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DEVIATION FROM STANDARD QED AT LARGE DISTANCES: INFLUENCE OF TRANSVERSE DIMENSIONS OF COLLIDING BEAMS ON BREMSTRAHLUNG

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Deviation from standard QED at large distances: influence of transverse dimensions of colliding beams on bremstrahlung

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Abstract

The radiation at collision of high-energy particles is formed over a rather long distances and therefore is sensitive to an environment. In particular the smallness of the transverse dimensions of the colliding beams leads to suppression of bremsstrahlung cross section for soft photons. This beam-size effect was discovered and investigated at INP, Novosibirsk around 1980. At that time an incomplete expression for the bremsstrahlung spectrum was calculated and used because a subtraction associated with the extraction of pure fluctuation process was not performed. Here this procedure is done. The complete expression for the spectral-angular distribution of incoherent bremsstrahlung probability is obtained. The case of Gaussian colliding beams is investigated in details. In the case of flat beams the expressions for the bremsstrahlung spectrum are simplified essentially. Comparison of theory with VEPP4 and HERA data is performed. Possible application of the effect to linear e^+e^- collider tuning is discussed.

1 Introduction

The formation of the bremsstrahlung process of high-energy particles occurs with extremely small momentum transfers. In the space-time picture this means that the process takes place at the large (macroscopic) distances. The longitudinal length (with respect to the direction of the initial momentum) of formation of the radiation usually is called the coherence (formation) length l_f. For emission of a photon with energy ω the coherence length is $l_f(\omega) \sim \varepsilon(\varepsilon-\omega)/m^2\omega$, where ε and m is the energy and mass of the emitting particle (here the system $\hbar = c = 1$ is used). If the particle experiences some action in this length, the radiation pattern changes (in the case when the action is the multiple scattering of the emitting particle one observes the famous Landau-Pomeranchuk effect [1]).

A different situation exists in the bremsstrahlung process at the collision of electron and electron (positron) in colliding beams experiments. The point is that the external factors act differently on the radiating particle and on the recoil particle. For the radiating particle the criterion of influence of external factors is the same both at an electron scattering from a nucleus and at a collision of particles. For the recoil particle the effect turns out to be enhanced by the factor ε^2/m^2, which is due to the fact that the main contribution to the bremsstrahlung cross section give emitted by the recoil particle virtual photons with very low energy

$$q_0 \sim \frac{m^2\omega}{\varepsilon(\varepsilon-\omega)}, \tag{1.1}$$

so that the formation length of virtual photon is

$$L_v(\omega) = l_f(q_0) = \frac{4\varepsilon^3(\varepsilon-\omega)}{m^4\omega}. \tag{1.2}$$

This means that the effect for the recoil particles appears much earlier than for the radiating particles. For example, the Landau-Pomeranchuk effect distorted the whole bremsstrahlung spectrum in a TeV range (for heavy elements) while it turns out that the action on the recoil particle can be important for contemporary colliding beam facilities in GeV range [2].

There are a few factors which could act on the recoil electron. One of them is the presence of an external magnetic field in the region of collision of particles [2-4] If the formation length of virtual photon L_v turns out to be larger than the formation length $l_H(\omega)$ of a photon with energy ω in a magnetic field H than the magnetic field will limit the region of minimal momentum transfers, which will lead to a decrease of the bremsstrahlung cross section and a change of its spectrum. Another effect can appear due to the smallness of the linear interval l where the collision occurs in comparison with $L_v(\omega)$ (see (1.2)). This was pointed out in [5].

In the experiment [6] devoted to study of the bremsstrahlung spectrum $d\sigma_\gamma(\omega)$ carried out at the electron-positron colliding beam facility VEPP-4 of Institute of Nuclear Physics at an energy $\varepsilon = 1.84$ GeV, a deviation of the bremsstrahlung spectrum from the standard QED spectrum was observed. This was attributed to the smallness of the transverse size of the colliding beams. Theoretically the problem of finite transverse dimensions was investigated in [7] were the bremsstrahlung spectrum at e^+e^- collision was calculated to within the power accuracy (the neglected terms are of the order $1/\gamma = m/\varepsilon$) . Later the problem was analyzed in [8], [9], [10] where the found bremsstrahlung spectra coincide with obtained in [7].

It should be noted that in [7] (as well as in all other papers mentioned above) an incomplete expression for the bremsstrahlung spectrum was calculated. One has to perform the subtraction associated with the extraction of pure fluctuation process. Let us discuss this item in some details. The momentum transfer $\mathbf{q}$ at collision is important for the radiation process (the cross section contains factor $\mathbf{q}^2$ at $\mathbf{q}^2 \ll m^2$). At the beam collision the momentum transfer may arise due to interaction of the emitting particle with the opposite beam as a whole (due to coherent interaction with averaged field of the beam) and due to interaction with an individual particle of the opposite beam. Here we are considering *the incoherent* process only (connected with the incoherent fluctuation of density) and so we have to subtract the coherent contribution. The expression for the bremsstrahlung spectrum found in [7] contains the mean value $< \mathbf{q}^2 >$, while the coherent contribution contains $< \mathbf{q} >^2$ and this term has to be subtracted. We encountered with an analogous problem in analysis of incoherent processes in the oriented crystals [11] where it was pointed out (see p.407) that the subtraction has to be done in the spectrum calculated in [7]. Without the subtraction the results for the incoherent processes in oriented crystals would be qualitatively erroneous.

In Sec.2 a qualitative analysis of the incoherent radiation process is given. In Sec.3 the general formulas for the spectral-angular distributions of incoherent bremsstrahlung are derived. The incoherent bremsstrahlung spectrum for the Gaussian beams is calculated in Sec.4 in the form of double integrals. In specific case of narrow beams (the size of beam is much smaller than the characteristic impact parameter) the formulas are simplified essentially (Sec.5). The experimental studies of effect were performed with flat beams (the beam vertical size is much smaller than horizontal one). This specific case is analyzed in Sec.6, while comparison with data is given in Sec.7. In Sec.8 the possible application to the linear e^+e^- collider tuning is discussed.

2 General analysis of probability of incoherent radiation

In this section we discuss in detail the conditions under which we consider the incoherent radiation. One can calculate the photon emission probability in the target rest frame, since the entering combinations ω/ε and $\gamma\vartheta$ (γ is the Lorentz factor $\gamma = \varepsilon/m$, ϑ is the angle of photon emission) are invariant (within a relativistic accuracy) and a transfer to any frame is elementary. We use the operator quasiclassical method [12], [13]. Within this method the photon formation length (time) is

$$
\begin{aligned}
& l_f = \frac{\varepsilon'}{\varepsilon kv} = \frac{\varepsilon'}{\varepsilon\omega(1-\mathbf{nv})} \simeq \frac{l_{f0}}{\zeta}; \\
& l_{f0} = \frac{1}{q_{min}} = \frac{2\varepsilon\varepsilon'}{\omega m^2} = \frac{4\varepsilon'\gamma_c\varepsilon_r}{\omega m^2}; \\
& \zeta = 1 + \gamma^2\vartheta^2, \quad \varepsilon' = \varepsilon - \omega, \qquad (2.1)
\end{aligned}
$$

where $p_\mu = \varepsilon v_\mu$ ($v_\mu = (1, \mathbf{v})$) is the 4-momentum of radiating particle, $\gamma_c = \varepsilon_c/m_c$, ε_c is the energy of target particle in the laboratory frame, m_c is its mass, ε_r is the energy of radiating particle in the laboratory frame; $k_\mu = (\omega,\ \omega\mathbf{n})$) is the photon 4-momentum, ϑ is the angle between vectors $\mathbf{n}$ and $\mathbf{v}$.

In the case when the transverse dimension of beam σ is $\sigma \gg l_{f0}$ the impact parameters $\varrho \leq \varrho_{max} = l_{f0}$ contribute. One can put that the particle density in the target beam is a constant, so that the standard QED formulas are valid. Note that the value ϱ_{max} is the relativistic invariant which is defined by the minimal value of square of the invariant mass of the intermediate photon $|q^2|$. In the case when the characteristic size of beams is smaller the value ϱ_{max} the lower value of $|q^2|$ is defined by this size.

In the target rest frame the scattering length of emitting particle is of order of the impact parameter ϱ. This length is much smaller than the longitudinal dimension of the target $\gamma_c l$ (l is the length of target beam in the laboratory frame). So one can neglect a variation of configuration of the beam during the scattering time. A possible variation of particle configuration in the beam during a long time one can take into account in the adiabatic approximation.

Another limitation is connected with the influence of value of transverse momentum arising from the electromagnetic field $E = |\mathbf{E}|$ of colliding (target) beam on the photon formation length. This value should be smaller than the characteristic transverse momentum transfer $m\sqrt{\zeta}$ in the photon emission process:

$$\begin{aligned} \frac{eEl_f}{m\sqrt{\zeta}} &\sim \frac{\alpha N_c}{(\sigma_z+\sigma_y)l\gamma_c}\frac{1}{m\sqrt{\zeta}}\frac{4\varepsilon'\gamma_c\varepsilon_r}{\omega\zeta m^2} \\ &\sim \frac{2\alpha N_c}{(\sigma_z+\sigma_y)l}\frac{1}{m\sqrt{\zeta}}\frac{2\varepsilon'\varepsilon_r}{\omega\zeta m^2} = \frac{4\alpha N_c\gamma_r\lambda_c^2\varepsilon'}{(\sigma_z+\sigma_y)l\zeta^{3/2}\omega} \ll 1, \end{aligned} \tag{2.2}$$

here $\alpha = 1/137$, N_c is the number of particles in the target beam, σ_z and σ_y are the vertical and horizontal transverse dimensions of target beam. Note that the ratio γ/l is the relativistic invariant. This condition can be presented in invariant form

$$\frac{2\chi}{u\zeta^{3/2}} \ll 1, \tag{2.3}$$

where $\chi = \dfrac{\gamma}{E_0}|\mathbf{E}_\perp + \mathbf{v}\times\mathbf{H}|,\ u = \dfrac{\omega}{\varepsilon'},\ E_0 = \dfrac{m^2}{e} = 1.32\cdot 10^{16}\mathrm{V/cm}.$ Since the main contribution to the spectral probability of radiation gives angles $\vartheta \sim 1/\gamma$ ($\zeta \sim 1$) this condition takes the form $\chi/u \ll 1$. For the case $\chi/u \gg 1$ the condition (2.3) can be satisfied for the large photon emission angles $\zeta \simeq \gamma^2\vartheta^2 > (\chi/u)^{2/3} \gg 1$. Under these conditions the formation length $l_f = l_{f0}/\zeta$ decreases as $(\chi/u)^{2/3}$. The same inhibition factor acquires the bremsstrahlung probability [14].

We consider now the spectral distribution of radiation probability in the case $\chi \ll 1$ (this condition is fulfilled in all existing installations), so

$$\chi \sim \alpha N_c\gamma\frac{\lambda_c^2}{(\sigma_z+\sigma_y)l} \ll 1. \tag{2.4}$$

Only the soft photons ($\omega \leq \chi\varepsilon \ll \varepsilon$) contribute to the *coherent* radiation ("beamstrahlung") while the hard photon region $\omega \gg \chi\varepsilon$ is suppressed exponentially as it is known from the classical radiation theory. As it was mentioned in the soft photon region ($\omega \leq \chi\varepsilon \ll \varepsilon$), the spectral probability of *bremsstrahlung* is suppressed by the factor $(\omega/\varepsilon\chi)^{2/3}$ only. On the contrary, the spectral probability of the bremsstrahlung is negligible comparing with the beamstrahlung taking into consideration that the mean square of multiple scattering angle during all time of beam collisions is small comparing with the value $1/\gamma^2$:

$$\gamma^2 < \vartheta_s^2 >= \frac{< q_s^2 >}{m^2} \simeq \frac{8\alpha^2 N_c \lambda_c^2}{\sigma_z \sigma_y} L \ll 1, \tag{2.5}$$

where L is the characteristic logarithm of scattering problem (in the typical experimental condition $L \sim 10$).

It was supposed in the above estimations of beamstrahlung probability that the radiation formation length is shorter than the target beam length

$$\frac{l_f}{l} \sim \frac{1}{u}\left(1 + \frac{\chi}{u}\right)^{-2/3} \frac{\gamma\lambda_c}{l} < 1. \tag{2.6}$$

Besides it was supposed that one can neglect a variation of the impact parameter $\boldsymbol{\varrho}$ and therefore of the transverse electric field $\mathbf{E}_\perp(\boldsymbol{\varrho})$ during the beam collision. This is true when disruption parameter is enough small

$$D_i = \frac{2\alpha N_c \lambda_c l}{\gamma_r \sigma_i(\sigma_z + \sigma_y)} \ll 1, \quad (i = z, y) \tag{2.7}$$

So, we consider the incoherent bremsstrahlung under following conditions:

$$\chi \ll 1, \quad \frac{\chi}{u} \ll 1, \quad D_i \ll 1. \tag{2.8}$$

3 Spectral-angular distribution of the incoherent bremsstrahlung probability

In this section we derive the basic expression for the incoherent bremsstrahlung probability at collision of two beams with bounded transverse dimensions.

We consider first the photon emission at collision of an electron with one particle with the transverse coordinate $\mathbf{x}$. We select the impact parameter

$\varrho_0 = |\boldsymbol{\varrho}_0|$ which is small comparing with the typical transverse beam dimension σ but which is large comparing with the electron Compton length λ_c ($\lambda_c \ll \varrho_0 \ll \sigma$). In the interval of impact parameters $\varrho = |\mathbf{r}_\perp - \mathbf{x}| \geq \varrho_0$, where $\mathbf{r}_\perp$ is the transverse coordinate of emitting electron, the probability of radiation summed over the momenta of final particle can be calculated using the classical trajectory of particle. Indeed,one can neglect by the value of commutators $|[\hat{p}_{\perp i}, \varrho_j]| = \delta_{ij}$ comparing with the value $p_\perp \varrho$ in this interval ($p_\perp \varrho \geq m\varrho_0 \gg 1$). In this case the expression for the probability has the form (see [13], Eqs.(7.3) and (7.4))

$$dw = |M(\boldsymbol{\varrho})|^2 w_r(\mathbf{r}_\perp) d^2r_\perp d^3k, \tag{3.1}$$

where

$$M(\boldsymbol{\varrho}) = \frac{e}{2\pi\sqrt{\omega}} \int_{-\infty}^{\infty} R(t) \exp(ik'x(t))dt, \quad k' = \frac{\varepsilon}{\varepsilon'}k. \tag{3.2}$$

Here $w_r(\mathbf{r}_\perp)d^2r_\perp$ is the probability to find the emitting particle with the impact parameter $\boldsymbol{\varrho} = \mathbf{r}_\perp - \mathbf{x}$ in the interval $d^2\varrho = d^2r_\perp$, $R(t) = R(\mathbf{p}(t))$, $kx(t) = \omega t - \mathbf{kr}(t)$ (for details see [13], Sec. 7.1). Integrating by parts in the last equation and taking into account that $|\mathbf{q}_\perp(\boldsymbol{\varrho})| \leq 1/\varrho_0 \ll m$ we find

$$M(\boldsymbol{\varrho}) \simeq \frac{ie}{2\pi\sqrt{\omega}} \int_{-\infty}^{\infty} \exp(ik'vt) \frac{d}{dt} \frac{R(t)}{k'v(t)} dt \simeq \frac{ie}{2\pi\sqrt{\omega}} \mathbf{m}(\boldsymbol{\varrho}) \frac{\partial}{\partial \mathbf{p}_\perp} \frac{R(\mathbf{p}_\perp)}{k'v}, \tag{3.3}$$

where

$$\begin{aligned}
&\mathbf{p}_\perp = \mathbf{p} - \mathbf{n}(\mathbf{np}) \simeq \varepsilon(\mathbf{v} - \mathbf{n}), \\
&\mathbf{m}(\boldsymbol{\varrho}) = \int_{-\infty}^{\infty} \exp(ik'vt)\dot{\mathbf{q}}(\boldsymbol{\varrho}, t)dt = -\frac{\partial}{\partial \boldsymbol{\varrho}} \int_{-\infty}^{\infty} \exp\left(\frac{it}{l_f}\right) V(\sqrt{\varrho^2 + t^2})dt \\
&= \frac{2\alpha}{l_f} \frac{\boldsymbol{\varrho}}{\varrho} K_1(\frac{\varrho}{l_f}) = 2\alpha q_{min}\zeta K_1(\varrho q_{min}\zeta) \frac{\boldsymbol{\varrho}}{\varrho},
\end{aligned} \tag{3.4}$$

for the Coulomb potential, $K_1(z)$ is the modified Bessel function (the Macdonald function), $R(\mathbf{p}_\perp)$ has a form of matrix element for the free particles:

$$\begin{aligned}
&R(\mathbf{p}_\perp) = \varphi_{s'}^+(A + i\boldsymbol{\sigma}\mathbf{B})\varphi_s; \quad A \simeq \frac{m(\varepsilon + \varepsilon')}{2\varepsilon\varepsilon'}(\mathbf{e}^*\mathbf{u}), \\
&\mathbf{B} \simeq \frac{m\omega}{2\varepsilon\varepsilon'}(\mathbf{e}^* \times (\mathbf{n} - \mathbf{u})); \quad \mathbf{u} = \frac{\mathbf{p}_\perp}{m}, \ \zeta = 1 + \mathbf{u}^2, \ k'v = q_{min}\zeta \, ,
\end{aligned} \tag{3.5}$$

where the vector $\mathbf{e}$ describes the photon polarization and the spinors φ_s and $\varphi_{s'}$ describe the polarization of the initial and final electrons correspondingly.

In the interval of impact parameters $\varrho \leq \lambda_c$ the expectation value of operator $< \boldsymbol{\varrho}|M^+M|\boldsymbol{\varrho} >$ cannot be written in the form (3.1) since the entering operators become noncommutative inside the expectation value. However because of the condition $\lambda_c \ll \sigma$ in this interval $w_r(\mathbf{r}_\perp) \simeq w_r(\mathbf{x}) + O(\frac{\lambda_c}{\sigma})$ one can neglect effect of the inhomogeneous distribution. For the same reason in the calculation of correction to the probability of photon emission, which is defined as the difference of $dw(\sigma)$ and the probability of photon emission in a inhomogeneous medium, one can extend the integration interval into region $\varrho \leq \varrho_0$.

In this paper we consider the incoherent bremsstrahlung which can be considered as the photon emission due to fluctuations of the potential V connected with uncertainty of a particle position in the transverse to its momentum plane. Because of this we have to calculate the dispersion of the vector $\mathbf{m}(\boldsymbol{\varrho})$ with respect to the transverse coordinate $\boldsymbol{\varrho}$:

$$
\begin{aligned}
< m_i m_j > - < m_i >< m_j >= \int m_i(\mathbf{r}_\perp - \mathbf{x}) m_j(\mathbf{r}_\perp - \mathbf{x}) w_c(\mathbf{x}) d^2x \\
- \int m_i(\mathbf{r}_\perp - \mathbf{x}) w_c(\mathbf{x}) d^2x \int m_j(\mathbf{r}_\perp - \mathbf{x}) w_c(\mathbf{x}) d^2x,
\end{aligned}
\tag{3.6}
$$

where $w_c(\mathbf{x})$ is the distribution function of target particles normalized to the unity.

As a result we obtain the following expression for the correction to the probability of photon emission connected with the restricted transverse dimensions of colliding beams of charged particles:

$$
\begin{aligned}
& dw_1 = \frac{\alpha}{(2\pi)^2} \frac{d^3k}{\omega} T_{ij}(\mathbf{e}, \mathbf{p}_\perp, s, s') L_{ij}, \ \ T_{ij} = \left[\frac{\partial}{\partial p_{\perp i}} \frac{R^*(\mathbf{p}_\perp)}{k'v}\right] \left[\frac{\partial}{\partial p_{\perp j}} \frac{R^*(\mathbf{p}_\perp)}{k'v}\right], \\
& L_{ij} = \int m_i(\boldsymbol{\varrho}) m_j(\boldsymbol{\varrho}) \left(w_r(\mathbf{x} + \boldsymbol{\varrho}) - w_r(\mathbf{x})\right) w_c(\mathbf{x}) d^2x d^2\varrho \\
& - \left(\int m_i(\boldsymbol{\varrho}) w_c(\mathbf{x} - \boldsymbol{\varrho}) d^2\varrho\right) \left(\int m_j(\boldsymbol{\varrho}) w_c(\mathbf{x} - \boldsymbol{\varrho}) d^2\varrho\right) w_r(\mathbf{x}) d^2x.
\end{aligned}
\tag{3.7}
$$

Averaging over the polarization of initial electrons and summing over the polarization of final electrons we find

$$
T_{ij} = \frac{l_f}{\varepsilon\varepsilon'} \Big[e_i e_j - \frac{2\mathbf{eu}}{\zeta}(e_i u_j + u_i e_j) + \frac{4(\mathbf{eu})^2}{\zeta^2} u_i u_j + \frac{\omega^2}{4\varepsilon\varepsilon'}\delta_{ij}\Big]. \tag{3.8}
$$

Note, that one can choose the real vector $\mathbf{e}$ since only the linear polarization could arise in the case of unpolarized electrons.

After summation in (3.8) over the polarization of emitted photon we have

$$T_{ij} = \frac{l_f}{2\varepsilon\varepsilon'}\left(v\delta_{ij} - \frac{8}{\zeta^2}u_i u_j\right), \quad v = \frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}, \quad \zeta = 1 + \gamma^2\vartheta^2. \tag{3.9}$$

Finally, averaging the last expression over the azimuth angle of emitted photon we obtain

$$T_{ij} = \frac{l_f}{2\varepsilon\varepsilon'}U(\zeta)\delta_{ij}, \quad U(\zeta) = v - \frac{4(\zeta-1)}{\zeta^2}. \tag{3.10}$$

Substituting the expression obtained into (3.7) we find the correction to the probability of photon emission connected with the restricted transverse dimensions of colliding beams of charged particles:

$$dw_1 = \frac{\alpha^3}{\pi m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}U(\zeta)F(\omega,\zeta)d\zeta, \tag{3.11}$$

where

$$\begin{aligned}
&F(\omega,\zeta) = F^{(1)}(\omega,\zeta) - F^{(2)}(\omega,\zeta),\\
&F^{(1)}(\omega,\zeta) = \frac{2\eta^2}{\zeta^2}\int K_1^2(\eta\varrho)\left(w_r(\mathbf{x}+\boldsymbol{\varrho}) - w_r(\mathbf{x})\right)w_c(\mathbf{x})d^2x d^2\varrho,\\
&F^{(2)}(\omega,\zeta) = \frac{2\eta^2}{\zeta^2}\int\left(\int K_1(\eta\varrho)\frac{\boldsymbol{\varrho}}{\varrho}w_c(\mathbf{x}-\boldsymbol{\varrho})d^2\varrho\right)^2 w_r(\mathbf{x})d^2x,
\end{aligned} \tag{3.12}$$

here $\eta = q_{min}\zeta$.

Using the integral

$$\int K_1^2(\eta\varrho)\varrho d\varrho = \frac{\varrho^2}{2}\left[K_1^2(\eta\varrho) - K_0(\eta\varrho)K_2(\eta\varrho)\right] \tag{3.13}$$

and integrating by parts we obtain

$$\begin{aligned}
F(\omega,\zeta) = \frac{\eta^2}{\zeta^2}\Bigg[&\int\left[K_0(\eta\varrho)K_2(\eta\varrho) - K_1^2(\eta\varrho)\right]\varrho\frac{d\Phi(\boldsymbol{\varrho})}{d\varrho}d^2\varrho\\
&-2\int\left(\int K_1(\eta\varrho)\frac{\boldsymbol{\varrho}}{\varrho}w_c(\mathbf{x}-\boldsymbol{\varrho})d^2\varrho\right)^2 w_r(\mathbf{x})d^2x\Bigg],
\end{aligned} \tag{3.14}$$

where

$$\Phi(\boldsymbol{\varrho}) = \int w_r(\mathbf{x}+\boldsymbol{\varrho})w_c(\mathbf{x})d^2x. \tag{3.15}$$

In the general case the axes of colliding beams are displaced with respect each other in the transverse plane by the vector $\mathbf{x}_0$ with coordinates z_0, y_0. In this case we have to consider

$$w_r(\mathbf{x}) \to w_r(\mathbf{x}+\mathbf{x}_0),\ F^{(1,2)}(\omega,\zeta) \to F^{(1,2)}(\omega,\zeta,\mathbf{x}_0),\ \Phi(\boldsymbol{\varrho}) \to \Phi(\boldsymbol{\varrho}+\mathbf{x}_0) \tag{3.16}$$

The first term in the expression for $F(\omega,\zeta)$ in (3.14) coincides with the function $F(\omega,\zeta)$ defined in [7], Eq.(13). The second (subtraction) term in (3.14) which naturally arises in this derivation was missed in Eq.(13), [7] as it was said above. The expression (3.11) is consistent with Eq.(21.6) in the book [13] (see also Eq.(2.2) in [11]) where another physical problem was analyzed. It is the incoherent bremsstrahlung in the oriented crystals.

Below we restrict ourselves to the case of unpolarized electrons and photons. Influence of bounded transverse size on the probability of process with polarized particles will be considered elsewhere.

4 Gaussian beams

For calculation of explicit expression for the bremsstrahlung cross section we have to specify the distributions of particles in the colliding beams. Here we consider the actual case of Gaussian beams. Using the Fourier transform we have

$$\begin{aligned} & w(\mathbf{x}) = \frac{1}{(2\pi)^2}\int d^2q \exp(-i\mathbf{q}\mathbf{x})w(\mathbf{q}); \\ & w_r(\mathbf{q}) = \exp\left[-\frac{1}{2}(q_z^2\Delta_z^2+q_y^2\Delta_y^2)\right],\ w_c(\mathbf{q}) = \exp\left[-\frac{1}{2}(q_z^2\sigma_z^2+q_y^2\sigma_y^2)\right], \end{aligned} \tag{4.1}$$

where as above the index r relates to the radiating beam and the index c relates to the target beam, Δ_z and Δ_y (σ_z and σ_y) are the vertical and horizontal transverse dimensions of radiating (target) beam. Substituting (4.1) into Eq.(3.15) we find

$$\begin{aligned} \Phi(\boldsymbol{\varrho}) &= \frac{1}{(2\pi)^2}\int d^2q \exp(-i\mathbf{q}\boldsymbol{\varrho})\exp\left[-\frac{q_z^2}{4\Sigma_z^2}-\frac{q_y^2}{4\Sigma_y^2}\right] \\ &= \frac{\Sigma_z\Sigma_y}{\pi}\exp[-\varrho_z^2\Sigma_z^2-\varrho_y^2\Sigma_y^2]; \\ \Sigma_z^2 &= \frac{1}{2(\sigma_z^2+\Delta_z^2)},\quad \Sigma_y^2=\frac{1}{2(\sigma_y^2+\Delta_y^2)}, \end{aligned} \tag{4.2}$$

Below we consider the general situation when the axes of colliding beams are displaced with respect each other in the transverse plane by the vector $\mathbf{x}_0$ with the coordinates z_0, y_0. This displacement has essential influence on the luminosity. For the processes for which the short distances are essential only (e.g. double bremsstrahlung [2]) the probability of process is the product of the cross section and luminosity. The geometrical luminosity per bunch, not taking into account the disruption effects, is given by

$$\mathcal{L} = N_c N_r \Phi(\mathbf{x}_0), \tag{4.3}$$

where as above N_r and N_c are the number of particles in the radiating and target beams correspondingly. We will use the same definition for our case. Then we have

$$dw_\gamma = \Phi(\mathbf{x}_0) d\sigma_\gamma, \quad d\sigma_1 = \Phi^{-1}(\mathbf{x}_0) dw_1, \tag{4.4}$$

where dw_1 is defined in Eq.(3.11).

We calculate first the function $F^{(1)}(\omega, \zeta)$ in Eq.(3.12) for the case of coaxial beams when $\mathbf{x}_0 = 0$. Passing on to the momentum representation with the help of formula (4.1) we find

$$F^{(1)}(\omega, \zeta) = -\frac{1}{2\pi\zeta^2} \int w_r(\mathbf{q}) w_c(\mathbf{q}) F_2\left(\frac{q}{2\eta}\right) q dq d\varphi, \tag{4.5}$$

where $\eta = q_{min}\zeta$ is introduced in (3.12),

$$F_2\left(\frac{q}{2\eta}\right) = \frac{\eta^2}{\pi} \int K_1^2(\eta\varrho)(1 - \exp(-i\mathbf{q}\boldsymbol{\varrho}))d^2\varrho,$$
$$F_2(x) = \frac{2x^2+1}{x\sqrt{1+x^2}} \ln(x + \sqrt{1+x^2}) - 1, \quad q_{min} = m^3\omega/4\varepsilon^2\varepsilon', \tag{4.6}$$

here value q_{min} is defined in c.m.s. of colliding particles. The function $F_2(x)$ encounters in the radiation theory. To calculate the corresponding contribution into the radiation spectrum we have to substitute (4.5) into Eq.(3.11) and take the integrals. After substitution of variable in (4.5)

$$w = \frac{q}{2q_{min}\zeta}, \tag{4.7}$$

we obtain the integral over ζ in Eq.(3.11):

$$\int_1^\infty \left(v - \frac{4}{\zeta} + \frac{4}{\zeta^2}\right) \exp(-s^2\zeta^2) d\zeta \equiv f(s)$$
$$= \frac{\sqrt{\pi}}{2s}(v - 8s^2)\mathrm{Erfc}(s) + 4e^{-s^2} + 2\mathrm{Ei}(-s^2), \tag{4.8}$$

where

$$s = wrq_{min}, \quad r^2 = \Sigma_z^{-2}\cos^2\varphi + \Sigma_y^{-2}\sin^2\varphi \tag{4.9}$$

Making use of Eq.(4.4) we find for the spectrum

$$\begin{aligned} d\sigma_1^{(1)} &= \frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}f^{(1)}(\omega), \\ f^{(1)}(\omega) &= -\frac{1}{\pi\Sigma_z\Sigma_y}\int_0^{2\pi}\frac{d\varphi}{\Sigma_z^{-2}\cos^2\varphi + \Sigma_y^{-2}\sin^2\varphi}\int_0^\infty F_2(z)f(s)sds, \\ z^2 &= \frac{s^2}{q_{min}^2}\frac{1}{\Sigma_z^{-2}\cos^2\varphi + \Sigma_y^{-2}\sin^2\varphi}. \end{aligned} \tag{4.10}$$

This formula is quite convenient for the numerical calculations.

In the case $\mathbf{x}_0 \neq 0$ we will use straightforwardly Eqs.(3.12) and (4.4). Taking into account (4.2) we have for the difference

$$\begin{aligned} &\Delta^{(1)}(\mathbf{x}_0) \equiv \frac{1}{2\pi}(\Phi^{-1}(\mathbf{x}_0)F^{(1)}(\mathbf{x}_0) - \Phi^{-1}(0)F^{(1)}(0)) \\ &= \frac{\eta^2}{\pi\zeta^2}\int K_1^2(\eta\varrho)\exp\left[-\varrho_z^2\Sigma_z^2 - \varrho_y^2\Sigma_y^2\right]\{\exp\left[-2\varrho_z z_0\Sigma_z^2 - 2\varrho_y y_0\Sigma_y^2\right] - 1\}d^2\varrho, \end{aligned} \tag{4.11}$$

where the function $F^{(1)}(\mathbf{x}_0)$ is defined in Eqs.(3.12), (3.16).

Using the Macdonald's formula (see e.g.[15], p.53)

$$2K_1^2(\eta\varrho) = \int_0^\infty \exp\left[-\varrho^2 t - \frac{\eta^2}{2t}\right]K_1\left(\frac{\eta^2}{2t}\right)\frac{dt}{t} \tag{4.12}$$

and taking the Gaussian integrals over ϱ_z and ϱ_y we get

$$\Delta^{(1)}(\mathbf{x}_0) = \frac{1}{\zeta^2}\int_0^\infty \frac{\exp\left(-\frac{\eta^2}{2t}\right)K_1\left(\frac{\eta^2}{2t}\right)}{\sqrt{t+\Sigma_z^2}\sqrt{t+\Sigma_y^2}}\left\{\exp\left[\frac{z_0^2\Sigma_z^4}{t+\Sigma_z^2} + \frac{y_0^2\Sigma_y^4}{t+\Sigma_y^2}\right] - 1\right\}\frac{\eta^2 dt}{2t}. \tag{4.13}$$

For the correction to the cross section (see Eqs.(4.4) and (4.10)) we have correspondingly

$$d\sigma_1^{(1)} = \frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}[f^{(1)}(\omega) + J^{(1)}(\omega, \mathbf{x}_0)], \tag{4.14}$$

where

$$J^{(1)}(\omega, \mathbf{x}_0) = \int_1^\infty U(\zeta)\Delta^{(1)}(\mathbf{x}_0)d\zeta. \tag{4.15}$$

Now we pass over to the calculation of the second (subtraction) term $F^{(2)}(\omega,\zeta)$ in Eq.(3.12). Using Eq.(4.1) we get

$$\mathbf{I} = \eta \int K_1(\eta\varrho)\frac{\boldsymbol{\varrho}}{\varrho}w_c(\mathbf{x}-\boldsymbol{\varrho})d^2\varrho = \frac{\eta}{(2\pi)^2}\int S(q)\frac{\mathbf{q}}{q}\exp(-i\mathbf{q}\mathbf{x})w_c(\mathbf{q})d^2q, \tag{4.16}$$

where

$$\begin{aligned} S(q) &= \int K_1(\eta\varrho)\frac{\mathbf{q}\boldsymbol{\varrho}}{q\varrho}\exp(i\mathbf{q}\boldsymbol{\varrho})d^2\varrho \\ &= 2\pi i\int K_1(\eta\varrho)J_1(q\varrho)\varrho d\varrho = 2\pi i\,\frac{q}{\eta}\frac{1}{q^2+\eta^2}. \end{aligned} \tag{4.17}$$

Using the exponential parametrization

$$\frac{1}{q^2+\eta^2} = \frac{1}{4}\int_0^{\infty}\exp\left[-\frac{s}{4}(q^2+\eta^2)\right]ds \tag{4.18}$$

and taking the Gaussian integrals over q_z and q_y we obtain

$$\begin{aligned} \mathbf{I} &= \int_0^{\infty}\exp\left[-\frac{\eta^2 s}{4} - \frac{z^2}{s+2\sigma_z^2} - \frac{y^2}{s+2\sigma_y^2}\right] \\ &\times\left[\frac{z\mathbf{e}_z}{s+2\sigma_z^2} + \frac{y\mathbf{e}_y}{s+2\sigma_y^2}\right]\frac{ds}{\sqrt{s+2\sigma_z^2}\sqrt{s+2\sigma_y^2}}, \end{aligned} \tag{4.19}$$

where $\mathbf{e}_z$ and $\mathbf{e}_y$ are the unit vectors along axes z and y. Substituting (4.19) into Eq.(3.12), taking the Gaussian integrals over z and y and using Eq.(4.4) we get the correction to the cross section

$$d\sigma_1^{(2)} = -\frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}J^{(2)}(\omega,\mathbf{x}_0), \tag{4.20}$$

where

$$\begin{aligned} J^{(2)}(\omega,\mathbf{x}_0) &= \frac{\sqrt{ab}}{\Sigma_z\Sigma_y}\exp(z_0^2\Sigma_z^2 + y_0^2\Sigma_y^2)\int_0^{\infty}ds_1\int_0^{\infty}ds_2 g\left(\frac{q_{min}\sqrt{s}}{2}\right)G(s_1,s_2,\mathbf{x}_0), \\ G(s_1,s_2,\mathbf{x}_0) &= \left(\frac{a_1a_2b_1b_2}{AB}\right)^{1/2}\left[\frac{a_1a_2}{A}\left(\frac{1}{2}+\frac{z_0^2a^2}{A}\right) + \frac{b_1b_2}{B}\left(\frac{1}{2}+\frac{y_0^2b^2}{B}\right)\right] \\ &\times\exp\left[-\frac{z_0^2a}{A}(a_1+a_2) - \frac{y_0^2b}{B}(b_1+b_2)\right]. \end{aligned} \tag{4.21}$$

Here the function g appears as a result of integration over ζ:

$$
\begin{aligned}
g(q) &= \int_1^{\infty}\left(v-\frac{4}{\zeta}+\frac{4}{\zeta^2}\right)\exp(-q^2\zeta^2)\frac{d\zeta}{\zeta^2} \\
&= \left(v-\frac{2}{3}\right)\exp(-q^2)-2q^2\int_1^{\infty}\left(v-\frac{2}{\zeta}+\frac{4}{3\zeta^2}\right)\exp(-q^2\zeta^2)d\zeta \\
&= \left(v-\frac{2}{3}\right)\exp(-q^2)-2q^2\left[\frac{\sqrt{\pi}}{2q}\left(v-\frac{8}{3}q^2\right)\mathrm{Erfc}(q)\right. \\
&\quad \left. +\ \frac{4}{3}e^{-q^2}+\mathrm{Ei}(-q^2)\right]. \qquad (4.22)
\end{aligned}
$$

In (4.21) we introduced the following notations

$$
\begin{aligned}
& a=\frac{1}{2\Delta_z^2}, \quad b=\frac{1}{2\Delta_y^2}, \quad a_{1,2}=\frac{1}{s_{1,2}+2\sigma_z^2}, \quad b_{1,2}=\frac{1}{s_{1,2}+2\sigma_y^2}, \\
& A=a_1+a_2+a, \quad B=b_1+b_2+b, \quad s=s_1+s_2. \qquad (4.23)
\end{aligned}
$$

5 Narrow beams

This is the case when the ratio $q_{min}/(\Sigma_z+\Sigma_y)\ll 1$, so that the main contribution to the integral (4.10) gives the region $q\sim\zeta\sim 1,\ z\gg 1$. Using the asymptotics of function $F_2(z)$ at $z\gg 1$

$$
F_2(z)\simeq \ln(2z)^2-1 \qquad (5.1)
$$

and the following integrals

$$
\begin{aligned}
&\frac{1}{2\pi\Sigma_z\Sigma_y}\int_0^{2\pi}\frac{d\varphi}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}=1, \\
&\frac{1}{2\pi\Sigma_z\Sigma_y}\int_0^{2\pi}\frac{d\varphi}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi} \\
&\times\ \ln\frac{4}{\Sigma_z^{-2}\cos^2\varphi+\Sigma_y^{-2}\sin^2\varphi}=\ln(\Sigma_z+\Sigma_y)^2, \\
&\int_1^{\infty}dq^2\left(\alpha-\beta\ln q^2\right)\int_1^{\infty}\left(v-\frac{4}{\zeta}+\frac{4}{\zeta^2}\right)\exp(-q^2\zeta^2)d\zeta \\
&=\left(v-\frac{2}{3}\right)\left[\alpha+\beta(2+C)\right]+\frac{2}{9}\beta, \qquad (5.2)
\end{aligned}
$$

where C is Euler's constant $C = 0.577...$, we get for the function $f^{(1)}(\omega)$ (4.14) the following expression

$$f^{(1)}(\omega) \simeq \left(v - \frac{2}{3}\right)\left(2\ln\frac{q_{min}}{\Sigma_z + \Sigma_y} + 3 + C\right) + \frac{2}{9}, \qquad q_{min} \ll (\Sigma_z + \Sigma_y). \quad (5.3)$$

This expression agrees with Eq.(24) of [7].

Under the assumption used in (5.3) and the additional condition $q_{min}(z_0 + y_0) \ll 1$ the main contribution to the integral in (4.13) gives the region $t \gg \eta^2$. In this case one can use the asymptotic expansion $K_1(z) \simeq 1/z (z \ll 1)$. Then we have for the function $J^{(1)}(\omega, \mathbf{x}_0)$ in Eq.(4.14) the following expression

$$J^{(1)}(\omega, \mathbf{x}_0) \simeq \left(v - \frac{2}{3}\right) J\,,$$

$$J = \int_0^\infty \left[\exp\left(\frac{z_0^2\Sigma_z^4}{t + \Sigma_z^2} + \frac{y_0^2\Sigma_y^4}{t + \Sigma_y^2}\right) - 1\right] \frac{dt}{\sqrt{t + \Sigma_z^2}\sqrt{t + \Sigma_y^2}}. \quad (5.4)$$

The expression (5.4) is consistent with Eq.(26) of [7].

In the case $(\mathbf{x}_0^2 + \sigma_z^2 + \sigma_y^2)q_{min}^2 \ll 1$ the main contribution to the integral in (4.21) gives the interval $sq_{min}^2 \sim (\mathbf{x}_0^2 + \sigma_z^2 + \sigma_y^2)q_{min}^2 \ll 1$. Keeping the main term of expansion over q^2 in Eq.(4.22) we get

$$g\left(\frac{q_{min}\sqrt{s}}{2}\right) \simeq v - \frac{2}{3}. \quad (5.5)$$

The same result can be obtained if one neglects the term containing η^2 in the exponent of integrand in Eq.(4.19).

Summing the cross section $d\sigma = d\sigma_1^{(1)} + d\sigma_1^{(2)}$ with the standard QED bremsstrahlung cross section

$$d\sigma_0 = \frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left(v - \frac{2}{3}\right)\left(\ln\frac{m^2}{q_{min}^2} - 1\right), \quad (5.6)$$

we get the cross section for the case of interaction of narrow beams

$$d\sigma_\gamma = d\sigma_0 + d\sigma_1 = \frac{2\alpha^3}{m^2}\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\Bigg\{\left(v - \frac{2}{3}\right)\Bigg[2\ln\frac{m}{\Sigma_z + \Sigma_y} + C + 2$$

$$+J - J_-\Bigg] + \frac{2}{9}\Bigg\}, \quad v = \frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}, \quad \varepsilon' = \varepsilon - \omega, \quad (5.7)$$

where J is given in (5.4),

$$J_{-} = \frac{\sqrt{ab}}{\Sigma_z\Sigma_y}\exp(z_0^2\Sigma_z^2 + y_0^2\Sigma_y^2)\int_0^\infty ds_1 \int_0^\infty ds_2 G(s_1, s_2, \mathbf{x}_0), \tag{5.8}$$

where entering functions defined in Eqs.(4.21) and (4.23).

In the case of coaxial beams $\mathbf{x}_0 = 0,\ J = 0$ one can take the integral in (5.8) over one of variables (for definiteness over s_2) using the formula

$$\int_0^\infty \frac{dx}{(a_z + b_z x)^{3/2}(a_y + b_y x)^{1/2}} = \frac{2}{a_z\sqrt{b_z b_y} + b_z\sqrt{a_z a_y}}. \tag{5.9}$$

After this we have the simple integral over $s \equiv s_1$

$$\begin{aligned} J_{-}(0) &= \sqrt{1+\delta_z}\sqrt{1+\delta_y}(J_z + J_y), \\ J_{z,y} &= \int_0^\infty D_{z,y}(s)ds, \quad D_{z,y} = \frac{1}{a_{z,y}\sqrt{b_z b_y} + b_{z,y}\sqrt{a_z a_y}}, \end{aligned} \tag{5.10}$$

where

$$a_{z,y} = s(1+\delta_{z,y}) + 2\sigma_{z,y}^2(2+\delta_{z,y}),\ \ b_{z,y} = \frac{s}{2\Delta_{z,y}^2} + 1 + \delta_{z,y},\ \ \delta_{z,y} = \frac{\sigma_{z,y}^2}{\Delta_{z,y}^2}, \tag{5.11}$$

The cross section (5.7) differs from Eq.(24) of [7] because the subtraction term J_{-} is included. Without this term, generally speaking, the bremsstrahlung cross section would be qualitatively erroneous. In particular an appearance of the term J_{-} violates, generally speaking, the symmetry of radiation cross section in opposite directions in e^-e^- (e^-e^+) collisions.

To elucidate the qualitative features of narrow beams bremsstrahlung process we consider the case of round beams where the calculation becomes more simple:

$$\begin{aligned} &\sigma_z = \sigma_y = \sigma, \quad \Delta_z = \Delta_y = \Delta, \quad \Sigma_z^2 = \Sigma_y^2 = \Sigma^2 = \frac{1}{2(\sigma^2+\Delta^2)}, \\ &b = a, \quad b_{1,2} = a_{1,2}, \quad B = A, \quad \delta = \frac{\sigma^2}{\Delta^2}. \end{aligned} \tag{5.12}$$

We consider first the case of coaxial beams ($\mathbf{x}_0 = 0,\ J = 0$),

$$\begin{aligned} J_{-} &= (1+\delta)\int_0^\infty \frac{ds}{[s(1+\delta)+2+\delta][s\delta+1+\delta]} \\ &= (1+\delta)\ln\frac{(1+\delta)^2}{\delta(2+\delta)}. \end{aligned} \tag{5.13}$$

In the limiting cases the function J_- has the form

$$J_-(\delta \gg 1) \simeq \frac{1}{\delta}, \quad J_-(\delta = 1) = 2\ln\frac{4}{3}, \quad J_-(\delta \ll 1) \simeq \ln\frac{1}{2\delta}. \tag{5.14}$$

In the first case the subtraction term J_- is small. For the beams of the same size the subtraction term J_- contributes to the constant entering into the expression for the cross section. The subtraction term J_- modifies essentially the cross section in the case when the radius of target beam is much smaller than the radius of radiating beam. In this case the cross section (5.7) contains the combination

$$\ln\frac{m^2}{4\Sigma^2} - J_- \simeq \ln\frac{m^2\Delta^2}{2} - \ln\frac{\Delta^2}{2\sigma^2} = \ln(m\sigma)^2. \tag{5.15}$$

So in the all cases considered above the cross section defines the transverse dimension of target beam.

When the axes of round beams are displaced with respect each other in the transverse plane the integral in (5.4) is

$$\begin{aligned} J &= \int_0^\infty \left[\exp\left(\frac{d}{x+1}\right) - 1\right]\frac{dx}{x+1} = \mathrm{Ei}(d) - C - \ln d, \\ d &= \mathbf{x}_0^2\Sigma^2 = \frac{x_0^2 + y_0^2}{2(\Delta^2+\sigma^2)}. \end{aligned} \tag{5.16}$$

It is convenient in this case to calculate the function J_- using straightforwardly Eq.(4.19) where we omit the term with η^2 in the exponent of integrand

$$\mathbf{I}_{cr} = \boldsymbol{\varrho}\int_0^\infty \exp\left(-\frac{\boldsymbol{\varrho}^2}{s+2\sigma^2}\right)\frac{ds}{(s+2\sigma^2)^2} = \frac{\boldsymbol{\varrho}}{\boldsymbol{\varrho}^2}\left[1 - \exp\left(-\frac{\boldsymbol{\varrho}^2}{2\sigma^2}\right)\right] \tag{5.17}$$

Substituting this expression ($\mathbf{I}$ is defined in Eq.(4.16)) into the subtraction term Eq.(3.12) and using the exponential parametrization

$$\frac{1}{\boldsymbol{\varrho}^2} = \int_0^\infty \exp(-\boldsymbol{\varrho}^2 s)ds$$

we obtain

$$
\begin{aligned}
J_- &= \frac{ae^d}{\pi\Sigma^2}\int_0^\infty ds \int d^2\varrho \exp(-\boldsymbol{\varrho}^2 s)\left[1-\exp\left(-\frac{\boldsymbol{\varrho}^2}{2\sigma^2}\right)\right]\exp(-a(\boldsymbol{\varrho}+\mathbf{x}_0)^2) \\
&= \frac{ae^{d-d_1}}{\Sigma^2}\int_0^\infty \left[\frac{1}{s+a}\exp\left(\frac{d_1 a}{s+a}\right)\right. \\
&-2\frac{1}{s+a+\sigma^{-2}/2}\exp\left(d_1\frac{a}{s+a+\sigma^{-2}/2}\right) \\
&\left.+\frac{1}{s+a+\sigma^{-2}}\exp\left(d_1\frac{a}{s+a+\sigma^{-2}}\right)\right]ds \\
&= \frac{ae^{d-d_1}}{\Sigma^2}\left[\mathrm{Ei}(d_1)-2\mathrm{Ei}\left(d_1\frac{\sigma^2}{\sigma^2+\Delta^2}\right)+\mathrm{Ei}\left(d_1\frac{\sigma^2}{\sigma^2+2\Delta^2}\right)\right], \\
d_1 &= a\mathbf{x}_0^2 = \frac{z_0^2+y_0^2}{2\Delta^2}.
\end{aligned}
\tag{5.18}
$$

In the limit $d_1 \to 0$ the last expression goes over to Eq.(5.13).

When the displacement of the axes of colliding beams is large enough ($\mathbf{x}_0^2 \gg \sigma^2+\Delta^2$) one use the asymptotic expansion of the function $\mathrm{Ei}(z)$ in (5.18):

$$
\mathrm{Ei}(z) \simeq \frac{e^z}{z}\left(1+\frac{1}{z}\right), \quad z \gg 1. \tag{5.19}
$$

In this case the main terms in the difference $J - J_-$ in Eq.(5.7) are canceled:

$$
J - J_- \simeq \frac{e^d}{d}\left(\frac{1}{d}-\frac{1}{d_1}\right) = \frac{2e^d}{d}\frac{\sigma^2}{\mathbf{x}_0^2}. \tag{5.20}
$$

The compensation of the main terms in (5.19) is due to the fact that the incoherent scattering originates on the fluctuations of the potential of the target (scattering) beam. Correspondingly we have for the mean square of the momentum transfer dispersion at the large distance from the target beam

$$
\begin{aligned}
&< \mathbf{q}^2(\boldsymbol{\varrho}) > - < \mathbf{q}(\boldsymbol{\varrho}) >^2 \propto \left\langle \frac{1}{(\mathbf{x}_0+\boldsymbol{\varrho})^2}-\frac{1}{\mathbf{x}_0^2}\right\rangle \\
&\simeq \left\langle \frac{4(\mathbf{x}_0\boldsymbol{\varrho})^2}{\mathbf{x}_0^6}-\frac{\boldsymbol{\varrho}^2}{\mathbf{x}_0^4}\right\rangle = \frac{<\boldsymbol{\varrho}^2>}{\mathbf{x}_0^4} = \frac{2\sigma^2}{\mathbf{x}_0^4}.
\end{aligned}
\tag{5.21}
$$

Substituting (5.20) into Eq.(5.7) and multiplying the result by the luminosity (4.3):

$$
\mathcal{L} = N_c N_r \frac{\Sigma^2}{\pi}\exp(-\mathbf{x}_0^2\Sigma^2) \tag{5.22}
$$

we have for the probability of bremsstrahlung of round beams moving apart at large distance

$$dw_\gamma \simeq 4N_cN_r\frac{\alpha^3}{\pi}\lambda_c^2\Sigma^2\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left(v-\frac{2}{3}\right)$$
$$\times\left[\exp(-\mathbf{x}_0^2\Sigma^2)\ln\frac{m}{\Sigma}+\frac{\sigma^2\Sigma^2}{(\mathbf{x}_0^2\Sigma^2)^2}+O(\exp(-\mathbf{x}_0^2\Sigma^2))\right],$$
$$\Sigma^2=\frac{1}{2(\Delta^2+\sigma^2)},\ \ \mathbf{x}_0^2\Sigma^2=\frac{z_0^2+y_0^2}{2(\Delta^2+\sigma^2)}\gg 1,\ \ q_{min}^2(z_0^2+y_0^2)\ll 1. \quad (5.23)$$

According to (5.23) when $\mathbf{x}_0^2$ increases so that one can neglect the first term in square brackets, the probability of bremsstrahlung of the round beams diminishes as a power of distance between beams ($\propto \sigma^2/\mathbf{x}_0^4$). The cross section Eq.(5.7) in this case grows exponentially as e^d/d^2. Let us note that without the subtraction term one has erroneous qualitative behaviour of probability ($\propto 1/\mathbf{x}_0^2$). These circumstances explain also Eq.(5.15) for the coaxial beams: at integration over $d^2\varrho$ the region contributes where $<\mathbf{q}^2(\boldsymbol{\varrho})>-<\mathbf{q}(\boldsymbol{\varrho})>^2\propto 1/\boldsymbol{\varrho}^2$, so that $\varrho\leq\sigma$.

Let us consider now the general case $\Sigma_z\neq\Sigma_y$ for enough large displacement of beams $\mathbf{x}_0^2\gg\Sigma_{z,y}^{-2}$. In this case the main contribution into the integral $\mathbf{I}(\mathbf{x})$ (for $\eta^2=0$) in Eqs.(4.16),(4.19) at large $|\mathbf{x}|\simeq|\mathbf{x}_0|$ (see Eq.(3.12)) are given by large values $s\sim\mathbf{x}_0^2\gg\sigma_{z,y}^2$. Expanding the integrand over the powers $\sigma_{z,y}^2/s$ and keeping after integration the two main terms of the decomposition over $1/\mathbf{x}^2$ we get

$$\mathbf{I}^2(\mathbf{x})\simeq\frac{1}{\mathbf{x}^2}\left[1+\frac{2}{(\mathbf{x}^2)^2}(y^2-z^2)(\sigma_y^2-\sigma_z^2)\right]. \quad (5.24)$$

Expanding the function $1/(\mathbf{x}_0+\boldsymbol{\xi})^2$ over the powers ξ/x_0 at the integration over $\boldsymbol{\xi}=\mathbf{x}-\mathbf{x}_0$ in Eq.(3.12)) we find

$$\int\mathbf{I}^2(\mathbf{x}_0+\boldsymbol{\xi})w_r(\boldsymbol{\xi})d^2\xi\simeq\frac{1}{\mathbf{x}_0^2}\left[1+\frac{4}{(\mathbf{x}_0^2)^2}(z_0^2\Delta_z^2+y_0^2\Delta_y^2)\right.$$
$$\left.-\frac{\boldsymbol{\Delta}^2}{\mathbf{x}_0^2}+\frac{2}{(\mathbf{x}_0^2)^2}(y_0^2-z_0^2)(\sigma_y^2-\sigma_z^2)\right],\quad \boldsymbol{\Delta}^2=\Delta_z^2+\Delta_y^2. \quad (5.25)$$

In this case the region $t\sim 1/\mathbf{x}_0^2\ll\Sigma_{z,y}^2$ contributes into the integral J Eq.(5.4)). Expanding the integrand over the powers $t\Sigma_{z,y}^{-2}$ and keeping the two main terms of decomposition over $1/\mathbf{x}_0^2$ we have

$$J \simeq \frac{1}{\Sigma_z\Sigma_y\mathbf{x}_0^2}\exp(z_0^2\Sigma_z^2 + y_0^2\Sigma_y^2)\left\{1 - \frac{\boldsymbol{\sigma}^2 + \boldsymbol{\Delta}^2}{\mathbf{x}_0^2}\right.$$
$$\left. + \frac{4}{(\mathbf{x}_0^2)^2}\left[z_0^2(\sigma_z^2 + \Delta_z^2) + y_0^2(\sigma_y^2 + \Delta_y^2)\right]\right\}, \quad \boldsymbol{\sigma}^2 = \sigma_z^2 + \sigma_y^2. \quad (5.26)$$

For the difference $J - J_-$ we obtain finally

$$J - J_- = \frac{1}{\Sigma_z\Sigma_y}\exp(z_0^2\Sigma_z^2 + y_0^2\Sigma_y^2)\frac{\boldsymbol{\sigma}^2}{(\mathbf{x}_0^2)^2}. \quad (5.27)$$

6 Narrow flat beams ($\sigma_z \ll \sigma_y, \Delta_z \ll \Delta_y$)

Let us begin with the coaxial beams. We consider first the case where the size of radiating beam is much larger than size of target beam ($\delta_{z,y} \ll 1$). In this case one can neglect the terms $\propto \delta_{z,y}, \sigma_z^2, \Delta_y^{-2}$ in the functions $a_{z,y}$ and $b_{z,y}$ in the integral in Eq.(5.10). Within this accuracy

$$a_z \simeq s, \quad a_y \simeq s + 4\sigma_y^2, \quad b_z \simeq \frac{s}{2\Delta_z^2} + 1, \quad b_y \simeq 1. \quad (6.1)$$

After substitution in the integral J_y in Eq.(5.10) $s \to 4\sigma_y^2 s$ one gets

$$J_y(\kappa) = \int_0^\infty \frac{ds}{\sqrt{s+1}(\sqrt{s} + \sqrt{s+1}\sqrt{1+2\kappa s})}, \quad \kappa = \frac{\sigma_y^2}{\Delta_z^2}. \quad (6.2)$$

After substitution in the integral J_z in Eq.(5.10) $s \to 2\Delta_z^2/s$ one gets $J_z = J_y$ so that

$$J_-(\kappa) = 2\sqrt{1+\delta_z}\sqrt{1+\delta_y}J_y(\kappa) \simeq 2J_y(\kappa),$$
$$J_-(\kappa \ll 1) \simeq \ln\frac{8}{\kappa}, \quad J_-(\kappa \gg 1) \simeq \pi\sqrt{\frac{2}{\kappa}}. \quad (6.3)$$

It is seen from the last equation that at $\Delta_z \ll \sigma_y$ the contribution of the term J_- into the cross section Eq.(5.7) is relatively small. In the opposite case $\Delta_z \gg \sigma_y$ this contribution leads to change of the logarithm argument in Eq.(5.7)

$$2\ln\frac{m}{(\Sigma_z + \Sigma_y)} - \ln\frac{8}{\kappa} \simeq 2(\ln(\sqrt{2}m\Delta_z) - \ln(2\sqrt{2}\frac{\Delta_z}{\sigma_y})) = 2\ln\frac{m\sigma_y}{2}. \quad (6.4)$$

This is a new qualitative result.

In the opposite case when the size of radiating beam is smaller or is of the order of size of target beam ($\delta_{z,y} \geq 1$) the contribution into the integral J_z in Eq.(5.10) gives the region $s \sim \sigma_z^2$ and into the integral J_y the region $s \sim \sigma_y^2$. Performing in the integral J_z the substitution $s \to 2\sigma_z^2 s$ and in the integral J_y the substitution $s \to 2\sigma_y^2/s$ one gets

$$
\begin{aligned}
J_z &\simeq \frac{\sigma_z}{\sqrt{2+\delta_y}\sigma_y}\int_0^\infty \frac{ds}{((s+1)\delta_z+1)\sqrt{s(1+\delta_z)+2+\delta_z}} \\
&= \frac{2}{\sqrt{2+\delta_y}}\frac{\Delta_z}{\sigma_y}\arctan\frac{1}{\sqrt{\delta_z(2+\delta_z)}}; \\
J_y &\simeq \frac{\Delta_z}{\sigma_y}\int_0^\infty \frac{ds}{((s+1)(\delta_y+1)+s)\sqrt{(s+1)\delta_y+s}} \\
&= \frac{2}{\sqrt{2+\delta_y}}\frac{\Delta_z}{\sigma_y}\arctan\frac{1}{\sqrt{\delta_y(2+\delta_y)}}; \\
J_- &= \sqrt{1+\delta_z}\sqrt{1+\delta_y}(J_z+J_y) \\
&= \frac{2\sqrt{1+\delta_z}\sqrt{1+\delta_y}}{\sqrt{2+\delta_y}}\frac{\Delta_z}{\sigma_y}\left(\arctan\frac{1}{\sqrt{\delta_z(2+\delta_z)}}+\arctan\frac{1}{\sqrt{\delta_y(2+\delta_y)}}\right).
\end{aligned} \tag{6.5}
$$

In the case $\delta_{z,y} \ll 1,\ \Delta_z \ll \sigma_y$ this formula is consistent with Eq.(6.3).

Now we go over to the case of the displaced beams. For enough large displacement of the beams the formulas (5.7) and (5.27) are valid. So the intermediate case is of interest. As an example we consider the case $\sigma_y^2 \gg z_0^2 \gg \sigma_z^2 + \Delta_z^2,\ y_0^2 \ll \sigma_y^2$. In this case the contribution to the integral in (5.4) gives the interval $\Sigma_y^2 \ll t \sim z_0^{-2} \ll \Sigma_z^2$. Keeping the main terms of decomposition over $t\Sigma_z^{-2} \ll 1$ and $t\Sigma_y^{-2} \gg 1$ we have

$$
J \simeq \frac{1}{\Sigma_z}\int_0^\infty \exp\left(z_0^2\Sigma_z^2 - z_0^2 t\right)\frac{dt}{\sqrt{t}} = \frac{\sqrt{\pi}}{z_0\Sigma_z}\exp(z_0^2\Sigma_z^2). \tag{6.6}
$$

Under these conditions ($\mathbf{x}_0^2 \ll \sigma_y^2$) the contribution into the integral for J_- in (5.8)of the term in the function $G(s_1, s_2, \mathbf{x}_0)$ Eq.(4.21) in the square brackets containing $b_1 b_2/B$ is defined by the function J_y in Eq.(6.5) to within the terms $\sim z_0/\sigma_y$. In the term containing $a_1 a_2/A$ (which we denote by $J_-^{(z)}$) the main contribution gives the summand $z_0^2 a^2/A^2$ in the interval $\sigma_y^2 \gg s_{1,2} \sim z_0^2 \gg \sigma_z^2$ where

$$
a_{1,2} \simeq \frac{1}{s_{1,2}}, \quad b_{1,2} \simeq \frac{1}{2\sigma_y^2}, \quad A \simeq a, \quad B \simeq \frac{1}{\sigma_y^2} + \frac{1}{2\Delta_y^2}. \tag{6.7}
$$

As a result we obtain

$$
\begin{aligned}
& J_-^{(z)}(\mathbf{x}_0) \simeq \frac{z_0^2}{2\Sigma_z\Sigma_y\sigma_y^2}\sqrt{\frac{b}{B}}e^{d_z}\int_0^\infty \frac{ds_1}{s_1^{3/2}}\int_0^\infty \exp\left(-z_0^2\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\right)\frac{ds_2}{s_2^{3/2}} \\
& = \pi\frac{\Delta_z}{\sigma_y}\frac{\sqrt{1+\delta_z}\sqrt{1+\delta_y}}{\sqrt{2+\delta_y}}e^{d_z}, \\
& J - J_- \simeq \sqrt{\frac{\pi}{d_z}}e^{d_z}h(z_0), \quad d_z = z_0^2\Sigma_z^2, \\
& h(z_0) = 1 - \frac{\sqrt{\pi(1+\delta_y)}}{\sqrt{2(2+\delta_y)}}\frac{z_0}{\sigma_y}\left(1+\frac{2}{\pi}\arctan\frac{1}{\sqrt{\delta_y(2+\delta_y)}}\right). \qquad (6.8)
\end{aligned}
$$

It should be noted that for the flat beams the probability of radiation as a function of distance between beams (for the considered interval) decreases more slowly $\propto 1/\sqrt{d_z}$ than for the round beams given in Eq.(5.23)

$$
dw_\gamma^{fl} \simeq 4N_cN_r\frac{\alpha^3}{\pi}\lambda_c^2\Sigma_z\Sigma_y\frac{\varepsilon'}{\varepsilon}\frac{d\omega}{\omega}\left(v-\frac{2}{3}\right)\left[e^{-d_z}\ln\frac{m}{\Sigma_z}+\frac{1}{2}\sqrt{\frac{\pi}{d_z}}h(z_0)\right]. \qquad (6.9)
$$

Compensation in the difference $J - J_-$ begins in the region $z_0 \sim \sigma_y + \Delta_y$ were Eq.(6.8) is not valid and one have to use more accurate Eq.(5.8). In the region $z_0 \gg \sigma_y + \Delta_y$ the probability of radiation decreases as $1/z_0^4$ according to Eqs.(4.4), (5.7), (5.27) provided that one can neglect the exponential term in the square brackets in Eq.(6.9) (compare with Eq.(5.23))

$$
dw_\gamma^{fl}(z_0) \simeq 2N_cN_r\frac{\alpha^3}{\pi}\frac{\lambda_c^2\sigma_y^2}{z_0^4}\frac{\varepsilon'}{\varepsilon}\left(v-\frac{2}{3}\right)\frac{d\omega}{\omega}, \quad z_0 \gg y_0. \qquad (6.10)
$$

7 Observation of beam-size effect

Above we calculated the incoherent bremsstrahlung spectrum at collision of electron and positron beams with finite transverse dimensions. This spectrum differs from spectrum found previously in [7], [8], [9] because here (in contrast to previous papers) we subtract the coherent contribution. In general expression for correction to the probability of photon emission (3.11) the subtraction term is $F^{(2)}(\omega, \zeta)$. For the coaxial beams for numerical calculation it is convenient to use Eqs.(4.10), (4.20) and (4.21). In the last equation one have to put $y_0 = z_0 = 0$. In the case of collision of narrow beams the subtraction term in the bremsstrahlung spectrum (5.7) is J_-.

The dimensions of beams in the experiment [6] were $\sigma_z = \Delta_z = 24\ \mu m$, $\sigma_y = \Delta_y = 450\ \mu m$, so this is the case of flat beams. The estimate for this case (6.5) gives $J_- \simeq (4/3\sqrt{3})\pi\sigma_z/\sigma_y \ll 1$. This term is much smaller than other terms in (5.7). This means that for this case the correction to the spectrum calculated in [7] is very small.

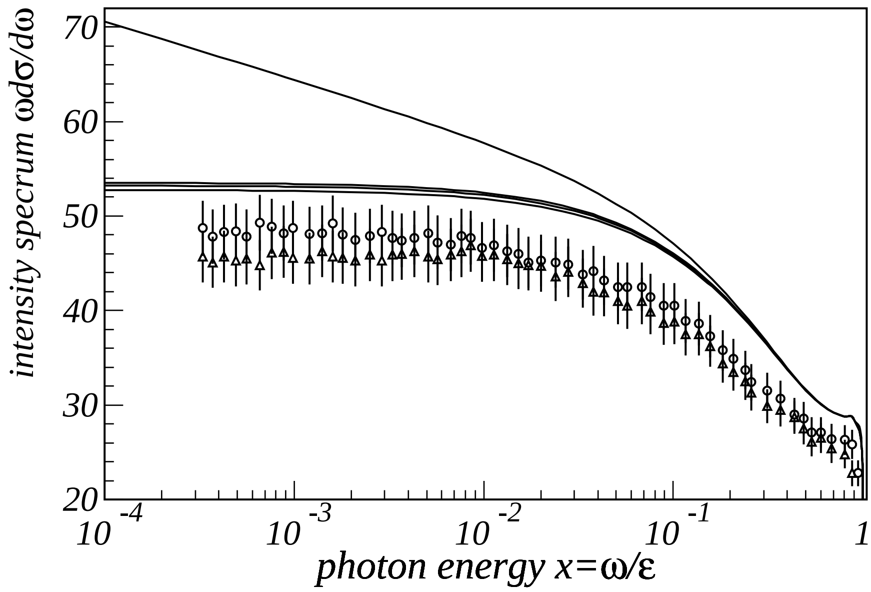


Figure 1: The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units $2\alpha r_0^2$ versus the photon energy in units of initial electron energy ($x = \omega/\varepsilon$) for VEPP4 experiment. The upper curve is the standard QED spectrum, the three close curves below are calculated for the different vertical dimensions of colliding beams (equal for two colliding beams $\sigma = \sigma_z = \Delta_z$): $\sigma = 20\ \mu m$ (bottom), $\sigma = 24\ \mu m$ (middle), $\sigma = 27\ \mu m$ (top). The data measured in [6] are presented as circles (the experiment in 1980) and as triangles (the experiment in 1981) with 6% systematic error as obtained in [6].

The result of calculation and VEPP4 (INP, Novosibirsk) data are presented in Fig.1 where the bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ is given in units $2\alpha r_0^2$ versus the photon energy in units of initial electron energy ($x = \omega/\varepsilon$). The upper curve is the standard QED spectrum, the three close curves below are calculated using Eqs.(4.10) and (4.20) for the different vertical dimensions of colliding beams (equal for two colliding beams $\sigma = \sigma_z = \Delta_z$): $\sigma = 20\ \mu m$ (bottom), $\sigma = 24\ \mu m$ (middle), $\sigma = 27\ \mu m$ (top) (this is just the 1-sigma dispersion for the beams used in the experiment). We want to emphasize that all the theoretical curves are calculated to within the relativistic accuracy (the discarded terms are of the order m/ε). It is seen that the effect of the small transverse dimensions is very essential in soft part of spectrum (at $\omega/\varepsilon = 10^{-4}$ the spectral curve diminishes in 25%), while for $\omega/\varepsilon > 10^{-1}$ the effect becomes negligible. The data measured in [6] are presented as circles (experiment in 1980) and as triangles (experiment in 1981) with 6% systematic error as obtained in [6] (while the statistical errors are negligible). This presentation is somewhat different from [6]. It is seen that the data points are situated systematically below the theory curves but the difference is not exceed the 2-sigma level [6]. It should be noted that this is true also in the hard part of spectrum where the beam-size effect is very small.

The last remark is connected with the radiative corrections (RC). The RC to the spectrum of double bremsstrahlung [16] (this was the normalization process) are essential (of the order 10%) and were taken into account. The RC to the bremsstrahlung spectrum [17] are very small (less than 0.4%) and may be neglected. It should be noted that the RC to the bremsstrahlung spectrum are insensitive to the effect of small transverse dimensions.

The dependence of bremsstrahlung spectrum on beams characteristics was measures specifically in [6]. The first is the dependence of bremsstrahlung spectrum on vertical sizes of beams σ_z. It is calculated using Eqs.(4.10) and (4.20) for $\omega/\varepsilon = 10^{-3}$. The result is shown in units $2\alpha r_0^2$ in Fig.2. The data is taken from Fig.7 in [6]. The second is the measurement of dependence of bremsstrahlung spectrum on the vertical displacement of beams z_0. It is calculated using Eqs.(5.4) and (5.8) for $\omega/\varepsilon = 10^{-3}$. Because of displacement it is necessary to normalize the spectrum on the luminosity

$$\mathcal{L} = N_c N_r \frac{\Sigma_z \Sigma_y}{\pi} \exp(-z_0^2 \Sigma_z^2),$$

see Eq.(4.3). This means that when we compare the bremsstrahlung process (where the beam-size effect is essential) with some other process like double bremsstrahlung used in [6] (which is insensitive to the effect) we have to multiply the cross section of the last process by the luminosity $\mathcal{L}$. This is seen in estimate Eq.(6.9): after taking out the exponent e^{-d_z} we have the luminosity as the external factor and in expression for ratio $N_\gamma/N_{2\gamma}$ (which was observed in [6]) the cross section of double bremsstrahlung will be multiplied by the luminosity. After this operation the second term in square brackets

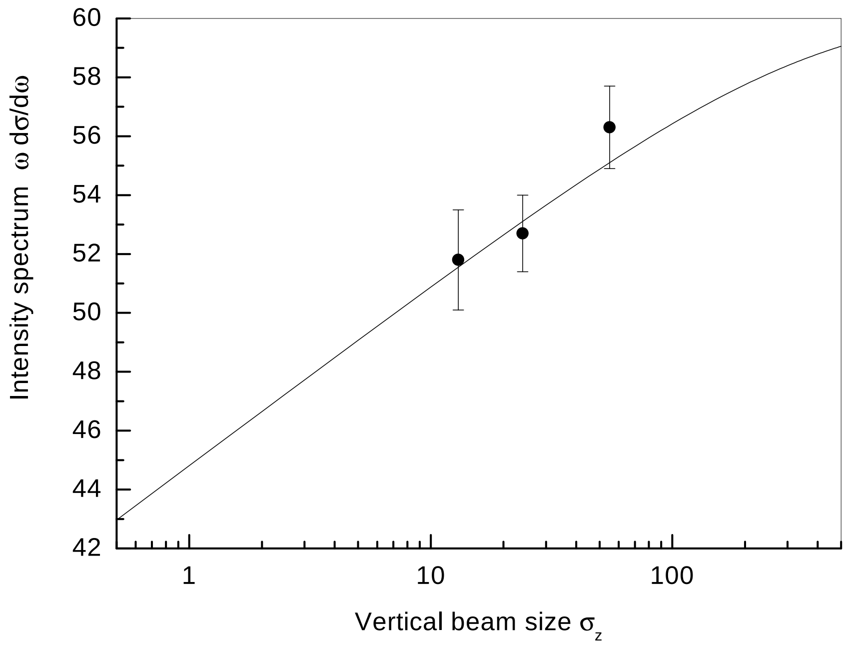


Figure 2: The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units $2\alpha r_0^2$ versus the vertical sizes of beams σ_z (in μm). The data taken from [6].

will contain the combination $e^{d_z}h(z_0)/\sqrt{d_z}$ which grows exponentially with the displacement z_0 increase. The normalized bremsstrahlung spectrum is shown in units $2\alpha r_0^2$ in Fig.3. So, the very fast (exponential) increase with z_0 is due to fast decrease with z_0 of the double bremsstrahlung probability for the displaced beams. The data is taken from Fig.8 in [6]. It should be noted that in soft part of spectrum the dependence on photon energy ω is very weak. It is seen in these figures that there is quite reasonable agreement between theory and data just as in [6]. This means that contribution of J_- term which is calculated only in the present paper is relatively small.

One more measurement of beam-size effect was performed at HERA electron-proton collider (DESY, Germany) [18]. The electron beam energy was $\varepsilon = 27.5$ GeV, the proton beam energy was $\varepsilon_p = 820$ GeV. The standard bremsstrahlung spectrum for this case is given by Eq.(5.6) where q_{min} should

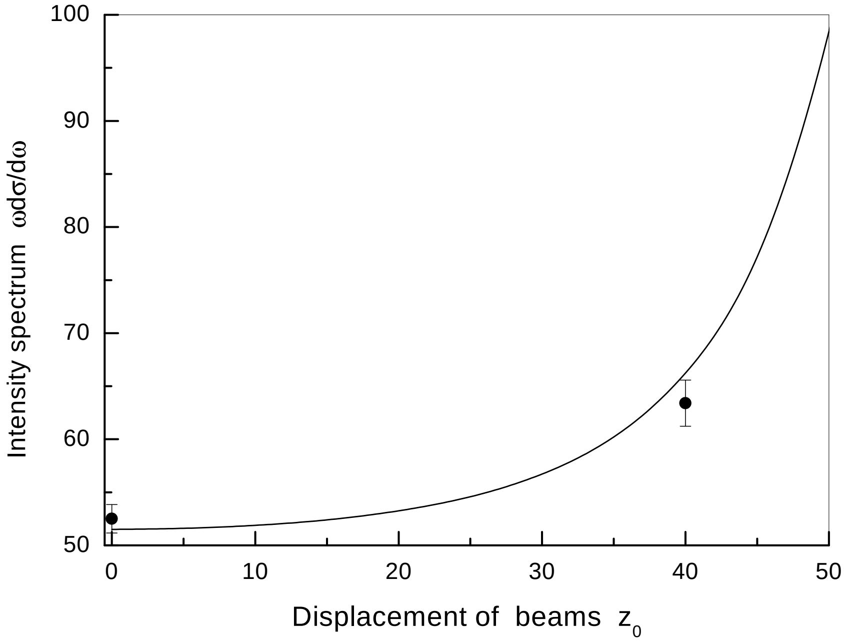
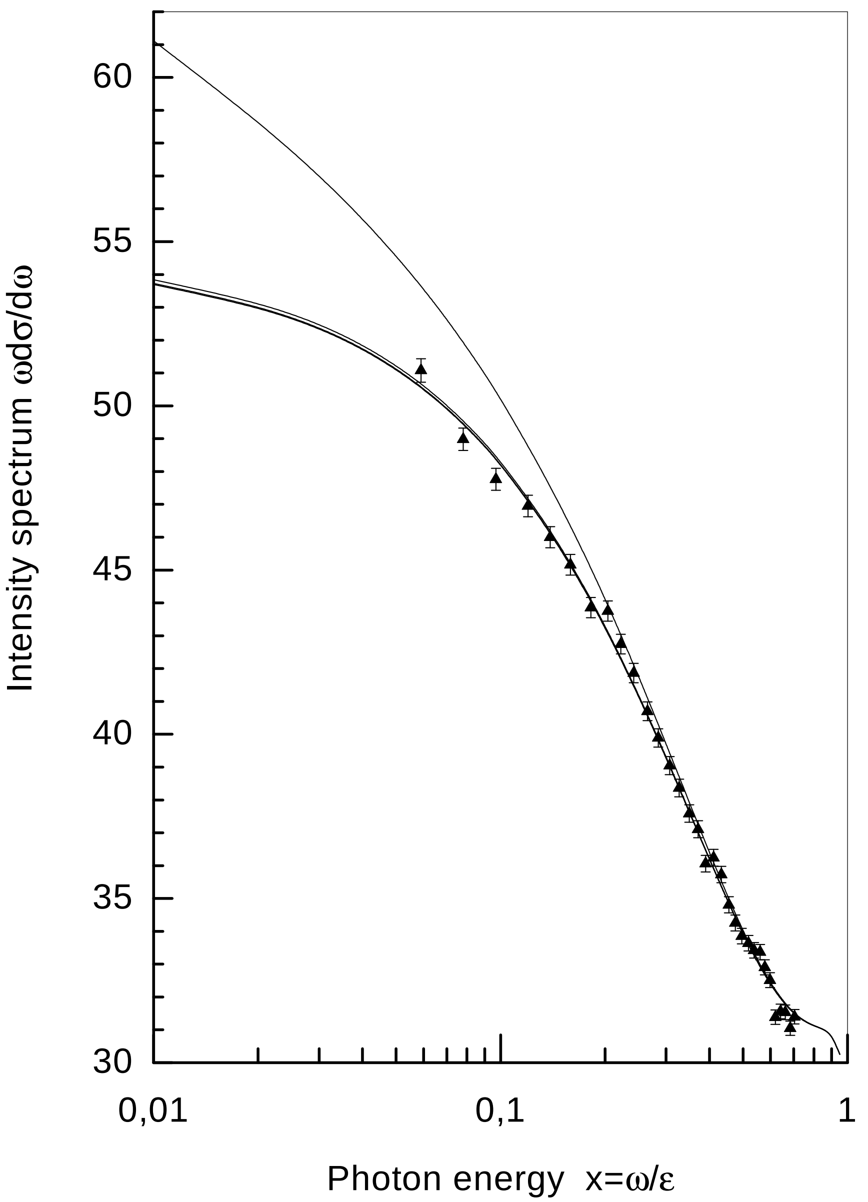


Figure 3: The normalized to luminosity $\mathcal{L}$ the bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units $2\alpha r_0^2$ versus the vertical displacement of beams z_0 (in μm). The data taken from [6].

be substituted:

$$q_{min} \to q^D_{min} = \frac{\omega m^2 m_p}{4\varepsilon_p \varepsilon \varepsilon'}, \tag{7.1}$$

here m_p is the proton mass. In this situation the characteristic length is $l^D_{f0} = 1/q^D_{min}$ and at the photon energy $\omega = 1$ GeV one has $l^D_{f0} \sim 2$ mm. Since the beam sizes at HERA are much smaller than this characteristic length, the beam-size effect can be observed at HERA. The parameters of beam in this experiment were (in our notation): $\sigma_z = \Delta_z = (50 \div 58)\mu m$, $\sigma_y = \Delta_y = (250 \div 290)\mu m$. In part of runs the displaced beams were used with $z_0 = 20\ \mu m$ and $y_0 = 100\ \mu m$. The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units $2\alpha r_0^2$ versus the photon energy in the units of initial electron energy ($x = \omega/\varepsilon$) for the HERA experiment is given in Fig.4. The upper curve is the standard QED spectrum. We calculated the spectrum



Intensity spectrum ωdσ/dω
60
55
50
45
40
35
30
0,01
0,1
1
Photon energy x=ω/ε

Figure 4: The bremsstrahlung intensity spectrum $\omega d\sigma/d\omega$ in units $2\alpha r_0^2$ versus the photon energy in units of initial electron energy ($x = \omega/\varepsilon$) for the HERA experiment. The upper curve is the standard QED spectrum, the two close curves below are calculated with the beam-size effect taken into account: the bottom curve is actually two merged curves for sets 1 and 2 (the set 1 is $\sigma_z = \Delta_z = 50\,\mu m$, $\sigma_y = \Delta_y = 250\,\mu m$, $z_0 = y_0 = 0$, set 2 is $\sigma_z = \Delta_z = 50\,\mu m$, $\sigma_y = \Delta_y = 250\,\mu m$, $z_0 = 20\,\mu m$, $y_0 = 0$); while the top curve is for set 3 ($\sigma_z = \Delta_z = 54\,\mu m$, $\sigma_y = \Delta_y = 250\,\mu m$, $z_0 = y_0 = 0$). The data taken from Fig.5c in [18].

with beam-size effect taken into account for three sets of beams parameters; the set 1: $\sigma_z = \Delta_z = 50\,\mu m$, $\sigma_y = \Delta_y = 250\,\mu m$, $z_0 = y_0 = 0$, the set 2: $\sigma_z = \Delta_z = 50\,\mu m$, $\sigma_y = \Delta_y = 250\,\mu m$, $z_0 = 20\,\mu m$, $y_0 = 0$, the set 3: $\sigma_z = \Delta_z = 54\,\mu m$, $\sigma_y = \Delta_y = 250\,\mu m$, $z_0 = y_0 = 0$. The result of calculation is seen as two close curves below, the top curve is for the set 3, while the bottom curve is actually two merged curves for the sets 1 and 2. Since the ratio of the vertical and the horizontal dimensions is not very small, the general formulas were us ed in calculation: for coaxial beams Eqs. (4.11) and (4.20),and for displaced beams Eqs. (4.14) and (4.20). It should be noted that the contribution of subtraction term (Eq.(4.20)) is quite essential (more than 10%) for the beam parameters used at HERA. The data are taken from Fig.5c in [18]. The errors are the recalculated overall systematic error given in [18]. It is seen that there is a quite satisfactory agreement of theory and data. The final data are given in [18] also as the averaged relative difference $\delta = (d\sigma_{QED} - d\sigma_{bs})/d\sigma_{QED}$ (where $d\sigma_{QED}$ is the standard QED spectrum, σ_{bs} is the result of calculation with the beam-size effect taken into account) over the whole interval of photon energies ($2 - 8$ GeV), e.g. for the set 1 $\delta_{ex} = (3.28 \pm 0.7)\%$, for the set 2 $\delta_{ex} = (3.57 \pm 0.7)\%$, for the set 3 $\delta_{ex} = (3.06 \pm 0.7)\%$, [18]. The averaged $<\delta>$ over the interval $0.07 \leq x \leq 0.28$ (or $1.95 \leq \omega \leq 7.7$ GeV) in our calculation are for the set 1 is $<\delta>$=2.69%, for the set 2 is $<\delta>= 2.65\%$, for the set 3 is $<\delta>= 2.54\%$. So, for these data there is also a satisfactory agreement of data and theory (at the 1-sigma level, except set 2 where the difference is slightly larger).

So, the beam-size effect discovered at BINP (Novosibirsk) was confirmed at DESY (Germany). Of course, more accurate measurement is desirable to verify that we entirely understand this mechanism of deviation from standard QED.

8 Conclusion

Above the influence of the finite transverse size of the colliding beams on the incoherent bremsstrahlung process is investigated. Previously (see papers [7], [8], [9], [10]) for analysis of this effect an incomplete expression for the bremsstrahlung intensity spectrum was used because in it the subtraction was not fulfilled. It is necessary to carry out this subtraction for the extraction of pure fluctuation process which is just the incoherent bremsstrahlung. We implement this procedure in the present paper. We indicated the cases where the results without the subtraction term are qualitatively erroneous. The first this is the case when the transverse sizes of scattering beam are much smaller than the corresponding sizes of radiating beam. For coaxial round beams see e.g. Eq.(5.15) and for flat beams Eq.(6.4). In contrast to previous papers here we draw a conclusion that the bremsstrahlung cross section is determined by the transverse sizes of scattering beam.

The new qualitative result is deduced for the case when the displacement of beams is enough large. Then the square of momentum transfer dispersion, which determines the bremsstrahlung cross section, decreases with displacement increase faster than mean value the momentum transfer squared (see Eqs.(5.21), (5.27)). As it was noted in Sec.7, it is necessary to normalize the spectrum on the luminosity for displaced beams. Then the bremsstrahlung cross section grows exponentially with displacement increase. This very fast (exponential) increase with z_0 is due to fast decrease with z_0 normalization process probability for displaced beams.

For Gaussian beams the expression for the bremsstrahlung spectrum is obtained in the form of double integrals convenient for numerical calculations (see Eqs.(4.10), (4.20) and (4.21)).For soft part of spectrum we deduced the general expression for spectrum which is independent of minimal momentum transfer q_{min} and is defined only by transverse size of beams (see Eqs.(5.3), (5.4) and (5.7)-(5.11)).

The important feature of the considered beam-size effect is smooth decrease of radiation probability with growth of displacement of beams. For the flat beams we see in Eqs. (6.9), (6.10) that the main (logarithmic) term in expression for the probability decreases exponentially ($\propto \exp(-z_0^2\Sigma_z^2)$ as luminosity), but there is the specific long-range term $\propto 1/z_0$ which results in quite appreciable radiation probability even in the case when beam the displacement is large. This phenomenon may be helpful for tuning of high-energy electron-positron colliders. As an example we consider the "typical" collider were the beam energy is $\varepsilon = 500$ GeV and the beam dimensions are equal and $\sigma_z = 5nm$ and $\sigma_y = 100nm$. The beam-size effect in this collider is

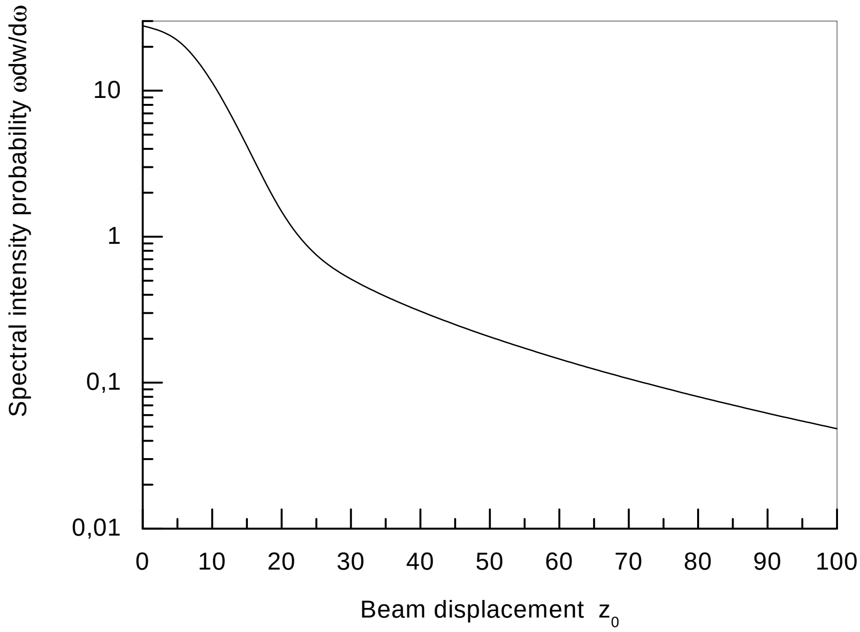


Figure 5: The spectral intensity probability $\omega dw_\gamma/d\omega$ normalized to one particle in the beam in units $2\alpha r_0^2\Sigma_z\Sigma_y/\pi$ versus the vertical displacement of beams z_0 (in nm).

very strong and for $x = 10^{-3}$ the intensity spectrum is only ~ 0.3 part of the standard $\omega d\sigma_{QED}(\omega)/d\omega$. The dependence of bremsstrahlung probability on the displacement distance z_0 (in nm) is shown in Fig.5. It is calculated using Eqs. (5.6)-5.8) for soft photons with $x = 10^{-3}$ (the asymptotic formulas (6.9) are (6.10) are not enough accurate in this case). Actually the dependence on photon energy is contained in the external factor $(1-x)(v(x)-2/3)$. The curve in Fig.5 reflects the main features mentioned above. One can see that even for z_0=100 ($z_0 = 20\ \sigma_z$) the cross section is ~ 0.002 part of very large bremsstrahlung probability at head-on collision of beams. So, measuring the radiation for displaced beams one can estimate magnitude of displacement of beams. This information may be useful for beam tuning.

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Deviation from standard QED at large distances: influence of transverse dimensions of colliding beams on bremstrahlung

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Отклонение от стандартной квантовой электродинамики на больших расстояниях: влияние поперечных размеров сталкивающихся пучков на тормозное излучение

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