, in the discontinuities, are linear in large logarithms, i.e. we need only $n \leq 2$. But the bootstrap relations can easily be derived without this restriction, as it is done below. Since in the NLO we need to keep only the first two leading total powers of n, calculating the imaginary part of the discontinuity in any of the variables s_1, s_2 or s we can take only real parts of logarithms of the other variables. It means that with our accuracy

$$\Re \left[\frac{1}{-2\pi i} disc_{s_i} \left(\hat{\mathcal{S}} \ln^{n_1}(-s_1) \ln^{n_2}(-s_2) \ln^{n_3}(\pm s) \right) \right]$$

$$= \frac{1}{2} \frac{\partial}{\partial \ln s_i} \Re \left[\hat{\mathcal{S}} \ln^{n_1}(-s_1) \ln^{n_2}(-s_2) \ln^{n_3}(\pm s) \right] \ , \tag{3.19}$$

where s_i can be s_1, s_2 or s, and the partial derivative is taken at fixed $s_j \neq s_i$.

Therefore we have, for example,

$$\Re \left[\frac{1}{-2\pi i s} (disc_{s_1} + disc_s) \mathcal{A}_{AB}^{A'GB'} \right]$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial \ln s_1} + \frac{\partial}{\partial \ln s} \right) \Re \left[\frac{1}{s} \mathcal{A}_{AB}^{A'GB'}(s_1, s_2, s) \right] \ , \tag{3.20}$$

where on the R.H.S. the first derivative is taken at fixed s_2 and s, and the second one at fixed s_1 and s_2. Using the equality

$$\left(\frac{\partial}{\partial \ln s_1} + \frac{\partial}{\partial \ln s} \right) f(s_1, s_2, s) = \frac{\partial}{\partial \ln s_1} f(s_1, s_2, \frac{s_1 s_2}{\vec{k}^{\,2}}) \ , \tag{3.21}$$

we arrive at

$$\Re \left[\frac{1}{-2\pi i s} (disc_{s_1} + disc_s) \mathcal{A}_{AB}^{A'GB'} \right] = \frac{1}{2} \frac{\partial}{\partial \ln s_1} \Re \left[\frac{1}{s} \mathcal{A}_{AB}^{A'GB'} \right] \ , \tag{3.22}$$

where on the R.H.S. the amplitude is considered as a function of s_1, s_2 and $\vec{k}^{\,2}$.

The requirement of the Reggeized form (2.9) of the amplitude on the R.H.S. gives us the bootstrap relation:

$$\Re \left[\frac{1}{-2\pi i} \left(disc_{s_1} + disc_s \right) \mathcal{A}_{AB}^{A'GB'} \right] = \frac{1}{2}\omega(t_1)\Re \, \mathcal{A}_{AB}^{A'GB'} , \qquad (3.23)$$

which coincides with (2.14). In the same way we obtain

$$\Re \left[\frac{1}{-2\pi i} \left(disc_{s_2} + disc_s \right) \mathcal{A}_{AB}^{A'GB'} \right] = \frac{1}{2}\omega(t_2)\Re \, \mathcal{A}_{AB}^{A'GB'} . \qquad (3.24)$$

It is known [5] that the bootstrap relation (3.17) for elastic scattering in the NLO leads to the bootstrap conditions for the impact factors and kernel. As we will see in Section 5, relations (3.24) and (3.23) (actually they are equivalent, so that the consideration of only one of them is sufficient) require the strong form of these conditions. Besides this, these relations give a completely new condition, which involves new "impact factors". This new condition appears in a "weak" form, analogous to the form of the conditions for the impact factors and kernel obtained from the elastic bootstrap. So the bootstrap relations for one-gluon production play a two-fold role: they strengthen the conditions imposed by the elastic bootstrap, and they give a new one. One might expect that the history will repeat itself with the addition of each next gluon in the final state. If this would be so, we had to consider bootstrap relations for production of arbitrary number of gluons, and we would obtain an infinite number of bootstrap conditions. Fortunately, history is repeated only partly: it turns out that already the bootstrap relations for two-gluon production only strengthen the form of the new condition imposed by the bootstrap for one-gluon production, and so the production of two gluons does not require new conditions. Therefore it is sufficient to consider the bootstrap relations for amplitudes of two-gluon production. They are derived in a way similar to the one-gluon case, although there are complications related to the larger number of energy variables and the larger number of combinations of energy variables which do not grow with s. Applying the notations of Section 2 to the production of two gluons, i.e. $n = 2$, we have six energy variables $s_{i,j} = (k_i + k_j)^2$, $j > i = 0 \div 2$ $(s_{0,3} = s)$, and two squared transverse momenta of produced gluons $\vec{k}_1^{\,2}, \vec{k}_2^{\,2}$ with four relations between them:

$$\frac{s_{0,1}s_{1,2}}{s_{0,2}} = \frac{s_{0,1}s_{1,3}}{s_{0,3}} = \vec{k}_1^{\,2}, \quad \frac{s_{1,2}s_{2,3}}{s_{1,3}} = \frac{s_{0,2}s_{2,3}}{s_{0,3}} = \vec{k}_2^{\,2} \qquad (3.25)$$

(actually only three of these relations are independent, since ratios of the first two terms in the first equality is identically equal to analogous ratios

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for the second equality). Similar to the production of one gluon, in order to avoid the need of going beyond the logarithmic dependence of the production amplitude on the energy variables we have to take definite combinations of discontinuities in these variables. The combinations must satisfy the requirement that for functions of $\vec{k}_1{}^2, \vec{k}_2{}^2$ they give zero. Let us take, for example, the sum of discontinuities in the channels $s_{2,3}, s_{1,3}$ and s. From (3.25) it is readily seen that

$$(disc_{s_{2,3}} + disc_{s_{1,3}} + disc_s)F(\vec{k}_1{}^2, \vec{k}_2{}^2) = 0 . \tag{3.26}$$

Using this and (3.18) we conclude that calculating the sum of the amplitude discontinuities in the channels $s_{2,3}, s_{1,3}$ and s we can ignore the analytical properties of the amplitude in the variables $\vec{k}_1{}^2$ and $\vec{k}_1{}^2$. Then, quite analogous to (3.20), we obtain

$$\Re\left[\frac{1}{-2\pi i s}\left(disc_{s_{2,3}} + disc_{s_{1,3}} + disc_s\right)\mathcal{A}_{AB}^{A'G_1G_2B'}\right]$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial \ln s_{2,3}} + \frac{\partial}{\partial \ln s_{1,3}} + \frac{\partial}{\partial \ln s}\right)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'G_1G_2B'}\right] , \tag{3.27}$$

where the amplitude on the R.H.S. is considered as a function of $s_{2,3}, s_{1,3}, s$ and $\vec{k}_1{}^2, \vec{k}_2{}^2$. Passing on to $s_{2,3} \equiv s_3$, $s_{1,2} \equiv s_3$, $s_{0,1} \equiv s_3$ and $\vec{k}_1{}^2$, $\vec{k}_2{}^2$ as independent variables, and using the requirement of the Reggeized form (2.9) we arrive at (cf. (3.24)):

$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_{2,3}} + disc_{s_{1,3}} + disc_s\right)\mathcal{A}_{AB}^{A'G_1G_2B'}\right] = \frac{1}{2}\omega(t_3)\Re\,\mathcal{A}_{AB}^{A'G_1G_2B'} . \tag{3.28}$$

It is one of three independent bootstrap relations for the two-gluon production amplitude. The other two relations,

$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_{0,1}} + disc_{s_{0,2}} + disc_s\right)\mathcal{A}_{AB}^{A'G_1G_2B'}\right] = \frac{1}{2}\omega(t_1)\Re\,\mathcal{A}_{AB}^{A'G_1G_2B'} ,$$

$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_{1,2}} + disc_{s_{1,3}} - disc_{s_{0,1}}\right)\mathcal{A}_{AB}^{A'G_1G_2B'}\right] \tag{3.29}$$

$$= \frac{1}{2}(\omega(t_2) - \omega(t_1))\Re\,\mathcal{A}_{AB}^{A'G_1G_2B'} ,$$

are obtained in the same way. We finally note that the relations (3.28) and (3.29) can be derived easily also from a representation analogous to (2.12).

4 Calculation of s-channel discontinuities
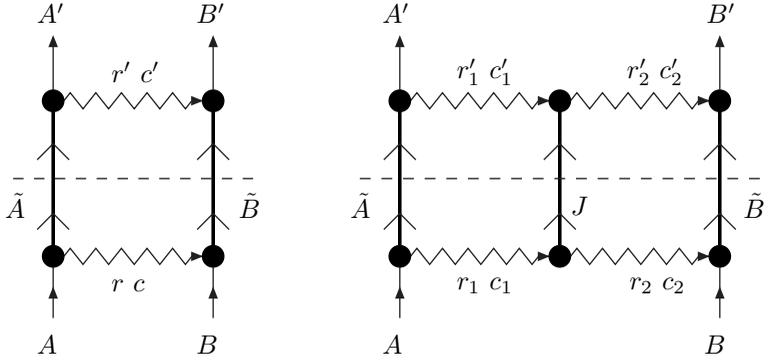
4.1 Discontinuity of elastic amplitudes

It is worth-while to start with the elastic amplitude $\mathcal{A}_{AB}^{A'B'}$, at least in order to introduce the notions of impact factors and of the BFKL kernel (of course, in the colour octet channel). After this the calculation of the discontinuity of the elastic amplitude can be generalized to inelastic amplitudes in a relatively simple way.

The s-channel discontinuity is calculated with the help of the unitarity relation, using on the R.H.S. of this relation the Reggeized form (2.9) of the amplitudes. In Eq. (2.9) only the real part of the amplitude is given, and we have omitted the symbol of the real part. In order to simplify the notations, in the following we shall omit this symbol without further notice. Fortunately, within the NLO accuracy only real parts are important for the calculation of the discontinuity. We will content ourselves with terms of the zeroth and first power of $\ln s$ in the discontinuity, since, as we shall see, this is sufficient for the derivation of the bootstrap conditions. In the NLA these terms can come from intermediate states with two, three and four jets (see Fig. 2). Indeed, in the LLA each additional gluon in the intermediate state gives a large logarithm, therefore for n jets in the intermediate state there are at least $n-2$ logarithms. Since we work in the NLA, in each order of perturbation theory in g^2 we have to retain only the leading and next-to leading terms. So it is clear that the five-jet contribution is irrelevant for us.

The unitarity relation with two jets in the intermediate state gives (see Fig. 2a))

$$\frac{1}{-2\pi i} disc_s^{(2\Lambda)} \mathcal{A}_{AB}^{A'B'} = -\frac{1}{2\pi} \sum_{\tilde{A}\tilde{B}} \mathcal{A}_{AB}^{\tilde{A}\tilde{B}}\, \mathcal{A}_{\tilde{A}\tilde{B}}^{A'B'}\, d\rho_{\tilde{A}\tilde{B}}\ , \qquad (4.30)$$

where the superscript Λ indicates that the squared invariant mass of the jet is less than s_Λ; $\tilde{A}$ and $\tilde{B}$ are the jets with momenta $p_{\tilde{A}}$ and $p_{\tilde{B}}$, respectively, $d\rho_{\tilde{A}\tilde{B}}$ denotes their phase space element. Amplitudes on the R.H.S. of Eq. (4.30) are of the form (2.9). Remember that it is our aim to obtain bootstrap conditions by inserting the discontinuity into the relation (3.17). To compare the left and right hand sides of the relation (3.17) we need to use the same scale of energy on both sides. We choose the scale $-t = -q_\perp^2$, which is natural for the amplitude $\mathcal{A}_{AB}^{A'B'}$, but not for the amplitudes $\mathcal{A}_{AB}^{\tilde{A}\tilde{B}}$ and $\mathcal{A}_{\tilde{A}\tilde{B}}^{A'B'}$. Passing to this scale in Eq. (2.9), we obtain
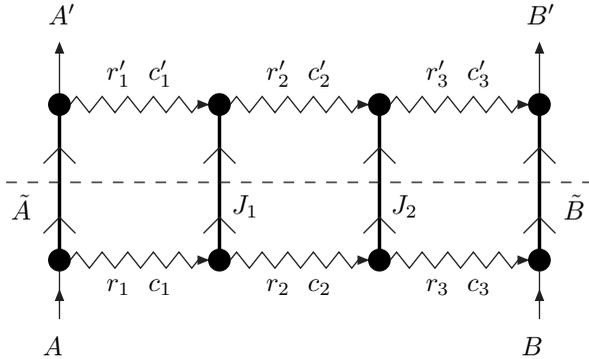
Figure 2: Schematic representation of the contributions to the s-channel discontinuity of the elastic amplitude $\mathcal{A}_{AB}^{A'B'}$: a) two-jet, b) three-jet and c) four-jet.

$$\mathcal{A}^{\tilde{A}\tilde{B}}_{AB} = \frac{2s}{r^2} \, \Gamma^c_{\tilde{A}A}(q_t) \left(\frac{s}{q_t^2} \right)^{\omega(r^2)} \Gamma^c_{\tilde{B}B}(q_t) \, , \qquad (4.31)$$

$$\mathcal{A}^{A'B'}_{\tilde{A}\tilde{B}} = \frac{2s}{r'^2} \, \Gamma^c_{A'\tilde{A}}(q_t) \left(\frac{s}{q_t^2} \right)^{\omega(r'^2)} \Gamma^c_{B'\tilde{B}}(q_t) \, .$$

Here $r = p_A - p_{\tilde{A}}$ and $r' = q - r$ are the transferred momenta. Remind that $q = p_A - p_{A'}$ and we can put $q = q_\perp$. Since in the unitarity relation essential transverse momenta are limited (i.e. they do not grow with s), we can put also $r = r_\perp$, $r' = r'_\perp$. We have introduced the notation $p_t \equiv \sqrt{-p_\perp^2}$ for any p, and the scattering vertices at the scale q_t. For any transition $P \to \tilde{P}$ with momentum transfer p these vertices are defined as

$$\Gamma^c_{\tilde{P}P}(q_t) = \Gamma^c_{\tilde{P}P} \left(\frac{q_t}{p_t} \right)^{\omega(p^2)} . \qquad (4.32)$$

In order to write formulas in a simple form we do not perform an explicit expansion in the coupling g^2. In practice, however, the expansion is always assumed, and only the NLA accuracy is needed. Therefore we have

$$\Gamma^c_{\tilde{P}P}(q_t) = \Gamma^c_{\tilde{P}P} \left(\frac{q_t}{p_t} \right)^{\omega(p^2)} = \Gamma^c_{\tilde{P}P} \left[1 + \omega(p^2) \ln \left(\frac{q_t}{p_t} \right) \right]$$

$$= \Gamma^c_{\tilde{P}P} + \Gamma^{c(B)}_{\tilde{P}P} \omega(p^2) \ln \left(\frac{q_t}{p_t} \right) . \qquad (4.33)$$

Furthermore, since we are going to retain only the zeroth and the first power of $\ln s$ we can put

$$\left(\frac{s}{q_t^2} \right)^{\omega(r^2)} \left(\frac{s}{q_t^2} \right)^{\omega(r'^2)} = 1 + \Omega Y \, , \qquad (4.34)$$

where we have introduced the notations

$$\Omega = \omega(r^2) + \omega(r'^2) \, , \quad Y = \ln \left(\frac{s}{q_t^2} \right) \qquad (4.35)$$

used below.

It is convenient to write down the phase space element $d\phi_J$ for a jet J with total momentum k_J consisting of particles with momenta l_i:

$$d\phi_J = \frac{dk_J^2}{2\pi} \theta(s_\Lambda - k_J^2)(2\pi)^D \delta^D(k_J - \Sigma_i l_i) \prod_i \frac{d^{D-1}l_i}{(2\pi)^{D-1} 2\epsilon_i} \, . \qquad (4.36)$$

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For the phase space element for two produced jets $d\rho_{\tilde{A}\tilde{B}}$ in Eq. (4.30) we have

$$d\rho_{\tilde{A}\tilde{B}} = d\phi_{\tilde{A}}d\phi_{\tilde{B}}\frac{d^{D-2}r_{\perp}}{2s(2\pi)^{D-2}} \ . \tag{4.37}$$

In Eq. (4.30) the sum extends over colours and polarizations of the intermediate particles. It is performed independently for each jet. Note that the projection onto the antisymmetric colour octet state in the t-channel is always understood. Making this projection explicitly, we define the non-subtracted impact factors as

$$\Phi_{A'A}^{i(\Lambda)}(r_{\perp}, r'_{\perp}) = i\frac{f^{icc'}}{N_c}\sum_{\tilde{A}}\int\Gamma_{\tilde{A}A}^{c}(q_t)\Gamma_{A'\tilde{A}}^{c'}(q_t)d\phi_{\tilde{A}} \ ,$$

$$\Phi_{B'B}^{i(\Lambda)}(-r_{\perp}, -r'_{\perp}) = i\frac{f^{icc'}}{N_c}\sum_{\tilde{B}}\int\Gamma_{\tilde{B}B}^{c}(q_t)\Gamma_{B'\tilde{B}}^{c'}(q_t)d\phi_{\tilde{B}} \ . \tag{4.38}$$

In order to simplify the representation of our results for the discontinuities it is convenient to introduce operators in the transverse momentum representation. From the t-channel point of view we have to consider two interacting Reggeized gluons (see Fig. 2) with "coordinates" $\vec{r}$ and $\vec{q} - \vec{r}$ in the transverse momentum space ($\vec{q}$ is the total transverse momentum in the t-channel). Let us introduce $\hat{\vec{r}}$ as the operator of "coordinate" of one of the Reggeized gluons in the transverse momentum space: $\hat{\vec{r}}|\vec{q}_i\rangle = \vec{q}_i|\vec{q}_i\rangle$. The total transverse momentum $\vec{q}$ is considered as a c-number. With the normalization $\langle\vec{q}_1|\vec{q}_2\rangle = \vec{q}_1^{\,2}(\vec{q}_1 - \vec{q})^{\,2}\delta^{(D-2)}(\vec{q}_1 - \vec{q}_2)$ we define

$$\langle\Psi_2|\Psi_1\rangle = \int\frac{d^{D-2}r}{\vec{r}^{\,2}(\vec{q}-\vec{r})^2}\langle\Psi_2|\vec{r}\rangle\langle\vec{r}|\Psi_1\rangle \ . \tag{4.39}$$

In this formalism the impact factors $\Phi_{A'A}$ and $\Phi_{B'B}$ appear as the "wave functions" of the t-channel states $\langle A'A|$ and $|B'B\rangle$, respectively, and the BFKL kernel $\mathcal{K}(\vec{r}_2, \vec{r}_1, \vec{q})$ as the "matrix element" $\langle\vec{r}_1|\hat{\mathcal{K}}|\vec{r}_2\rangle$. Since the t-channel is assumed to be in a colour octet state, the impact factors carry a colour index. For simplicity, we have omitted this index, and in the following we often do the same whenever possible (the same applies to colour indices of Reggeon vertices). It is worthwhile to mention that impact factors are assumed to be symmetric under the exchange of the two gluon momenta: $r_{\perp} \leftrightarrow r'_{\perp}$. For the quark and gluon impact factors this property is fulfilled automatically. It is not so in more complicated cases; therefore, in the general

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case, symmetrization with respect to the exchange $r_\perp \leftrightarrow r'_\perp$ on the R.H.S. of Eq. (4.38) is understood.

With these definitions we can write

$$\frac{1}{-2\pi i} disc_s^{(2\Lambda)} \mathcal{A}_{AB}^{A'B'} = \frac{2sN_c}{(2\pi)^{D-1}} \langle A'A^{(\Lambda)}|1 + \hat{\Omega}Y|B'B^{(\Lambda)}\rangle \ , \qquad (4.40)$$

where $\hat{\Omega} = \omega(\hat{r}_\perp^2) + \omega((q - \hat{r})_\perp^2)$ and the states $\langle A'A^{(\Lambda)}|$, and $|B'B^{(\Lambda)}\rangle$ are determined by the relations

$$\langle A'A^{(\Lambda)}|r_\perp\rangle = \Phi_{A'A}^{(\Lambda)}(r_\perp, r'_\perp) \ , \quad \langle r_\perp|B'B^{(\Lambda)}\rangle = \Phi_{B'B}^{(\Lambda)}(-r_\perp, -r'_\perp) \ . \quad (4.41)$$

Note that, strictly speaking, in Eq. (4.40) the use of the equality sign is incorrect, because Y^2-terms are omitted, despite the fact that Y-terms with coefficients of the same order in g^2 are kept. The Y^2-terms are omitted because we want to compare the first two terms of the expansion in Y on both sides of the relation (3.17). In the following we also shall use the equality sign in this sense for discontinuities.

Let us turn to the contribution of intermediate states with three jets (see Fig. 2b):

$$\frac{1}{-2\pi i} disc_s^{(3\Lambda)} \mathcal{A}_{AB}^{A'B'} = -\frac{1}{2\pi} \sum_{\tilde{A}J\tilde{B}} \mathcal{A}_{AB}^{\tilde{A}J\tilde{B}} \ \mathcal{A}_{\tilde{A}J\tilde{B}}^{A'B'} d\rho_{\tilde{A}J\tilde{B}} \ , \qquad (4.42)$$

where $\tilde{A}$ and $\tilde{B}$ are the jets produced in the fragmentation regions of the particles A and B, respectively, and J is the jet with total momentum k produced in the central region (it can be a single gluon, or two gluons, or a $q\bar{q}$ pair). The amplitudes in Eq. (4.42) have the form (2.9)). Passing to the scale q_t at the PPR vertices we have

$$\mathcal{A}_{AB}^{\tilde{A}J\tilde{B}} = 2s \ \Gamma_{\tilde{A}A}^a(q_t)\frac{1}{r_1^2} \left(\frac{s_1}{q_t k_t}\right)^{\omega(r_1^2)} \gamma_{ab}^J(r_1, r_2)\frac{1}{r_2^2} \left(\frac{s_2}{k_t q_t}\right)^{\omega(r_2^2)} \Gamma_{\tilde{B}B}^b(q_t) \ ,$$

$$\mathcal{A}_{\tilde{A}J\tilde{B}}^{A'B'} = 2s \ \Gamma_{A'\tilde{A}}^a(q_t)\frac{1}{r_1'^2} \left(\frac{s_1}{q_t k_t}\right)^{\omega(r_1'^2)} \left(\gamma_{ab}^J(r_1', r_2')\right)^* \frac{1}{r_2'^2} \left(\frac{s_2}{k_t q_t}\right)^{\omega(r_2'^2)} \Gamma_{B'\tilde{B}}^b(q_t) \ ,$$
$$(4.43)$$

where $r_1 = p_A - p_{\tilde{A}}$, $r_2 = p_{\tilde{B}} - p_B$ and $r'_{1,2} = q - r_{1,2}$. Note that $q = p_A - p_{A'}$ and that we can put $q = q_\perp$. The Sudakov decomposition for the other momenta can be written as

$$k = \beta p_1 + \alpha p_2 + k_\perp \ , \quad s\alpha\beta = k_t^2 + k^2 \ , \quad r_1 = \beta p_1 + q_{1\perp} \ , \quad r_2 = -\alpha p_2 + q_{2\perp} \ , \qquad (4.44)$$

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so that $s_1 = (k_t^2 + k^2)/\beta$ and $s_2 = s\beta$. The phase space element has the form

$$d\rho_{\bar{A}J\bar{B}} = d\phi_{\bar{A}} \frac{d\phi_J}{2(2\pi)^{D-1}} d\phi_{\bar{B}} \frac{d^{D-2}r_{1\perp}}{2s(2\pi)^{D-2}} d^{D-2}r_{2\perp} \frac{d\beta}{\beta} , \qquad (4.45)$$

and the limits of integration over β are defined by the conditions $s_2 \geq s_\Lambda$, $s_1 \geq s_\Lambda$. In the NLA we can put

$$\frac{k_t^2}{s_\Lambda} \geq \beta \geq \frac{s_\Lambda}{s} , \qquad (4.46)$$

since an exact value of the limits of integration is important only when the jet J consists of a single gluon. Passing to the integration variable $y = \ln\left(\beta s/(q_t k_t)\right)$, we have

$$Y - y_\Lambda \geq y \geq y_\Lambda , \quad y_\Lambda \equiv \ln\left(\frac{s_\Lambda}{q_t k_t}\right) . \qquad (4.47)$$

Again, in Eq. (4.42) the sums are performed independently for each jet. The un-subtracted contribution to the colour octet kernel from the production of real particles is defined as

$$\langle r_{1\perp}|\hat{\mathcal{K}}_r^{(\Lambda)}|r_{2\perp}\rangle = \mathcal{K}_r^{(\Lambda)}(r_{1\perp}, r_{2\perp}; q_\perp) \qquad (4.48)$$
$$= \frac{f_{c_1 c_2' c} f_{c_2 c_2' c}}{N_c(N_c^2 - 1)} \sum_J \int \gamma_{c_1 c_2}^J (q_1, q_2) \left(\gamma_{c_1' c_2'}^J (r_1', r_2')\right)^* \frac{d\phi_J}{2(2\pi)^{D-1}}.$$

Here and below the subscript r denotes the contribution coming from real particle production.

Since the kernel depends on transverse momenta only, the dependence on β in Eq. (4.42) is contained only in the phase space element (4.45) and in the Regge factors of the amplitudes (4.43). In our approximation, the product of these factors is reduced to

$$\left(\frac{s_1}{q_t k_t}\right)^{\omega(r_1^2) + \omega(r_1'^2)} \left(\frac{s_2}{k_t q_t}\right)^{\omega(r_2^2) + \omega(r_2'^2)} = 1 + \Omega_1(Y - y) + \Omega_2 y , \qquad (4.49)$$

where $\Omega_i = \omega(r_i^2) + \omega(r_i'^2)$. With NLO accuracy we get

$$\int_{y_\Lambda}^{Y - y_\Lambda} dy \, (1 + \Omega_1(Y - y) + \Omega_2 y) = Y + (\Omega_1 + \Omega_2)\frac{Y^2}{2} - y_\Lambda(1 + \Omega_1 Y) - y_\Lambda(1 + \Omega_2 Y).$$
$$(4.50)$$

Omitting the irrelevant Y^2-terms in Eq. (4.50) we obtain

$$\frac{1}{-2\pi i} disc_s^{(3\Lambda)} \mathcal{A}_{AB}^{A'B'} \tag{4.51}$$

$$= \frac{2sN_c}{(2\pi)^{D-1}} \int \frac{d^{D-2}r_1}{r_{1\perp}^2 r_{1\perp}'^2} \frac{d^{D-2}r_2}{r_{2\perp}^2 r_{2\perp}'^2} \Phi_{A'A}^{i(\Lambda)}(r_{1\perp}, r_{1\perp}') \mathcal{K}_r^{(\Lambda)}(r_{1\perp}, r_{2\perp}; q_\perp)$$

$$\times [Y - y_\Lambda(1 + \Omega_1 Y) - y_\Lambda(1 + \Omega_2 Y)] \Phi_{B'B}^{i(\Lambda)}(-r_{2\perp}, -r_{2\perp}') .$$

Taking into account that the terms with y_Λ are sub-leading, so that in the NLA they have to go together with the LO impact factors and kernel, we see that their contributions can be combined with the terms of Eq. (4.40) leading to a subtraction in the impact factors. This subtraction is necessary in order to make the impact factors independent on s_Λ. Note that, as usually, any subtraction is "scheme dependent". In fact, here we have fixed already the scheme by the choice of the energy scale q_t. So we define

$$\Phi_{A'A}^i(r_\perp, r_\perp') = \Phi_{A'A}^{i(\Lambda)}(r_\perp; r_\perp') - \int \frac{d^{D-2}r_1}{r_{1\perp}^2 r_{1\perp}'^2} \Phi_{A'A}^{i(B)}(r_{1\perp}, r_{1\perp}')$$

$$\times \mathcal{K}_r^{(B)}(r_{1\perp}, r_\perp; q_\perp) \ln\left(\frac{s_\Lambda}{(r_1 - r)_t q_t}\right) , \tag{4.52}$$

where $r_{1\perp}' = q_\perp - r_{1\perp}$, and the superscript (B) refers to the Born approximation. In this approximation the impact factor and the kernel are given by Eqs. (4.38) and (4.48), respectively, where the jets consist of one particle (for the kernel it is a gluon). Making use of the hermiticity property of the vertices $\left(\Gamma_{\tilde{P}P'}^i\right)^* = \Gamma_{P'\tilde{P}}^i$ one can see from Eqs. (4.38) and (4.52) that, apart from the coefficient $i/\sqrt{N_c}$, our definition of impact factors coincides with the one given in Ref. [5] for the case $s_0 = q_t^2$.

Since we work in the LLA, NLO terms in the discontinuities can appear only once. Therefore, with the definition (4.52) the sum of Eqs. (4.40) and (4.51) can be written as

$$\frac{1}{-2\pi i} disc_s^{(2\Lambda+3\Lambda)} \mathcal{A}_{AB}^{A'B'} \tag{4.53}$$

$$= \frac{2sN_c}{(2\pi)^{D-1}} \left(\langle A'A|1 + \hat{\Omega}Y|B'B\rangle + \langle A'A^{(\Lambda)}|\hat{\mathcal{K}}_r^{(\Lambda)}Y|B'B^{(\Lambda)}\rangle \right) .$$

Consider now four jets in the unitarity relation. Since the leading contributions from such intermediate states contain at least Y^2, we need to take

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only the sub-leading piece, coming from the integration over rapidities of jets produced in the central regions. Actually these jets can contain only one gluon each; the amplitudes entering in the unitarity condition have the form (2.9), with the Regge factors being omitted, and the vertices being taken in the Born approximation. After the summation over the discrete quantum numbers of the produced particles we recover in the discontinuity the Born impact factors and kernels. The integration over the rapidities of the produced gluons with momenta k_1 and k_2 is performed by taking into account the limitations

$$(p_{\tilde{A}} + k_1)^2 = \frac{k_{1t}^2}{\beta_1} \geq s_\Lambda \ , \quad (k_1 + k_2)^2 = \frac{\beta_1 k_{2t}^2}{\beta_2} \geq s_\Lambda \ , \quad (k_2 + p_{\tilde{B}})^2 = s\beta_2 \geq s_\Lambda \ .$$
(4.54)

With the NLO accuracy the result of the integration is

$$\int \frac{d\beta_1}{\beta_1} \frac{d\beta_2}{\beta_2} = \frac{Y^2}{2} - Y(y_{1\Lambda} + y_{2\Lambda} + y_{3\Lambda}) \ ,$$
(4.55)

where

$$y_{1\Lambda} = \ln\left(\frac{s_\Lambda}{q_t k_{1t}}\right) \ , \quad y_{2\Lambda} = \ln\left(\frac{s_\Lambda}{k_{1t} k_{2t}}\right) \ , \quad y_{3\Lambda} = \ln\left(\frac{s_\Lambda}{k_{2t} q_t}\right) \ .$$
(4.56)

The first term is irrelevant for us; the terms with $y_{i\Lambda}$ are necessary for subtractions in the second term in Eq. (4.53): here the first one and the last one serve for subtractions in the impact factors of $A \to A'$ and $B \to B'$ transitions, respectively, the second one in the kernel. After the subtraction the kernel becomes

$$\mathcal{K}_r(q_{1\perp}, {}_{2\perp}; q_\perp) = \mathcal{K}_r^{(\Lambda)}(q_{1\perp}, q_{2\perp}; q_\perp) - \int \frac{d^{D-2}r}{r_\perp^2 (q - r)_\perp^2} \mathcal{K}_r^{(B)}(q_{1\perp}, r_\perp; q_\perp)$$

$$\times \mathcal{K}_r^{(B)}(r_\perp, q_{2\perp}; q_\perp) \ln\left(\frac{s_\Lambda}{(q_1 - r)_t (q_2 - r)_t}\right) \ .$$
(4.57)

It is worthwhile to note that the kernel is symmetric in its first two arguments.

As a result we see that the discontinuity can be written as

$$\frac{1}{-2\pi i} disc_s \mathcal{A}_{AB}^{A'B'} = \frac{2sN_c}{(2\pi)^{D-1}} \langle A'A | 1 + \hat{\mathcal{K}} Y | B'B \rangle \ ,$$
(4.58)

where

$$\hat{\mathcal{K}} = \hat{\mathcal{K}}_r + \hat{\Omega}$$
(4.59)

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is the total colour octet kernel. As stated before, we keep only the first two terms of the expansion in Y. Moreover, since we work in the NLA, only leading and next-to-leading orders in the coefficients of the expansion are under controll.

4.2 Discontinuities of one-gluon production amplitudes

We now turn to the amplitude $\mathcal{A}_{AB}^{A'GB'}$ for the production of a gluon G with momentum k in the MRK, for which we have

$$k = q_1 - q_2 \ , \quad q_1 = p_A - p_A' \ , \quad q_2 = p_B' - p_B \ ,$$

$$k = \beta p_1 + \alpha p_2 + k_\perp \ , \quad s\alpha\beta = -k_\perp^2 = k_t^2 \ , \quad \alpha \ll 1 \ ; \quad \beta \ll 1 \ . \tag{4.60}$$

Consequently we obtain

$$s_1 \equiv (p_{A'} + k)^2 = s\alpha = \frac{k_t^2}{\beta} \ , \quad s_2 \equiv (p_{B'} + k)^2 = s\beta \ , \tag{4.61}$$

and can put

$$q_1 = \beta p_1 + q_{1\perp} \ , \quad q_2 = -\alpha p_2 + q_{2\perp} \ , \quad t_{1,2} = q_{1,2}^2 = q_{1,2\perp}^2 = -\vec{q}_{1,2}^{\,2} \ . \tag{4.62}$$

The Reggeized form of the production amplitude (2.9) now reads

$$\mathcal{A}_{AB}^{A'GB'} = 2s\Gamma_{A'A}^a \left(\frac{s_1}{k_t q_{1t}} \right)^{\omega(t_1)} \gamma_{ab}^G(q_1, q_2) \left(\frac{s_2}{k_t q_{2t}} \right)^{\omega(t_2)} \Gamma_{B'B}^b \ . \tag{4.63}$$

The bootstrap relations (3.23) and (3.24) contain discontinuities in s_1, s_2 and s. For brevity, we use the term discontinuities, though actually we need and will calculate only their imaginary parts. In analogy with the elastic case they are calculated with the help of unitarity relations. In the expressions for the discontinuities, given by unitarity relations and calculated in NLA, only real parts of amplitudes are important, so we can use the form (2.9). Similar to the elastic case, in order to compare the left and right sides of the relations (3.23) and (3.24) we need to use the same scales of energies on both sides. The difference is that now we have two independent energy variables (remember that $s_1 s_2 = s k_t^2$) and two scales. Their choice is not unique; we prefer to use variables and scales shown in Eq. (4.63), i.e. s_1 and s_2 and the scales for them $q_{1t} k_t$ and $k_t q_{2t}$ respectively. We will use the notations $Y_1 = \ln(s_1/(q_{1t}k_t))$ and $Y_2 = \ln(s_2/(k_t q_{2t}))$ and, in analogy with the elastic case, we will restrict ourselves to terms that are linear in these variables.

s_2-channel discontinuity

The discontinuity can be found without large efforts, since its calculation is very similar to the calculation of the discontinuity of elastic amplitudes performed above. Indeed, in Eq. (4.63) the vertex $\Gamma^a_{A'A}$ and the energy factor of the s_1-channel are factorized, so that we may say we need to calculate the discontinuity of the amplitude of the process $R_1 + B \to G + B'$, where the Reggeon R_1 with momentum q_1 (see Fig. 3) plays the role of an incoming particle.
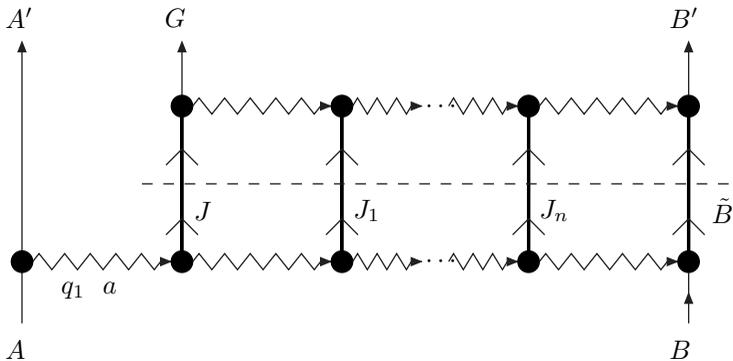
Figure 3: Schematic representation of the $(n+2)$-jet contribution to the s_2-channel discontinuity.

This discontinuity is calculated in the same way as in Eq. (4.58) with two minor differences. The first of them is evident: the PPR vertex is replaced by the RPR vertex. The second difference is related to the first one, but is not so evident. It results from the change of energy scales. Remember that when inserting our amplitudes into the unitarity relations we have to pass from the 'natural' energy scale of these amplitudes to 'external' scales. With a change of scales both the scattering vertices and the production vertices transform; their transformation laws, however, are different.

In order to make this point clear (although, for the advanced reader, it might be evident) let us consider the two-jet contribution in the s_2-channel

unitarity relation (see Fig. 3 with n=0)

$$\frac{1}{-2\pi i} disc^{(2\Lambda)}_{s_2} \mathcal{A}^{A'GB'}_{AB} = -\frac{1}{2\pi} \sum_{J\tilde{B}} \mathcal{A}^{A'J\tilde{B}}_{AB} \; \mathcal{A}^{GB'}_{J\tilde{B}} \, d\rho_{J\tilde{B}} \; . \qquad (4.64)$$

The amplitudes on the R.H.S. of this equation are defined in Eq. (2.9):

$$\mathcal{A}^{A'J\tilde{B}}_{AB} = 2s \, \Gamma^a_{A'A} \frac{1}{t_1} \left(\frac{s_1}{q_{1t}\tilde{k}_t}\right)^{\omega(t_1)} \gamma^J_{ab}(q_1,r) \left(\frac{s_2}{\tilde{k}_t r_t}\right)^{\omega(r^2)} \frac{1}{r^2}\Gamma^b_{\tilde{B}B} \; ,$$

$$\mathcal{A}^{GB'}_{J\tilde{B}} = \frac{2s_2}{r'^2} \, \Gamma^c_{GJ} \left(\frac{s_2}{r'^2_t}\right)^{\omega(r'^2)} \Gamma^c_{B'\tilde{B}} \; , \qquad (4.65)$$

where $\tilde{k} = q_1 - r$ is the J-jet momentum, and $r' = \tilde{k} - k = q_2 - r$; note that since transverse momenta are limited $\tilde{k}$ has the same component along p_1 as k , i.e. $\tilde{\beta} = \beta$. Passing here to new energy scales, we obtain

$$\mathcal{A}^{A'J\tilde{B}}_{AB} = 2s \, \Gamma^a_{A'A} \frac{1}{t_1} \left(\frac{s_1}{q_{1t}k_t}\right)^{\omega(t_1)} \gamma^J_{ab}(q_1,r;k_t) \left(\frac{s_2}{k_t q_{2t}}\right)^{\omega(r^2)} \frac{1}{r^2}\Gamma^b_{\tilde{B}B}(q_{2t}) \; ,$$

$$\mathcal{A}^{GB'}_{J\tilde{B}} = \frac{2s_2}{r'^2} \, \Gamma^c_{GJ}(q_{2t}) \left(\frac{s_2}{q^2_{2t}}\right)^{\omega(r'^2)} \Gamma^c_{B'\tilde{B}}(q_{2t}) \; , \qquad (4.66)$$

where the transformation of the PPR vertices Γ^c_{GJ}, $\Gamma^c_{\tilde{B}B}$ and $\Gamma^c_{B'\tilde{B}}$, imposed by the change of the energy scale, is defined in Eq. (4.32), and

$$\gamma^J_{ab}(q_1,r;k_t) = \left(\frac{k_t}{\tilde{k}_t}\right)^{\omega(t_1)} \gamma^J_{ab}(q_1,r) \left(\frac{k_t}{\tilde{k}_t}\right)^{\omega(r^2)} \; . \qquad (4.67)$$

Remember that, in order to write the formula in a compact way, the perturbative expansion is not performed explicitly. Actually the exponents in Eq. (4.67) should be expanded as in Eq. (4.33).

Now the difference with respect to the elastic case is clear. Therefore we do not show a detailed treatment of three- and four-jet contributions and only present the result, which in our approximation takes the form

$$\frac{1}{-2\pi i} disc_{s_2} \mathcal{A}^{A'GB'}_{AB} = \frac{2sN_c}{(2\pi)^{D-1}} \Gamma_{A'A} \frac{1}{t_1} < GR_1 | 1 + \omega(t_1)Y_1 + \hat{\mathcal{K}}Y_2 | B'B > \; . \qquad (4.68)$$

The state $< GR_1|$ describes the transition of the Reggeon R_1 with momentum q_1 into the gluon G, and it is determined by the equality

$$< GR_1|r_\perp >_{ij} = \ i\frac{f^{jbb'}}{N_c} \sum_J \int \gamma_{ib}^J(q_1, q_1 - \tilde{k}, k_t)\Gamma_{GJ}^{b'}(q_{2t})d\phi_J$$

$$- \int \frac{d^{D-2}r_1}{r_{1\perp}^2 r_{1\perp}'^2} < GR_1^{(B)}|r_{1\perp} >_{ij} \mathcal{K}_r^{(B)}(r_{1\perp}, r_\perp; q_{2\perp}) \ln \left(\frac{s_\Lambda}{(r_1 - r)_t k_t} \right) .$$
(4.69)

Here i and j are the colour indices in the t_1 and t_2 channels, respectively, $\tilde{k}$ is the J-jet momentum, $r_1' = q_2 - r_1$, and symmetrization with respect to $r \leftrightarrow r' = q_2 - r$ is understood. Note that in Eq. (4.68) the total momentum of two t-channel Reggeons (see Fig. 3) is $q_2 = p_{B'} - p_B$, so that $< r_{1\perp}|\hat{\mathcal{K}}|r_{2\perp} > = \mathcal{K}(r_{1\perp}, r_{2\perp}; q_{2\perp})$. The only feature of Eq. (4.69) that may require an explanation is the argument of the logarithm in the subtraction term. It can be understood easily. In the non-logarithmic terms the subtraction comes from the three-particle intermediate state (see Fig. 3 with n=1). The region of integration over rapidity of the jet J with momentum $\tilde{k}_1$ in the s_2-channel intermediate state is limited by the conditions $(\tilde{k}_1 + p_{\tilde{B}})^2 \geq s_\Lambda$, $(\tilde{k} + \tilde{k}_1)^2 \geq s_\Lambda$, which in the NLA (c.f. Eq. (4.46)) lead to:

$$\frac{\tilde{k}_{1t}^2}{s_\Lambda} \geq \frac{\tilde{\beta}_1}{\beta} \geq \frac{s_\Lambda}{s_2} .$$
(4.70)

For the integration variable $\tilde{y} = \ln \left(\tilde{\beta}_1 s/(q_{2t}\tilde{k}_{1t}) \right)$ this means (c.f. Eq. (4.47)):

$$Y_2 - \tilde{y}_\Lambda \geq \tilde{y} \geq y_{2\Lambda} , \quad \tilde{y}_\Lambda \equiv \ln \left(s_\Lambda/(k_t\tilde{k}_{1t}) \right) , \quad \tilde{y}_{2\Lambda} \equiv \ln \left(s_\Lambda/(q_{2t}\tilde{k}_{1t}) \right) .$$
(4.71)

The term with $y_{2\Lambda}$ is used for the subtraction in the impact factor for the $B \rightarrow B'$ transition (c.f. Eq. (4.52)). Therefore, the subtraction in $< GR_1|r_\perp >$ is performed by the term with $\tilde{y}_\Lambda$, and this explains the argument of the logarithm in Eq. (4.69).

s-channel discontinuity

The calculation of the s-channel discontinuity is more intricate because it contains more components. Note that, since $s' \equiv (p_{A'} + p_{B'})^2 \simeq s$, we have to include here discontinuities both properly in the s channel (Fig. 4) and in the s'-channel (Fig. 5).
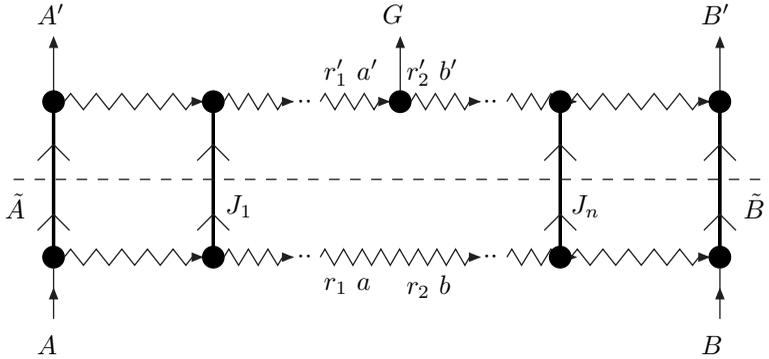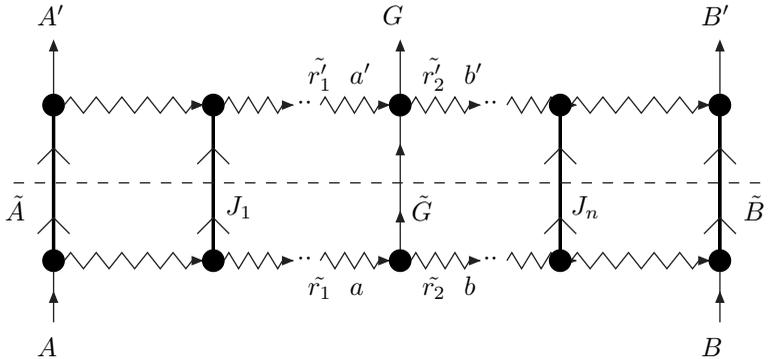
Figure 4: Schematic representation of contributions to the s-channel discontinuity: a) all produced jets are far away in rapidity space from the gluon G; b) the intermediate gluon $\tilde{G}$ is close to G.

We divide each of them into two parts. The first one takes into account contributions of those intermediate jets which in rapidity space are well seperated from the gluon G. Schematically this part is represented in Fig. 4a for the s-channel, and in Fig. 5a for the s'-channel. Here the momenta k_J of the produced jets are restricted by the condition $2kk_J \geq s_\Lambda$. In the second part, on the other hand, one of produced jets in the rapidity space is close to the gluon G. Actually in the NLA this jet consists of a single gluon. Its momentum k' is subject to the restriction $2kk' \leq s_\Lambda$.

With the experience of the preceeding calculations, instead of performing a long sequence of derivations, we can immediately write down the result. In our operator notation the terms of the discontinuity linear in $Y_{1,2}$ can be presented as

$$\frac{disc_s \mathcal{A}_{AB}^{A'GB'}}{-2\pi i} = \frac{2sN_c}{(2\pi)^{D-1}} \langle A'A|\hat{\mathcal{G}} + Y_1\hat{\mathcal{K}}\,\hat{\mathcal{G}} + \hat{\mathcal{G}}\hat{\mathcal{K}}Y_2|B'B\rangle \,, \qquad (4.72)$$
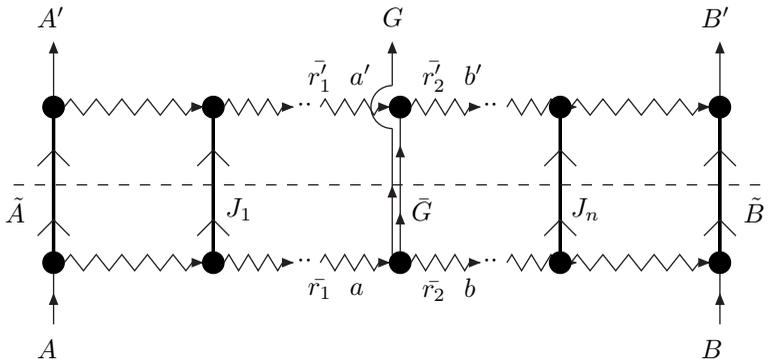
where $\hat{\mathcal{G}}$ is the operator of the gluon production. Note that it changes the total two-Reggeon state momentum from q_1 to q_2. With explicit reference to this fact the matrix element of this operator takes the form

$$\langle r_{1\perp}|\hat{\mathcal{G}}(q_1, q_2)|r_{2\perp}\rangle_{ij} \,. \qquad (4.73)$$

$$= \frac{f^{iaa'}f^{jbb'}}{N_c} \left[2\gamma_{a'b'}^G(q_1 - r_{1\perp}, q_2 - r_{2\perp})\delta^{D-2}(r_{1\perp} - r_{2\perp})r_{1\perp}^2 \delta_{ab} \right.$$

$$+ \int_{\frac{k_t^2}{s_\Lambda}}^{1 - \frac{k_t^2}{s_\Lambda}} \frac{dx}{2x(1-x)} \sum_{\tilde{G}} \frac{\gamma_{ab}^{\tilde{G}(\tilde{k})}(\tilde{r}_1, \tilde{r}_2)}{(2\pi)^{D-1}} \gamma_{a'b'}^{\{G(k)\tilde{G}(-\tilde{k})\}} (q_1 - \tilde{r}_1, q_2 - \tilde{r}_2) \right]$$

$$+ \int_{\frac{k_t^2}{s_\Lambda}}^{1 - \frac{k_t^2}{s_\Lambda}} \frac{dx}{2x(1-x)} \sum_{\bar{G}} \gamma_{ab}^{\{G(k)\bar{G}(\bar{k})\}} (\bar{r}_1, \bar{r}_2) \frac{\gamma_{a'b'}^{\bar{G}(-k')}(q_1 - \bar{r}_1, q_2 - \bar{r}_2)}{(2\pi)^{D-1}}$$

$$- \int \frac{d^{D-2}r_\perp \langle r_\perp|\hat{\mathcal{G}}(q_1, q_2)|r_{2\perp}\rangle_{ij}^{(B)}}{r_\perp^2 (r - q_1)_\perp^2} \mathcal{K}_r^{(B)} (r_{1\perp}, r_\perp; q_{1\perp}) \ln\left(\frac{s_\Lambda}{(r - r_1)_t k_t}\right)$$

$$- \int \frac{d^{D-2}r_\perp \langle r_{1\perp}|\hat{\mathcal{G}}(q_1, q_2)|r_\perp\rangle_{ij}^{(B)}}{r_\perp^2 (r - q_2)_\perp^2} \mathcal{K}_r^{(B)} (r_\perp, r_{2\perp}; q_{2\perp}) \ln\left(\frac{s_\Lambda}{(r - r_2)_t k_t}\right) \,.$$

The notations used here, apart from those of Eqs. (4.60) - (4.62), are the following: i and j are the colour indices in the t_1 and t_2 channels, $\tilde{k}$ and $\bar{k}$ are the momenta of the intermediate gluons $\tilde{G}$ and $\bar{G}$ in Fig. 4b and Fig. 5b,
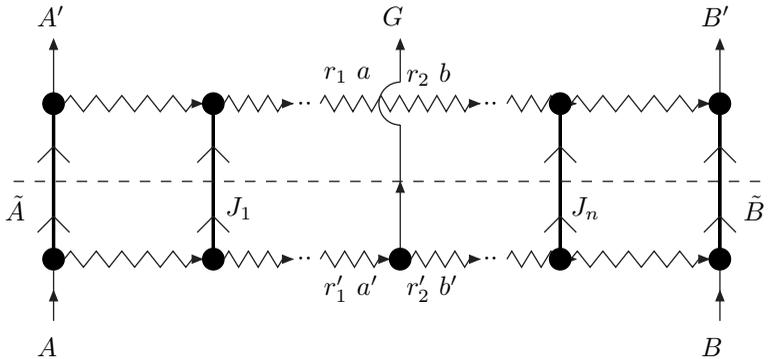
Figure 5: Schematic representation of contributions to the s'-channel discontinuity: a) all produced jets are far away in rapidity space from the gluon G; b) the intermediate gluon $\bar{G}$ is close to G.

respectively, and $\tilde{k} = \tilde{r}_1 - \tilde{r}_2$, $\bar{k} = \bar{r}_1 - k - \bar{r}_2$, where $\tilde{r}_i$ and $\bar{r}_i$ are the Reggeon momenta in these figures, with transverse components being equal to $r_{i\perp}$, $i = 1, 2$. Their Sudakov decomposition is

$$\tilde{r}_1 = \frac{x\beta}{1-x}p_1 + r_{1\perp} \ , \quad \tilde{r}_2 = \frac{\tilde{k}_\perp^2(1-x)}{x\beta s}p_2 + r_{2\perp} \ ,$$

$$\bar{r}_1 = \frac{\beta}{1-x}p_1 + r_{1\perp} \ , \quad r_2 = \left(\frac{\bar{k}_\perp^2(1-x)}{x\beta s} + \frac{k_\perp^2}{\beta s}\right)p_2 + r_{2\perp} \ . \qquad (4.74)$$

As always, the superscript (B) in Eq. (4.73) refers to the leading (Born) approximation; for $\langle r_{1\perp}|\hat{\mathcal{G}}|r_{2\perp}\rangle$ it is given by the first term where the gluon production vertex is taken in the LO.

Let us add a few necessary explanations. The first and the two last terms on the R.H.S of Eq. (4.73) belong to the contributions represented in Fig. 4a and and Fig. 5a, which evidently are equal (remember the signature). An interesting aspect is that, unlike in Eq. (4.69), in Eq. (4.73) the gluon production vertex enters with its natural scale. This can readily be seen from the two-jet contributions. Indeed, for example, the contribution to the discontinuity from Fig. 4a at $n = 0$ contains the product $\mathcal{A}_{AB}^{\tilde{A}\tilde{B}} \mathcal{A}_{\tilde{A}\tilde{B}}^{A'GB'}$. Using the equalities $s = s_1 s_2/k_t^2$ and $r_1 = r_2 \equiv r$ we can rewrite the Regge factor in the amplitude $\mathcal{A}_{AB}^{\tilde{A}\tilde{B}}$ in the form

$$\left(\frac{s}{r_t^2}\right)^{\omega(r^2)} = \left(\frac{s_1}{r_{1t}k_t}\right)^{\omega(r_1^2)} \left(\frac{s_2}{r_{2t}k_t}\right)^{\omega(r_2^2)} \ . \qquad (4.75)$$

Then, in the product $\mathcal{A}_{AB}^{\tilde{A}\tilde{B}} \mathcal{A}_{\tilde{A}\tilde{B}}^{A'GB'}$, the Regge factors depending on r_1 and $r_1' = q_1 - r_1$ can be rewritten as

$$\left(\frac{s_1}{r_{1t}'k_t}\right)^{\omega(r_1'^2)} \left(\frac{s_1}{r_{1t}k_t}\right)^{\omega(r_1^2)} = \left(\frac{q_{1t}}{r_{1t}'}\right)^{\omega(r_1'^2)} \left(\frac{q_{1t}}{r_{1t}}\right)^{\omega(r_1^2)} \left(\frac{s_1}{q_{1t}k_t}\right)^{\omega(r_1'^2)+\omega(r_1^2)} \ . \tag{4.76}$$

The first two factors on the R.H.S. of this equation are used for the transition (4.32) to the scale q_{1t} in the Reggeon scattering vertices entering the impact factor for the $A \to A'$ transition (see (Eq. 4.38)). After that in Eq. (4.76) we are left just with the "right" scale for s_1. The same procedure can be applied to the Regge factors depending on r_2. Therefore there are no additional factors which should be assigned to $\langle r_{1\perp}|\hat{\mathcal{G}}(q_1, q_2)|r_{2\perp}\rangle$.

The second and third term in Eq. (4.73) correspond to Fig. 4b and Fig. 5b, respectively. Note that their contributions are sub-leading, so that

in Eq. (4.72) in the NLA they have to go together with the LO impact factors and kernel.

Finally, the last two terms in Eq. (4.73) are subtraction terms. As before, they appear as the result of the limits of integration over rapidities of the produced jets (actually: gluons). Since these jets are separated in the rapidity space from the gluon G, the full integration region for the s-channel discontinuity is divided into two disconnected subregions which actually are the integration regions of the s_1- and s_2-channel discontinuities. Therefore we have two subtraction terms. They are quite analogous to the subtraction terms in the impact factors.

4.3 Discontinuities of two-gluon production amplitudes

The calculation of the discontinuities of the two-gluon production amplitudes is performed quite in the same way as the one-gluon case, so that we skip the description and only present the results. Again, for brevity, we use the term discontinuities, although, in reality, we calculate only their imaginary parts. We use the notations of Section 2. The energy variables are $s_i \equiv s_{i-1,i}$; their scales $k_{(i-1)t}k_{it}$, $k_{0t} \equiv q_{1t}$, $k_{3t} \equiv q_{3t}$; $Y_i = \ln\left(s_i/(k_{(i-1)t}k_{it})\right)$; in analogy to the elastic case, only terms linear in Y_i are kept.

For the s_3-channel discontinuity one obtains (cf. (4.68))

$$\frac{disc_{s_3}\mathcal{A}_{AB}^{A'G_1G_2B'}}{-2\pi i} \tag{4.77}$$

$$= \frac{2sN_c}{(2\pi)^{D-1}}\Gamma_{A'A}\frac{1}{t_1}\gamma^{G_1}(q_1,q_2)\frac{1}{t_2} < G_2R_2|1+\omega(t_1)Y_1+\omega(t_2)Y_2+\hat{\mathcal{K}}Y_3|B'B> ,$$

where $< G_2R_2|r_\perp >$ is given by (4.69) with the substitutions $G \to G_2$, $R \to R_2$, $q_1 \to q_2$, $q_2 \to q_3$; R_2 is the Reggeon with momentum q_2.

In the calculation of the s_{13}-channel discontinuity the peculiarities of the calculations of both the s_2- and of the s-channel discontinuities of the one-gluon production amplitudes are combined. However, it does not require any new ideas, and the calculation is straightforward. The result is

$$\frac{disc_{s_{13}}\mathcal{A}_{AB}^{A'G_1G_2B'}}{-2\pi i} \tag{4.78}$$

$$= \frac{2sN_c}{(2\pi)^{D-1}}\Gamma_{A'A}\frac{1}{t_1} < G_1R_1|\hat{\mathcal{G}}_2 + \omega(t_1)Y_1\hat{\mathcal{G}}_2 + Y_2\hat{\mathcal{K}}\,\hat{\mathcal{G}}_2 + \hat{\mathcal{G}}_2\hat{\mathcal{K}}Y_3|B'B> ,$$

where $< G_1 R_1 | r_\perp >$ is given by (4.69) with the substitution $G \to G_1$; $\hat{\mathcal{G}}_2$ is the operator for the production of the gluon G_2; it changes the total momentum of the two-Reggeon state from q_2 to q_3, so that the matrix elements of this operator are given by (4.73) with the substitutions $k \to k_2$, $q_1 \to q_2$, $q_2 \to q_3$.

The calculation of the s-channel discontinuity resembles the one-gluon case and gives

$$\frac{disc_s \mathcal{A}_{AB}^{A'G_1G_2B'}}{2\pi i} = \frac{2sN_c}{(2\pi)^{D-1}} \ \langle A'A|\hat{\mathcal{G}}_1\hat{\mathcal{G}}_2 + Y_1\hat{\mathcal{K}} \ \hat{\mathcal{G}}_1\hat{\mathcal{G}}_2 + \hat{\mathcal{G}}_1 Y_2\hat{\mathcal{K}} \ \hat{\mathcal{G}}_2 + \hat{\mathcal{G}}_1\hat{\mathcal{G}}_2\hat{\mathcal{K}}Y_3|B'B\rangle \ . \tag{4.79}$$

The discontinuities in the channels s_1 and s_{02} are obtained from (4.77) and (4.78), respectively, by suitable replacements. The s_2-channel discontinuity is

$$\frac{disc_{s_2} \mathcal{A}_{AB}^{A'G_1G_2B'}}{-2\pi i} \tag{4.80}$$

$$= \frac{2sN_c}{(2\pi)^{D-1}} \Gamma_{A'A} \frac{1}{t_1} \langle G_1 R_1 | 1 + \omega(t_1)Y_1 + \hat{\mathcal{K}}Y_2 + \omega(t_3)Y_3 | G_2 R_2 \rangle \frac{1}{t_3} \Gamma_{B'B} \ .$$

5 Bootstrap conditions

Use of Eq. (2.9) at $n = 0$ on the R.H.S. of the relation (3.17) gives an explicit form of the bootstrap condition for the elastic amplitudes:

$$\frac{1}{-2\pi i} disc_s \mathcal{A}_{AB}^{A'B'} = \frac{\omega(t)}{2} \frac{2s}{t} \Gamma_{A'A}^i \left(\frac{s}{q_t^2}\right)^{\omega(t)} \Gamma_{B'B}^i \ . \tag{5.81}$$

Taking into account Eq. (4.58) and comparing non-logarithmic terms in both sides of this equation, we obtain

$$\frac{2sN_c}{(2\pi)^{D-1}} \langle A'A|BB'\rangle = \frac{\omega(t)}{2} \frac{2s}{t} \Gamma_{A'A} \Gamma_{B'B} \ . \tag{5.82}$$

The terms proportional to Y give

$$\frac{2sN_c}{(2\pi)^{D-1}} \langle A'A|\hat{\mathcal{K}}|BB'\rangle = \frac{(\omega(t))^2}{2} \frac{2s}{t} \Gamma_{A'A} \Gamma_{B'B} \ . \tag{5.83}$$

Instead of using the last equation it is more convenient to consider the difference between (5.83) and (5.82), the latter being multiplied by $\omega(t)$; this leads to

$$\langle A'A|\hat{\mathcal{K}} - \omega(t)|BB'\rangle = 0 \ . \tag{5.84}$$

In the LO these equalities follow from so called strong bootstrap conditions for the impact factors and for the kernel:

$$\langle A'A^{(B)}| = \frac{g}{2}\Gamma^{(B)}_{A'A}\langle R^{(B)}_\omega| \ ,$$

$$|BB'^{(B)}\rangle = |R^{(B)}_\omega\rangle\frac{g}{2}\Gamma^{(B)}_{B'B} \ , \quad \left(\hat{\mathcal{K}}^{(B)} - \omega^{(1)}(t)\right)|R^{(B)}_\omega\rangle = 0 \ , \qquad (5.85)$$

where the superscripts (B) and (1) mean Born and one-loop approximations, respectively, and $|R^{(B)}_\omega\rangle$ is the universal (process independent) eigenfunction of the kernel with the eigenvalue $\omega^{(1)}(t)$. The normalization of $|R^{(B)}_\omega\rangle$ is determined by (5.82):

$$\frac{g^2 N_c t}{2(2\pi)^{D-1}}\langle R^{(B)}_\omega|R^{(B)}_\omega\rangle = \omega^{(1)}(t) \ . \qquad (5.86)$$

The fulfilment of Eq. (5.85) is known from Ref. [3]. Moreover, it is known that

$$\langle r_\perp|R^{(B)}_\omega\rangle = 1 \ . \qquad (5.87)$$

The conditions which, with the account of Eq. (5.85), follow from Eqs. (5.82) and (5.83) in the NLO are the following:

$$\frac{g N_c t}{(2\pi)^{D-1}} < A'A^{(1)}|R^{(B)}_\omega > = \omega^{(1)}(t)\Gamma^{(1)}_{A'A} + \frac{\omega^{(2)}(t)}{2}\Gamma^{(B)}_{A'A} \qquad (5.88)$$

and

$$\frac{g^2 N_c t}{2(2\pi)^{D-1}} < R^{(B)}_\omega|\hat{\mathcal{K}}^{(1)}|R^{(B)}_\omega > = \omega^{(1)}(t)\omega^{(2)}(t) \ , \qquad (5.89)$$

where the superscript (2) indicates the two-loop contribution. It is just the NLO bootstrap conditions for the octet impact factors and kernel derived in Ref. [5].

Let us now turn to the bootstrap relation (3.24) for one-gluon production amplitudes. Using Eq. (2.9) at $n = 1$ we obtain its explicit form:

$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_2} + disc_s\right)\mathcal{A}^{A'GB'}_{AB}\right] =$$

$$\frac{\omega(t_2)}{2} \ 2s \ \Gamma^i_{A'A}\frac{1}{t_1}\left(\frac{s_1}{q_{1t}k_t}\right)^{\omega(t_1)} \gamma^G_{ij}(q_1,q_2)\frac{1}{t_2}\left(\frac{s_2}{k_t q_{2t}}\right)^{\omega(t_2)} \Gamma^j_{B'B} \ . \qquad (5.90)$$

Comparing here non-logarithmic terms with the account of Eqs. (4.58) and (4.72) we have

$$\frac{2sN_c}{(2\pi)^{D-1}} \left[\langle A'A|\hat{\mathcal{G}}|BB'\rangle + \Gamma_{A'A}\frac{1}{t_1} \langle GR_1|BB'\rangle \right]$$

$$= \frac{\omega(t_2)}{2} \, 2s\Gamma^i_{A'A}\frac{1}{t_1}\gamma^G_{ij}(q_1,q_2)\frac{1}{t_2}\Gamma^j_{B'B} \; . \tag{5.91}$$

Comparison of terms proportional to Y_1 and Y_2 in (5.90) gives

$$\frac{2sN_c}{(2\pi)^{D-1}} \left[\langle A'A|\hat{\mathcal{K}}\,\hat{\mathcal{G}}|BB'\rangle + \Gamma_{A'A}\frac{\omega(t_1)}{t_1} \langle GR|BB'\rangle \right]$$

$$= \frac{\omega(t_2)}{2} \, 2s\Gamma^i_{A'A}\frac{\omega(t_1)}{t_1} \, \gamma^G_{ij}(q_1,q_2)\frac{1}{t_2}\Gamma^j_{B'B} \tag{5.92}$$

and

$$\frac{2sN_c}{(2\pi)^{D-1}} \left[\langle A'A|\hat{\mathcal{G}}\hat{\mathcal{K}}|BB'\rangle + \Gamma_{A'A}\frac{1}{t_1} \langle GR|\hat{\mathcal{K}}|BB'\rangle \right]$$

$$= \frac{\omega(t_2)}{2} \, 2s\Gamma^i_{A'A}\frac{1}{t_1} \, \gamma^G_{ij}(q_1,q_2)\frac{\omega(t_2)}{t_2}\Gamma^j_{B'B} \; , \tag{5.93}$$

respectively. Subtracting Eq. (5.91) (multiplied by $\omega(t_1)$) from Eq. (5.92) we obtain

$$\langle A'A| \left(\hat{\mathcal{K}} - \omega(t_1)\right) \hat{\mathcal{G}}|BB'\rangle = 0 \; . \tag{5.94}$$

Since $\langle r_\perp|\hat{\mathcal{G}}|BB'\rangle$ even in the Born approximation is a complicated function of $r_\perp$ depending on q_2 and the gluon momentum k, it can be considered as an arbitrary function. Therefore, in order to satisfy Eq. (5.94) the equality

$$\langle A'A| \left(\hat{\mathcal{K}} - \omega(t_1)\right) = 0 \tag{5.95}$$

must be fulfilled in the NLO. It is easy to see that the difference between Eq. (5.93) and Eq. (5.91) (multiplied by $\omega(t_2)$) is zero, assuming that

$$\left(\hat{\mathcal{K}} - \omega(t_1)\right)|BB'\rangle = 0 \; . \tag{5.96}$$

Actually Eq. (5.96) is equivalent to Eq. (5.95), so that we do not obtain a new relation.

Eq. (5.95) requires the fulfilment, in the NLO, of the strong bootstrap conditions for the impact factors and for the kernel:

$$\langle A'A| = \frac{g}{2}\Gamma_{A'A}\langle R_\omega| \ , \quad |BB'\rangle = |R_\omega\rangle\frac{g}{2}\Gamma_{B'B} \ , \quad \left(\hat{\mathcal{K}} - \omega(t)\right)|R_\omega\rangle = 0 \ .$$
(5.97)

Note that the normalization of $|R_\omega\rangle$ is fixed if we take into account (5.86) and (5.88):

$$\frac{g^2 t N_c}{2(2\pi)^{D-1}}\langle R_\omega|R_\omega\rangle = \omega(t) \ .$$
(5.98)

So the first important consequence derived from the bootstrap relations for one-gluon production is the strong form (5.97), (5.98) of the bootstrap conditions for impact factors and for the kernel in the NLO. Moreover, these relations give a new restriction on the Reggeon vertices and on the gluon trajectory.

In the leading order, using the equalities (5.85) and (5.86) (writing explicitly the total momenta of the two-Reggeon states, in order to avoid uncertainties, and denoting by R_i the Reggeons with momenta q_i), we obtain from (5.91)

$$\frac{gt_1}{2}\langle R_\omega^{(B)}(q_1)|\hat{\mathcal{G}}^{(B)}|R_\omega^{(B)}(q_2)\rangle_{ij} + \langle GR_1^{(B)}|R_\omega^{(B)}(q_2)\rangle_{ij}$$

$$= \gamma_{ij}^{G(B)}(q_1, q_2)\frac{g}{2}\langle R_\omega^{(B)}(q_2)|R_\omega^{(B)}(q_2)\rangle \ .$$
(5.99)

This equality is a particular case of the stronger version:

$$\frac{gt_1}{2}\langle R_\omega^{(B)}(q_1)|\hat{\mathcal{G}}^{(B)} + \langle GR_1^{(B)}| = \gamma^{G(B)}(q_1, q_2)\frac{g}{2}\langle R_\omega^{(B)}(q_2)| \ .$$
(5.100)

Indeed, Eq. (5.99) can be obtained from Eq. (5.100) by projection on the state $|R_\omega^{(B)}\rangle$. The fulfillment of Eq. (5.100) was demonstrated long ago [4], since it has been used in the proof of the gluon Reggeization in the LLA. Together with Eq. (5.91) it gives a new NLO bootstrap condition, which can be written as

$$\frac{gN_c t_2}{(2\pi)^{D-1}}\left[\frac{gt_1}{2}\langle R_\omega(q_1)|\hat{\mathcal{G}}|R_\omega(q_2)\rangle + \langle GR_1|R_\omega(q_2)\rangle\right] = \omega(t_2)\gamma_{ij}^G(q_1, q_2) \ .$$
(5.101)

Note that this condition has a "weak" form (it is a condition for matrix elements, not for state vectors). In this sense it is analogous to the conditions

for the impact factors and for the kernel, obtained from the elastic bootstrap. Since the bootstrap relations for one-gluon production lead to the "strong" form (5.97) of the conditions for the impact factors and for the kernel, it is natural to expect that (5.99) can be strengthened to the form

$$\frac{gt_1}{2}\langle R_\omega(q_1)|\hat{\mathcal{G}} + \langle GR_1| = \gamma^G(q_1, q_2)\frac{g}{2}\langle R_\omega(q_2)| \ , \qquad (5.102)$$

which is an evident generalization of (5.99) to the NLO, by consideration of the bootstrap relations for two-gluon production amplitudes. Moreover, one could expect that the bootstrap relations for the two-gluon production will give us a new restriction on the Reggeon vertices and trajectory. It turns out that the first expectation is correct; instead, the second one, fortunately, is not justified.

Indeed, let us turn to the relations (3.28) and (3.29) and consider the first of them, substituting the discontinuities (4.77)-(4.79) in the L.H.S. of the relation, and the Reggeized form (2.9) of the two-gluon production amplitude on the R.H.S. Comparing non-logarithmic terms on both sides of the bootstrap relation we obtain, with the help of (5.97),

$$\frac{gN_c t_3}{(2\pi)^{D-1}}\left[\frac{gt_1 t_2}{2}\langle R_\omega(q_1)|\hat{\mathcal{G}}_1\hat{\mathcal{G}}_2|R_\omega(q_3)\rangle\right.$$

$$\left. + \ t_2\langle G_1 R_1|\hat{\mathcal{G}}_2|R_\omega(q_3)\rangle + \gamma^{G_1}(q_1, q_2)\langle G_2 R_2|R_\omega(q_3)\rangle\right]$$

$$= \ \omega(t_3)\gamma^{G_1}(q_1, q_2)\gamma^{G_2}(q_2, q_3) \ . \qquad (5.103)$$

The terms proportional to Y_2 give

$$\frac{gN_c t_3}{(2\pi)^{D-1}}\left[\frac{gt_1 t_2}{2}\langle R_\omega(q_1)|\hat{\mathcal{G}}_1\hat{\mathcal{K}}\hat{\mathcal{G}}_2|R_\omega(q_3)\rangle\right.$$

$$\left. + \ t_2\langle G_1 R_1|\hat{\mathcal{K}}\hat{\mathcal{G}}_2|R_\omega(q_3)\rangle + \omega(t_2)\gamma^{G_1}(q_1, q_2)\langle G_2 R_2|R_\omega(q_3)\rangle\right]$$

$$= \ \omega(t_2)\omega(t_3)\gamma^{G_1}(q_1, q_2)\gamma^{G_2}(q_2, q_3) \ . \qquad (5.104)$$

It is easy to see that a comparison of the terms proportional to Y_1 and Y_3 does not lead to a new condition. Together with (5.97) the corresponding equations are reduced to (5.103).

Subtracting (5.103) (multiplied by $\omega(t_2)$) from (5.104) we obtain

$$\left(\frac{gt_1}{2}\langle R_\omega(q_1)|\hat{\mathcal{G}}_1 + \langle G_1 R_1|\right)\left(\hat{\mathcal{K}} - \omega(t_2)\right)\hat{\mathcal{G}}_2|R_\omega(q_3)\rangle = 0 \ , \qquad (5.105)$$

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which means that $(gt_1/2)\langle R_\omega(q_1)|\hat{\mathcal{G}}_1 + \langle G_1 R_1|$ is the eigenvector of the kernel with the eigenvalue $\omega(t_2)$, i.e. it must be proportional $\langle R_\omega(q_2)|$. Taking into account (5.101) we arrive at (5.102).

One can easily see that a double use of this bootstrap condition, together with the normalization of $|R_\omega\rangle$, guarantees that (5.103) is fulfilled. Moreover, it is not difficult to see that the relations (3.29) do not lead to new conditions.

6 Summary

The phenomenon of gluon Reggeization, which is very important for high energy QCD, has been proven in the leading logarithmic approximation, but it still remains a hypothesis in the next-to-leading approximation. The requirement of compatibility of the gluon Reggeization with s-channel unitarity imposes stringent restrictions on the gluon Regge trajectory and on the vertices of Reggeon interactions. The restrictions deduced from elastic scattering amplitudes in next-to-leading order were derived several years ago [5]. They are known as the bootstrap conditions for the color octet impact factors and for the BFKL kernel, and they were proven to be satisfied. Moreover, subsequently it was shown (see [6] and references therein) that the stronger conditions on the impact factors and kernel are also fulfilled.

In this paper we have considered restrictions on the Reggeon vertices and trajectory, which emerge from amplitudes of gluon production in multi-Regge kinematics. We have shown that the requirement of compatibility of the multi-Regge form of these amplitudesw with s-channel unitarity leads, in particular, to the strong bootstrap conditions on the colour octet impact factors and on the kernel that we have mentioned above. Besides this, a new bootstrap condition has been derived. The most urgent problem now is a proof of fulfillment of this new condition. It will provide the possibility to prove the hypothesis of gluon Reggeization in the NLA. Indeed, the bootstrap conditions are extraordinarily significant. The proof of the gluon Reggeization in the leading logarithmic approximation was constructed just on the basis of these conditions. An analogous proof can be constructed in the next-to-leading approximation as well [7].

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The bootstrap conditions
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