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MULTUBUNCH BEAM INSTABILITY IN THE CASE OF LONG BUNCH

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1 Introduction

In this paper, the problem of multipole synchrotron oscillations of the multibunch beam will be considered without restriction on the gaussian bunch length or on the width of the frequency range, in which the impedance of the surrounding structure is taken into account. This result has been provided at the cost of increasing the matrix order in comparison with [2], for example, where it was equal to the number of bunches. Now, this value is multiplied by the number of terms of the distribution function amplitude expansion. But, with choosen set of orthogonal functions for this expansion (for a gaussian distribution of the undisturbed bunch), the necessary number of terms is not very high.

This method is particularly useful for higher multipole synchrotron oscillations, because the main contribution for their instability is given by the harmonics with numbers m, for which $m\phi_0$ is in some band near $\sqrt{2n-1} \geq 1$, where n is the multipole number and ϕ_0 is the angular r.m.s. length of the bunch. It is clear that the condition of small bunch length ($m\phi_0 \ll 1$) is not valid for the most important harmonics.

The amplitude dependence of the synchrotron frequency (the first approach: $\Omega = \Omega_0(1 - \xi \frac{J}{J_0}))$ can be taken into account rather simply in this method, leading to the correction to the matrix elemets proportional to the small parameter ξ. There arises no necessity to solve the nonlinear equation with an unknown value under the integral as it was in [2]. One can show (section 4.2) that both methods give the same asymptotic in the case of big coherent shifts and small bunch length, which verifies the correctness of present derivations.

Note, however, that the number of terms of expansion to be taken into account depends on the ratio of growth rates σ to the spread of synchrotronous frequencies $\Delta\Omega$. If $\sigma \leq \Delta\Omega$, then, in order to get a correct result, the number of terms of amplitude expansion should be increased.

Further (section 2), the matrix problem is formulated in the general form for the case of long bunches.

In section 3, the motion with regard to the spread of synchrotron frequencies is considered, without coherent interaction with surrounding structure.

In section 4, the particular case of one type of multipole oscillations and one bunch is considered for simplicity, in order to compare with the method of [2] and to discuss the necessary number of terms to be taken into account.

In section 5, the matrix problem is formulated, taking into account coupling with neighbour types of multipole oscillations.

In section 6, the code MBIM2 is described, in which the proposed method of calculating the eigen modes and their eigen values for unsymmetric multibunch beam with long bunches is realized.

In Appendix 1, the set of formulae for Laguerre polinomials is given, which were used in present derivations.

In Appendix 2, the formulae for summing up the series with the Watson-Sommerfeld transformation are given for the most common case, which are used for summing up serties over harmonic numbers in the case of resonant impedance.

2 Derivation of the problem of longitudinal stability in the matrix form for the case of the big bunch length

2.1 Vlasov equation for longitudinal motion of the multibunch beam

As in [2], following to [3], we start from the linearized Vlasov equation, subjected to the Laplace transform (assuming that the multipole oscillations are small as compared with the undisturbed distribution function):

$$sF(\psi, J, s) - \tilde{f}(\psi, J, 0) + \Omega\frac{\partial F}{\partial \psi} + L[\dot{J}]\frac{\partial f_0}{\partial J} = 0\,. \qquad (1)$$

We use the following denotations: s is the Laplace variable; Ω is the synchrotron frequency; J and ψ are the action and synchrotron phase variables respectively; f_0 is the undisturbed distribution function, independent of the time in the rotating reference frame; $\tilde{f}(\psi, J, t)$ is the disturbance of the distribution function; $F(\psi, J, s)$ is its Laplace transform: $F(\psi, J, s) = L[\tilde{f}(\psi, J, t)]$.

In (5) a forcing term $L[\dot{J}]$ is determined from the equation

$$\dot{J} = -\frac{\partial H}{\partial \psi}, \;\; H = -e\int E_{z0}dz,$$

E_{z0} is the longitudinal component of the electric field on the equilibrium orbit in the reference system rotating with the equilibrium particle (calculated in the same way as in [2]), at $z = l - \omega_0 Rt$.

A distribution function in eq. (1) describes the whole multibunch beam. For our purposes, we rewrite the whole beam distribution function as a sum of separate distribution functions for all n_0 bunches in the beam (as in [2]):

$$F = \sum_{l=1}^{n_0} F^l.$$

As each bunch oscillates in its own separatrix, we can further use the space orthogonality of the distribution functions of different bunches.

Thus, instead of one equation (1) we get a system of equations for bunches distribution functions:

$$sF^l(\psi, J, s) - \tilde{f}^l(\psi, J, 0) + \Omega\frac{\partial F^l}{\partial \psi} + L[\dot{J}]\frac{\partial f_0^l}{\partial J} = 0, l = 1, ..., n_0. \qquad (2)$$

Note that in $L[\dot{J}]$ the currents of all bunches are summarized, therefore (2) is not splitted into n_0 independent equations, as it was done in [2].

The longitudinal coordinate of the particle in the l-th separatrix is

$$z = \theta_l R + z_0(J) sin(\psi) = \frac{2\pi l}{n_0} R + z_0(J) sin(\psi), \qquad (3)$$

hence, the distribution function can be spread into the Fourier series over the synchrotron phase ψ:

$$F(\psi, J, s) = \sum_n e^{in\psi} F_n(J, s), \quad \tilde{f}_0(\psi, J) = \tilde{f}(\psi, J, 0) = \sum_n e^{in\psi} f_{0n}(J). \qquad (4)$$

With all these denotations, in [2] there was obtained the system of integral equations for functions $F_n^l(J, s)$ (see [2], eqs. (6),(7)):

$$F_n^l(J, s) - eI_0 \sum_{j=1}^{n_0} \sum_q \int K_{qn}^{lj}(J, J', s) F_q^j(J', s) dJ' = \frac{f_{0n}^l(J)}{(s + in\Omega)}, \qquad (5)$$

where

$$K_{qn}^{lj}(J, J', s) =$$

$$= \frac{\partial f_0^l}{\partial J} \frac{I_j / I_0}{(s + in\Omega)} \sum_m \frac{n}{m} Z_m(s - im\omega_0) A_{mn}(J) A_{mq}^*(J') e^{im(\theta_l - \theta_j)}, \qquad (6)$$

$$Z_m(s - im\omega_0) = \sum_k Z_k(s - im\omega_0) |E_{zkm}|^2.$$

Further, in [2], this system was considered in approach of short bunches. Now, we shall consider the problem without this restriction, assuming only that the motion of particles is still sinusoidal ((3)).

2.2 Application of Laguerre polinomials to the Vlasov equation

Now, we suppose that the undisturbed distribution functions for all bunches are gaussian and equivalent for all bunches:

$$f_0^l(J) = f_0(J) = \frac{1}{2\pi J_0} e^{-J/J_0}.$$

Denote $\phi = \frac{1}{R}\sqrt{\frac{2J}{M\Omega}}$, $\phi_0 = \frac{1}{R}\sqrt{\frac{2J_0}{M\Omega}} = \frac{\sigma\sqrt{2}}{R}$ (see [2], App.2), then we can write, using (27):

$$A_{nm}(J) = J_n\left(\frac{m}{R}\sqrt{\frac{2J}{M\Omega}}\right) = J_n\left(2\frac{m\phi_0}{2}\frac{\phi}{\phi_0}\right) =$$

$$= \left(\frac{m\phi}{2}\right)^{|n|} e^{-\frac{m^2\phi_0^2}{4}} \cdot \sum_{k=0}^{\infty}\left(\frac{m\phi_0}{2}\right)^{2k} \frac{L_k^{(|n|)}(\frac{\phi^2}{\phi_0^2})}{\Gamma(n+k+1)} =$$

$$= e^{-\frac{m^2\phi_0^2}{4}} \cdot \sum_{k=0}^{\infty}\left(\frac{m\phi_0}{2}\right)^{2k+|n|} \frac{f_k^{|n|}(\frac{\phi^2}{\phi_0^2})}{\sqrt{\Gamma(|n|+k+1)k!}}. \qquad (7)$$

The functions F_n^l can be spread over orthogonal functions (23), with the weight function proportional to $f_0(J)$ (as it was made, for example, in [4]):

$$F_n^l(J) = W(x)\sum_{k\geq 0} a_k^{nl} f_k^{|n|}(x), \quad x = J/J_0, \quad W(x) = e^{-x}, \qquad (8)$$

$$a_k^{nl} = \int_0^{\infty} F_n^l(J) f_k^{|n|}(x)dx.$$

Substituting (6), (7) and (8) into eq. (5), dropping the r.h.s., we get:

$$(s+in\Omega)W(x)\sum_{k\geq 0} a_k^{nl} f_k^{|n|}(x) + \frac{eI_0 n}{2\pi J_0^2} e^{-x} \sum_{j=1}^{n_0} \frac{I_j}{I_0} \sum_m \frac{1}{m} Z_m(s-im\omega_0) e^{im(\theta_l-\theta_j)} \times$$

$$\times A_{mn}(J)\int\sum_q A^*_{mq}(J')F^j_q(J',s)dJ'=0$$

or, denoting for simplicity $Z^{lj}_m(s)=Z_m(s-im\omega_0)e^{im(\theta_l-\theta_j)}$,

$$(s+in\Omega)W(x)\sum_{k\geq 0}a^{nl}_k f^{|n|}_k(x)+\frac{eI_0n}{2\pi J_0}W(x)\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m\frac{Z^{lj}_m(s)}{m}\times$$

$$\times e^{-\frac{m^2\phi_0^2}{4}}\cdot\sum_{k=0}^{\infty}\left(\frac{m\phi_0}{2}\right)^{2k+|n|}\frac{f^{|n|}_k(x)}{\sqrt{\Gamma(|n|+k+1)k!}}\times$$

$$\times\int e^{-\frac{m^2\phi_0^2}{4}}\cdot\sum_q\sum_{k=0}^{\infty}\left(\frac{m\phi_0}{2}\right)^{2k+|q|}\frac{f^{|q|}_k(x')}{\sqrt{\Gamma(|q|+k+1)k!}}\times$$

$$\times w(x')\sum_{k'\geq 0}a^{qj}_{k'}f^{|q|}_{k'}(x')dx'=0 \qquad (9)$$

or, integrating with $f^{|n|}_{k''}(x)$,

$$\int(s+in\Omega)W(x)\sum_{k\geq 0}a^{nl}_k f^{|n|}_k(x)f^{|n|}_{k''}(x)dx+\frac{eI_0n}{2\pi J_0}\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m\frac{Z^{lj}_m(s)}{m}\times$$

$$\times e^{-\frac{m^2\phi_0^2}{4}}\cdot\left(\frac{m\phi_0}{2}\right)^{2k''+|n|}\frac{1}{\sqrt{\Gamma(|n|+k''+1)(k'')!}}\times$$

$$\times e^{-\frac{m^2\phi_0^2}{4}}\cdot\sum_q\sum_{k\geq 0}\left(\frac{m\phi_0}{2}\right)^{2k+|q|}\frac{1}{\sqrt{\Gamma(|q|+k+1)k!}}a^{qj}_k=0,$$

or

$$\int(s+in\Omega)W(x)\sum_{k\geq 0}a^{nl}_k f^{|n|}_k(x)f^{|n|}_{k''}(x)dx+$$

$$+\frac{eI_0n}{2\pi J_0}\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m\frac{Z^{lj}_m(s)}{m}\cdot e^{-\frac{m^2\phi_0^2}{2}}\times$$

$$\times\sum_q\sum_{k\geq 0}\frac{\left(\frac{m\phi_0}{2}\right)^{2k+|q|+2k''+|n|}}{\sqrt{\Gamma(|n|+k''+1)(k'')!\Gamma(|q|+k+1)k!}}a^{qj}_k=0, \qquad (10)$$

or, denoting,

$$B_{k''k}^{|n||q|}(m) = \frac{\left(\frac{m\phi_0}{2}\right)^{2k+|q|+2k''+|n|}}{\sqrt{\Gamma(|n|+k''+1)(k'')!\Gamma(|q|+k+1)k!}} \cdot \frac{4}{m\phi_0^2} \qquad (11)$$

$$\left(B_{00}^{11} = 1,\ B_{01}^{11} = B_{10}^{11} = \frac{m^2\phi_0^2}{4\sqrt{2}},\ 'B_{11}^{11} = \frac{m^4\phi_0^4}{32},\ ...\right),$$

we get

$$\int (s+in\Omega)W(x) \sum_{k\geq 0} a_k^{nl} f_k^{|n|}(x) f_{k''}^{|n|}(x)dx +$$

$$+\frac{eI_0 n\cdot\phi_0^2}{2\pi J_0\cdot 4}\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_q\sum_{k\geq 0} B_{k''k}^{|n||q|}(m)(a_k^{qj}) = 0.$$

Note that $J_0 = \sigma^2 M\Omega < 0$ for $M < 0$. Hence, one can denote

$$A = -\frac{eI_0 n\cdot\phi_0^2}{2\pi J_0\cdot 4} = \frac{eI_0 n}{2\pi\sigma^2|M|\Omega} = \frac{\Omega_0^2}{\Omega}\frac{I_0 n\cdot\phi_0^2}{4q_{rf}V_{rf}\sin\phi_{s0}}\frac{R^2}{\sigma^2} =$$

$$= \frac{\Omega_0^2}{\Omega}\frac{I_0 n}{2q_{rf}V_{rf}\sin\phi_{s0}}.$$

One can see that equations for a_k^{nl} with $n = 0$ are independent of all others and have the trivial solution, thus, the last equation for $n \neq 0$ can be rewritten as

$$\int (s+in\Omega)W(x) \sum_{k\geq 0} a_k^{nl} f_k^{|n|}(x) f_{k''}^{|n|}(x)dx -$$

$$-A\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\sum_{k\geq 0} B_{k''k}^{|n||q|}(m)(a_k^{qj} + a_k^{-qj}) = 0. \qquad (12)$$

Further, we will consider a linear dependence of synchrotron frequency on J: $\Omega(x) = \Omega_0(1-\xi x)$, $x = J/J_0$, $\xi = q_{rf}^2\phi_0^2\alpha(\phi_s)$, $\alpha(0) = 1/16$, $\alpha(\pi/4) = 1/6$ (see [2]). In this case we have:

$$(s+in\Omega_0)\sum_{k\geq 0} a_k^{nl}\delta_{kk''} - in\Omega_0\xi\sum_{k\geq 0} a_k^{nl}\int xW(x) f_k^{|n|}(x) f_{k''}^{|n|}(x)dx -$$

$$-A\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\sum_{k\geq 0}B_{k''k}^{|n||q|}(m)(a_k^{qj}+a_k^{-qj})=0.$$

Remember (see (28)) that

$$M_{km}^n=\int xW(x)f_k^{|n|}(x)f_m^{|n|}(x)dx=$$

$$=(|n|+2k+1)\delta_{km}-\sqrt{k(|n|+k)}\delta_{k-1,m}-\sqrt{m(|n|+m)}\delta_{k,m-1},$$

hence, the amplitude dependence of the synchrotron frequency adds the terms proportional to ξ to the diagonal matrix elements ($k=m$) and to their neighbours ($k=m\pm1$). With regard to this, the equation can be written (for $\pm n,\ n>0$) as

$$(s\pm in\Omega_0)\sum_{k\geq 0}a_k^{nl}\delta_{kk''}\mp in\Omega_0\xi\cdot\sum_{k\geq 0}M_{k''k}^n a_k^{nl}\mp \tag{13}$$

$$\mp A\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\sum_{k\geq 0}B_{k''k}^{|n||q|}(m)(a_k^{qj}+a_k^{-qj})=0.$$

With denotations $S_k^{nl}=(a_k^{nl}+a_k^{-nl})$, $D_k^{nl}=(a_k^{nl}-a_k^{-nl})$, the sum and the difference of these equations can be written as

$$s\vec{S}^{nl}+(in\Omega_0)(\hat{E}-\xi\hat{M}^n)\vec{D}^{nl}=0,$$

$$s\vec{D}^{nl}+(in\Omega_0)(\hat{E}-\xi\hat{M}^n)\vec{S}^{nl}-$$

$$-2A\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\hat{B}^{|n||q|}\vec{S}^{qj}=0.$$

Expressing $\vec{S}$ via $\vec{D}$, we get the system only for D_k^{nl}:

$$s^2\vec{D}^{nl}-(in\Omega_0)^2(\hat{E}-\xi\hat{M}^n)^2\vec{D}^{nl}-$$

$$-2A\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\hat{B}^{|n||q|}(-iq\Omega_0)(\hat{E}-\xi\hat{M}^q)\vec{D}^{qj}=0. \tag{14}$$

The last equation formulates the eigen value problem for the most common case of multibunch beam, for which all multipole synchrotron oscillations are considered simultaneously, with regard to their coupling.

3 Incoherent motion in the case of the synchrotron frequency spread along the bunch

Let us consider the equation in the absence of coherent interaction, i.e. when $I_0 = 0$. In this case, the multipole oscillations with different n are decoupled, for each n we have the independent matrix equation

$$s^2\vec{D}^{nl} - (in\Omega_0)^2(\hat{E} - \xi\hat{M}^n)^2\vec{D}^{nl} = 0, \tag{15}$$

or, keeping only linear dependence on ξ and denoting

$$\lambda = \frac{(s^2 - (in\Omega_0)^2)}{(-2in\Omega_0)},$$

$$\lambda\vec{D}^{nl} - (in\Omega_0)(\xi\hat{M}^n)\vec{D}^{nl} = 0.$$

This equation describes the uncoupled motion of particles with frequency spread: in a time $\tau_{incoh} \sim \pi/(\xi n\Omega_0)$ the primarily sinphase particles get phase shift of order π. The matrix of this equation can not be restricted in the common case, hence, the solution should contain all terms of amplitude expansion.

In the time domain, if at the moment $t = 0$ there is only one nonzeroth amplitude D^{n0}, for example, it will excite, in time τ_{incoh} the next modes as $D^{nk} \sim (t/\tau_{incoh})^k$ (at $g \ll \tau_{incoh}$).

Figs.1,2 show pictures of different expansion terms amplitudes in the time domain, when $N_r = 10$ and $N_r = 40$ expansion terms were taken into account. One can show that the sum of squares of the amplitudes of all terms is $\sum_{k=0}^{N_r} |D^{nk}|^2 = |D^{n0}|^2 = 1$, therefore the amplitudes of expansion terms decrease (in average) with the increase in their total number N_r.

Figs.3,4 show the contribution of the taken into account expansion terms into eigen modes for these two cases. One can see that beginning from the mode with the eigen value $\frac{\lambda_0}{i\xi n\Omega_0} \approx 10$ (which corresponds to the mode with the number $k \approx \sqrt{N_r \cdot 10/e}$, see further, Fig.6) the eigen vectors practically do not contain the lowest expansion terms and

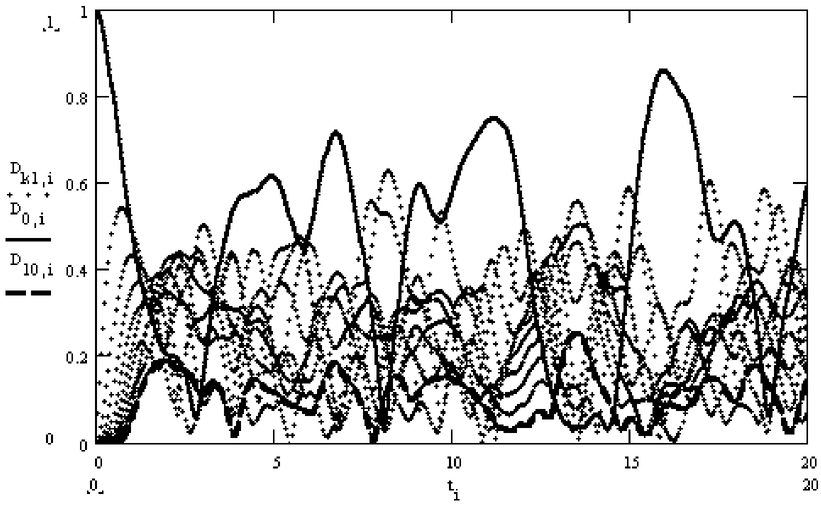

Figure 1: The expansion terms amplitudes in time domain for $N_r = 10$ expansion terms taken into account.

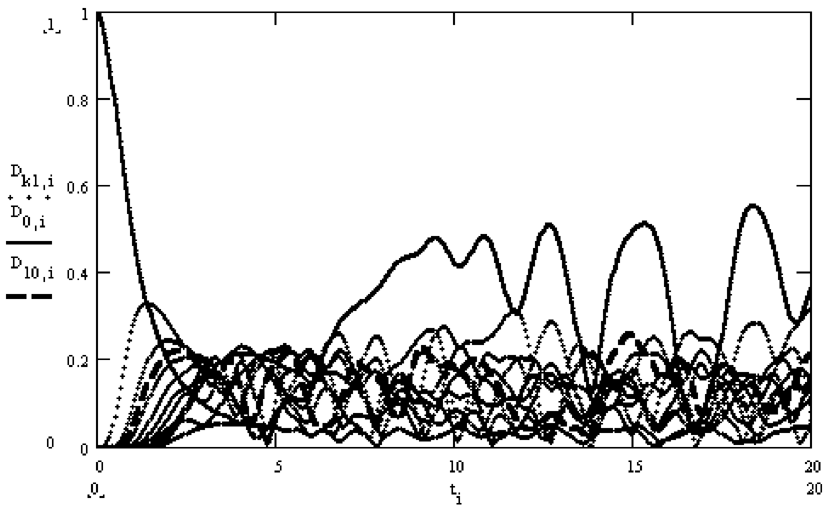

Figure 2: The same, for $N_r = 40$.

this fact is valid for all $N_r \geq 3$. It means that the primary excitation of the lowest expansion terms, at any N_r, excites in fact the eigen modes in limited frequency range, with $|\lambda| \leq |\lambda_0|$.

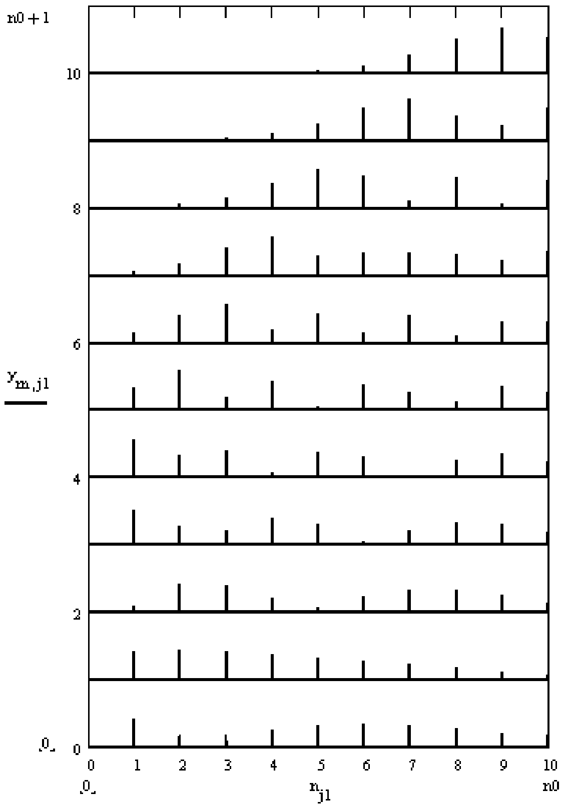

Figure 3: Contribution of taken into account expansion terms into eigen modes for $N_r = 10$.

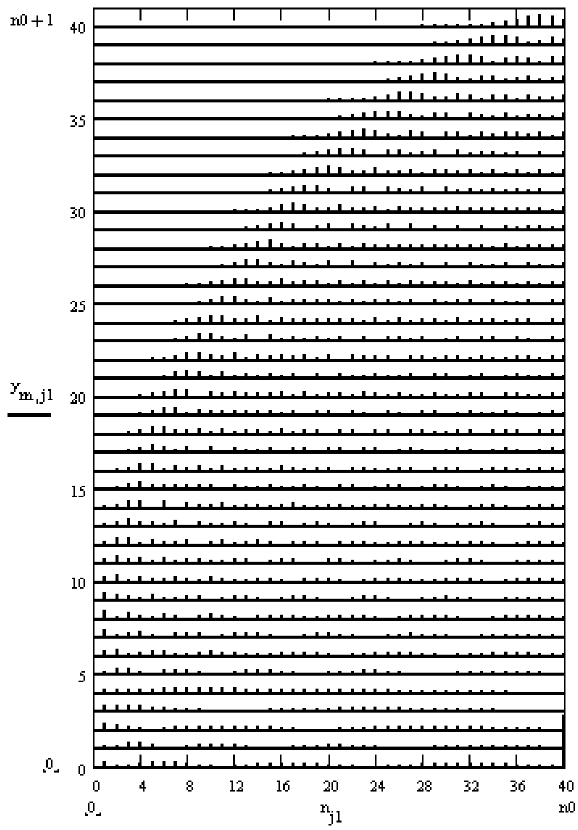

Figure 4: The same, for $N_r = 40$.

Fig.5 shows the set of eigenvalues versus the number of expansion terms taken into account N_r. The numerical calculations show that one can approximately describe these eigen values as $\frac{\lambda_{kN_r}}{i\xi n\Omega_0} \approx \frac{ek^2}{N_r}$. Increasing N_r leads to the more wide ($\frac{\lambda_{max}}{i\xi n\Omega_0} \approx eN_r$) and more dense spectrum of eigen frequencies. The limit $N_r \to \infty$ corresponds to the continuous spectrum in the infinite frequency range, which corresponds to the incoherent motion.

When we consider the coherent instabilities, one should compare the time of developing this instability ($\tau_{coh} \sim 1/\sigma$, where σ is the

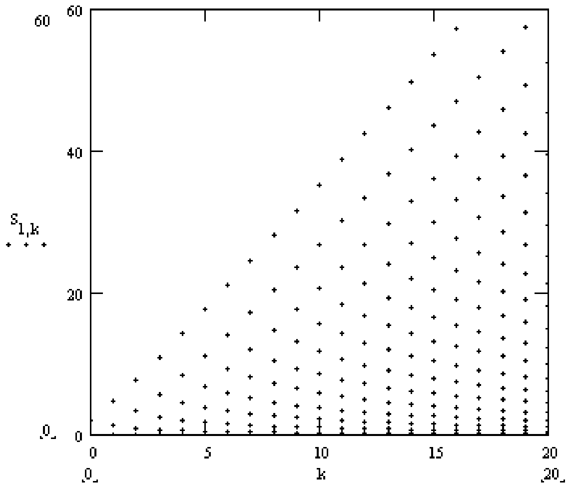

Figure 5: Eigenvalues versus the number of expansion terms.

growth rate) with τ_{incoh}. If $\tau_{incoh} \ll \tau_{coh}$, the coherent instability has no time to grow while the motion quits to be coherent. But if $\tau_{incoh} \geq \tau_{coh}$, i.e. when the coherent instability grows quicker than incoherent diverging of phases, this diverging only reduces the growth rate, and in order to estimate correctly this reduction, one must consider all expansion terms which can be excited in the time $\sim \tau_{coh}$.

4 Uncoupled multipole oscillations with different multipole numbers

Further, the multipole oscillations will be considered separately from each other, taking only $q = n$:

$$s^2 \vec{D}^{nl} - (in\Omega_0)^2 (\hat{E} - \xi \hat{M}^n)^2 \vec{D}^{nl} -$$

$$-2A \sum_{j=1}^{n_0} \frac{I_j}{I_0} \sum_m Z_m^{lj}(s) e^{-\frac{m^2 \phi_0^2}{2}} \hat{B}^{|n||n|} (-in\Omega_0)(\hat{E} - \xi \hat{M}^n) \vec{D}^{nj} = 0. \quad (16)$$

Let denote (for $s \approx -in\Omega_0$)

$$\lambda = \frac{(s^2 - (in\Omega_0)^2)}{(-2in\Omega_0)},$$

$$\tilde{Z}^{lj}_{k''k} = \sum_m Z^{lj}_m(s) e^{-\frac{m^2\phi_0^2}{2}} B^{|n||n|}_{kk''}(m).$$

With these denotations, the equation becomes (keeping only linear in ξ terms)

$$\lambda \vec{D}^{nl} - (in\Omega_0)\xi \hat{M}^n \vec{D}^{nl} - A \sum_{j=1}^{n_0} \frac{I_j}{I_0} \hat{\tilde{Z}}^{lj} (\hat{E} - \xi \hat{M}^n) \vec{D}^{nj} = 0. \qquad (17)$$

4.1 Necessary number of expansion terms for coherent oscillations

In the case of a single bunch (the simplest case, in order to see the effect of expansion terms), dropping subscripts n and $l, j = 1$, we have:

$$\left(\lambda \hat{E} - (in\Omega_0)\xi \hat{M} - A\hat{\tilde{Z}}(\hat{E} - \xi \hat{M}) \right) \vec{D} = 0. \qquad (18)$$

Here $\hat{E}$ is an unitary matrix, $\hat{M}$ is a 3-diagonal matrix, $\hat{\tilde{Z}}$ is the matrix depending on the impedance, with elements where the impedance is summed up over all harmonic numbers m with factors $B^{|n||n|}_{k''k}(m)$(see (11)).

In the case of small bunch length ($\xi = 0$ for simplicity), the matrix elements $\tilde{Z}_{kl}$ are proportional to $\phi_0^{2(k+l)}$ and decrease with increasing k, l. The first correction to the zero approach ($\lambda_0 = A\tilde{Z}_{00}$) is of order ϕ_0^4:

$$\lambda_1 \approx A\tilde{Z}_{00}(1 - (\frac{\tilde{Z}_{01}}{\tilde{Z}_{00}})^2), \ \frac{\tilde{Z}_{01}}{\tilde{Z}_{00}} \sim \phi_0^2.$$

The following corrections are of higher orders of ϕ_0 and can be dropped.

But in the case of big bunch length ($m_{max}\phi_0/2$ of order or greater than 1) one can not say that $\tilde{Z}_{kl}$ is proportional to ϕ_0^{k+l}. Instead of that, one can use the upper estimation of all elements of the sum.

One can see that for different $(|n|+k+k'')$ this factor has its maximum at different m_{max}, thus, different parts of the spectrum influence on different multipole modes of synchrotron oscillations: the function $f(N,x) = x^N \exp(-2x^2)$ (for $x = \frac{m\phi_0}{2}$, $N = 2(k+k''+|n|)-1)$) has its maximum at $x_{max} = \sqrt{N/4}$, $f(N, x_{max}) = (N/4e)^{N/2}$. Note that $f''(x) = 0$ at $x_{1,2} = \sqrt{(N+1 \mp \sqrt{3N+1})/4}$, hence, the spectrum band can be estimated as $x_2 - x_1 \approx \sqrt{3N+1}/\sqrt{N} \approx \sqrt{3}$ (for $N \gg 1$).

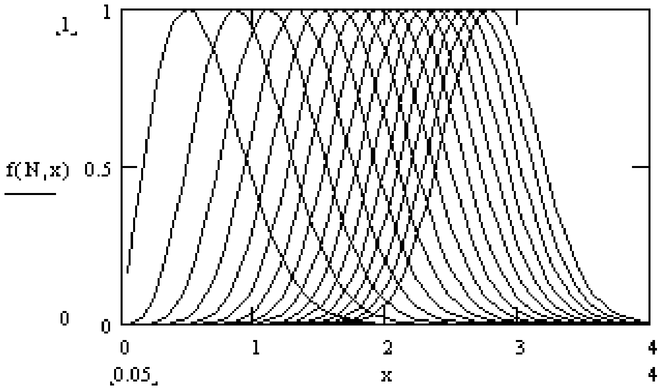

Figure 6: Normalized function $f(N,x) = x^N \exp(-2x^2)/f_{max}$, $f_{max} = (N/4e)^{N/2}$, for odd N=1,3,...,31.

Fig.6 shows normalized function $f(N,x)/f(N,x_{max})$, for odd N=1,3,...,31, i.e. for $k+k''+|n| = 1,2,...,16$. It is obvious that the modes with different N interact mainly with different regions of the spectrum, which are cut with the function f(N,x). The width of the spectrum band is approximately the same for all N.

On the other hand, the denominator of (11) increases very quickly

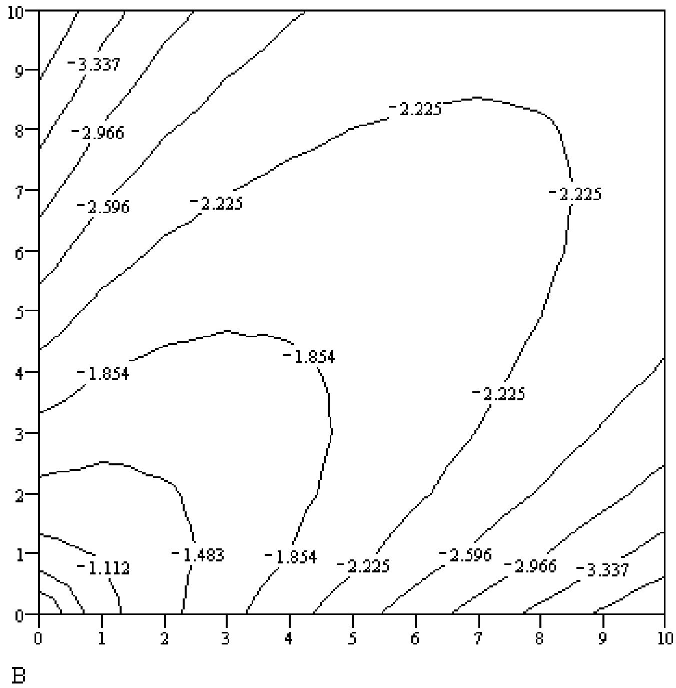

Figure 7: The level lines for $lg(A_{kk''}\frac{\phi_0^2}{4})$.

with increasing k, k''. Thus, the factors (11) have the upper estimation

$$A_{kk''} = \frac{(N/4e)^{N/2}}{\sqrt{\Gamma(|n|+k''+1)(k'')!\Gamma(|n|+k+1)k!}}\frac{4}{\phi_0^2},$$

where $N = 2(k + k'' + |n|) - 1)$. The elements of the matrix $\hat{A}$ decrease very quickly with increasing k, k''. Fig.7 shows the level lines for $lg(A_{kk''}\frac{\phi_0^2}{4})$ and Fig.8 demonstrates decreasing of the matrix elements along the diadonal $k = k''$.

These figures show that, in fact, only a small number of expansion terms should be taken into account, depending on n (for the case $\xi = 0$).

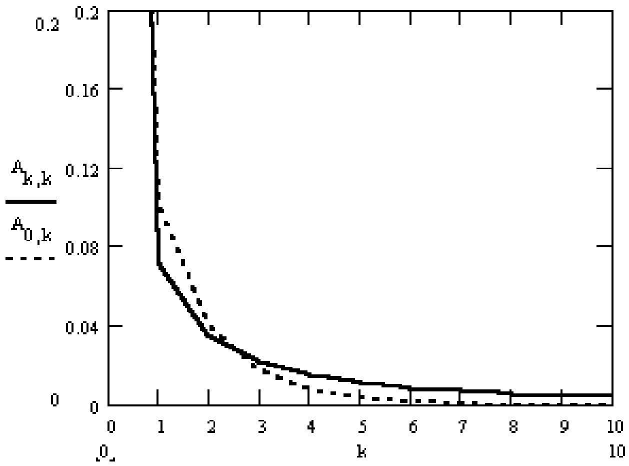

Figure 8: Diagonal elements A_{kk} and the first line elements A_{0k} versus k.

In each case, the necessary number of expansion terms to be taken into account depends on the product $m_{max}\phi_0$: one should consider all modes with $k, k'' \leq k_{max}$, for which $x_2 \approx x_{max} + (x_2 - x_1)/2 > m_{max}\phi_0/2$, i.e. $2k_{max} + |n| > [(m_{max}\phi_0 + \sqrt{3})^2 + 1]/2 = [(m_{max}\phi_0)^2 + 2\sqrt{3}m_{max}\phi_0 + 4]/2$, or, more rough estimation $m_{max}\phi_0 = 2x_{max}$, in order to overlap the whole region with "large" coefficients, below and above x_{max}, which gives

$2k_{max} + |n| > (m_{max}\phi_0)^2$.

Fig.9 shows an example of the dependence of eigen values on the number of expansion terms N_r taken into account, for different bunch lengths. The RF cavity spectrum contains one resonant mode with the resonant frequency $f = M \cdot f_{rev}$, $M = 2000 + \nu_z$, $\nu_z = 0.01$, $Q = 10000$, $f_{rev} = 3MHz$. The dipole synchrotron oscillations of single bunches with $\sigma = 0.01, 0.1, 1, 2, 3cm$ were considered. The graphics are normalized by the corresponding values limits at sufficiently large number of expansion terms taken into account. One can see that the bunch with $\sigma < 1cm$ can be considered (at this spectrum) as a short

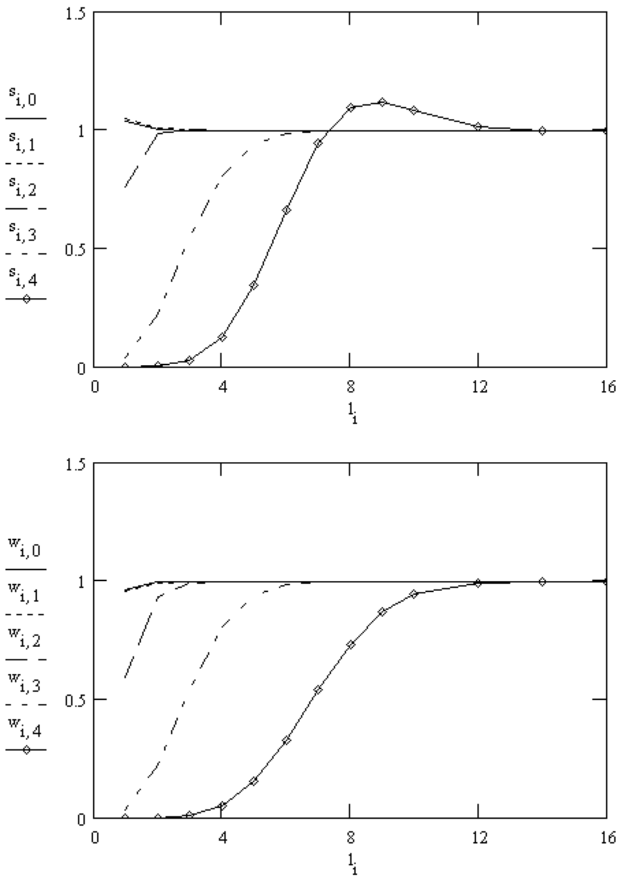

Figure 9: Dependence of the growth rates ($s_{i,k}$) and shifts of the synchrotron frequency ($w_{i,k}$) on the number of expansion terms taken into account (l_i), for different bunch length (σ =0.01, 0.1, 1,2,3 cm for k=0,1,...,4.)

bunch with 1 term of expansion. At $\sigma = 1cm$, the result obtained with $N_r = 1$ is ~ 1.5 times less than the limiting value with $N_r >> 1$, but the calculation with $N_r = 2$ already gives good approximation. And so on, for $\sigma = 2cm$ $N_r = 5$ and for $\sigma = 3cm$ $N_r = 12$ give good approximation. Note that calculations were made for bunch currents at which $\Delta\nu_z \ll \nu_z$, in order to separate effects of the number of expansion terms and that of nonlinear dependence of the matrix equation on ν_z.

4.2 Comparison of the asymptotics of two methods

In order to compare the present results with [2], where the eigen values are expressed only via $\tilde{Z}_{00}$, one can assume that the matrix $\hat{\tilde{Z}}$ has the only nonzeroth term $\tilde{Z}_{00}$ and all other terms are equal to zero (due to the spectrum or to the small bunch length).

Let us consider, for simplicity, the modes with $k, k'' \leq 3$. The determinant of the matrix of eq. (18) is

$$\begin{vmatrix} Z'_{00}(1-\xi M_{00}) + \xi M_{00} - \lambda' & -\xi Z'_{00} M_{01} + \xi M_{01} & 0 & 0 \\ \xi M_{10} & \xi M_{11} - \lambda' & \xi M_{12} & 0 \\ 0 & \xi M_{21} & \xi M_{22} - \lambda' & \xi M_{23} \\ 0 & 0 & \xi M_{32} & \xi M_{33} - \lambda' \end{vmatrix} = 0,$$

where $\lambda' = \frac{\lambda}{in\Omega_0}$, $Z'_{kj} = \frac{A}{in\Omega_0}\tilde{Z}_{kj}$.

If we keep only the term with zeroth indexes, then the frequency spread results only in the additional shift of the synchrotron frequency proportional to ξ while the growth rate remains unchanged:

$$\lambda'_0 = \xi M_{00} + Z'_{00}(1-\xi M_{00}) = \xi(n+1) + Z'_{00}(1-\xi(n+1)) = .$$

If we keep the terms with $i, j \leq 1$, then we get the next approach, keeping the terms of order ξ^2:

$$\lambda'_1 \approx \lambda'_0 + (\xi M_{01})^2 \frac{1 - Z'_{00}}{\lambda'_0 - \xi M_{11}} =$$

$$= \lambda'_0 + \xi^2 (n+1) \frac{1 - Z'_{00}}{\xi(n+1) + Z'_{00}(1-\xi(n+1)) - \xi(n+3)} \approx$$

$$\approx \lambda_0' + \xi^2(n+1)(\frac{1}{Z_{00}'} - 1).$$

One can show that the asymptotic of the solution obtained in [2] is the same. For that solution, there was a dependence of the eigenvalue z on the Z_{00}':

$$\frac{n!}{Z_{00}'} = \int_0^{+\infty} \frac{e^{-x}x^n(1-\xi x)}{z-\xi x}dx,$$

where $z = \frac{s^2-(in\Omega_0)^2}{2(in\Omega_0)^2}$. For $|z| \gg 1$ and $\xi \ll 1$ one can spread the expression under the integral:

$$\frac{(1-\xi x)}{z-\xi x} \approx \frac{1}{z}(1-\xi x)(1+\frac{\xi x}{z}+(\frac{\xi x}{z})^2+(\frac{\xi x}{z})^3+...) =$$

$$= \frac{1}{z}(1-\xi x(1-\frac{1}{z}) - \frac{(\xi x)^2}{z}(1-\frac{1}{z}) - ...)$$

Hence, the equation becomes to be

$$\frac{1}{AZ_{00}} = \frac{1}{n!}\int_0^{+\infty} e^{-x}x^n\frac{1}{z}(1-\xi x(1-\frac{1}{z}) - \frac{(\xi x)^2}{z}(1-\frac{1}{z}) - ...)dx =$$

$$= \frac{1}{z}(1-\xi(n+1)(1-\frac{1}{z}) - \frac{\xi^2(n+1)(n+2)}{z}(1-\frac{1}{z}) - ...)$$

or

$$z_1 = Z_{00}'(1-\xi(n+1)(1-\frac{1}{z_0}) - \frac{\xi^2(n+1)(n+2)}{z_0}(1-\frac{1}{z_0}) - ...),$$

where $z_0 = Z_{00}'(1-\xi(n+1)) + \xi(n+1)$ is the approach of order ξ, hence, the next approach (of order ξ^2) is

$$z_1 \approx Z_{00}'(1-\xi(n+1)(1-\frac{1}{Z_{00}'}(1+\xi(n+1))) - \frac{\xi^2(n+1)(n+2)}{Z_{00}'}(1-\frac{1}{Z_{00}'})) \approx$$

$$\approx Z_{00}'(1-\xi(n+1)) + \xi(n+1) + \xi^2(n+1)(\frac{1}{Z_{00}'} - 1).$$

In approach of order ξ^2, we get the solution, which coincides with that obtained above with Laguerre polinomials (for this case).

The numerical calculations show that by increasing the number of Laguerre polinomials taken into account, one can get the closer solutions in the region, where this solution exists for the previous method (see Figs.10 and 11). The region in upper semiplane, where there is no solution for the previous method, corresponds to the continuous spectrum, which describes nonperiodical motion due to the spread of synchrotron frequencies. Obviously, the previous method of searching isolated poles does not describe this region.

Moreover, our present consideration shows the way how the previous solution can be extended into the lower semiplane z: the solution should be a complex conjugate to that of the upper semiplane, with a cut along the positive real axis of the $z-$plane, and the region between the upper and lower edges of the cut corresponds to nonperiodical motion, with an infinite number of expansion terms necessary for its description and with no singular poles.

The following corrections should contain the terms Z_{jk} with j or k not equal to zero, which are absent in [2]. When these terms become appreciable, the approach of [2] becomes to be no more valid.

5 Coupling of different multipole modes

Turning to the equation (14), one can consider the coupling of neighbour multipole oscillations with multipole numbers from n_1 to n_2 (dropping the terms containing ξ^2):

$$s^2\vec{D}^{nl} - (in\Omega_0)^2(\hat{E} - 2\xi\hat{M}^n)\vec{D}^{nl} -$$

$$-2A\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\hat{B}^{|n||q|}(-iq\Omega_0)(\hat{E} - \xi\hat{M}^q)\vec{D}^{qj} = 0, \tag{19}$$

$$n = n_1, .., n_2.$$

In order to get the equations for solutions s near the given n_c-

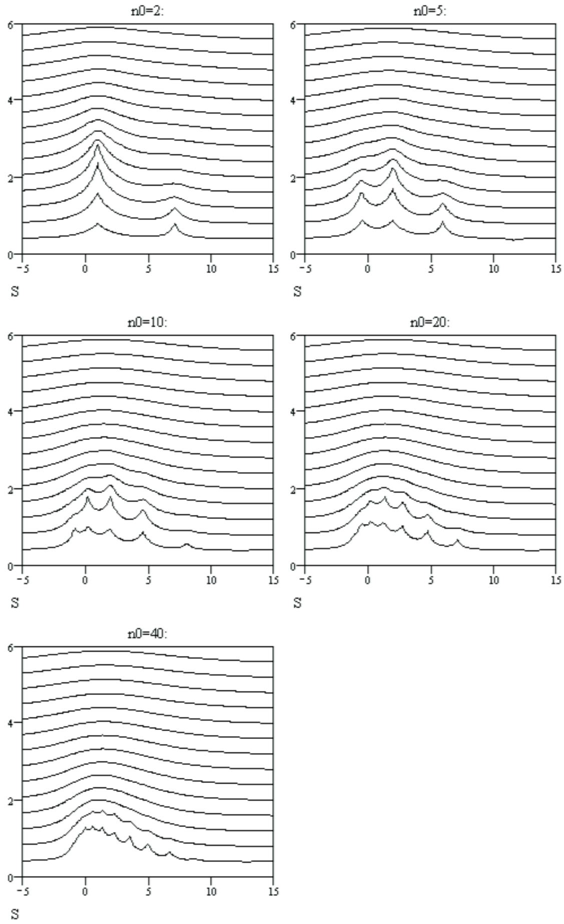

Figure 10: The level lines of maximal growth rate at the plane Z_{00} for different numbers of expansion terms taken into account (n=2,5,10,20,40).

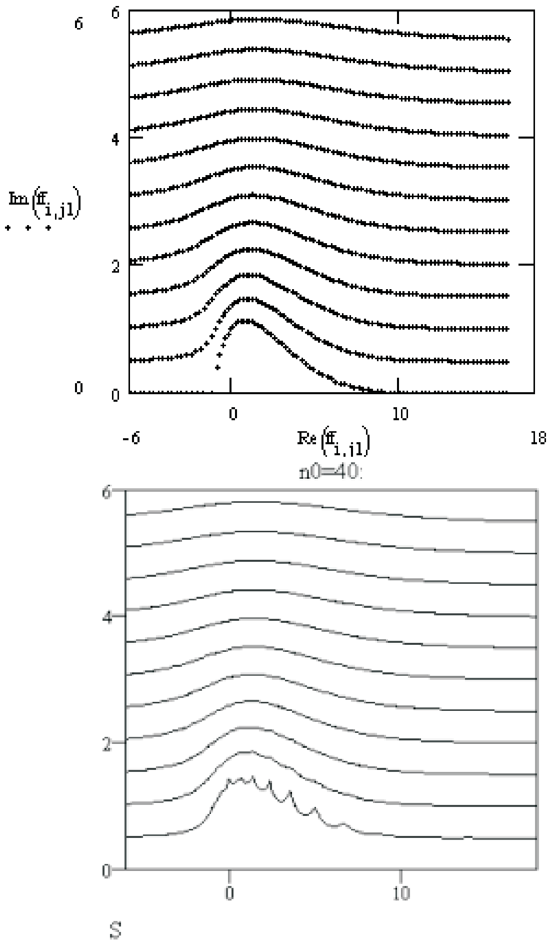

Figure 11: Comparison of two methods. The level lines of maximal growth rate at the plane Z_{00} for n=40 (below) and the same picture obtained with previous method (above).

th harmonic of synchrotron oscillations, with regard to coupling with neighbour harmonics from n_1 to n_2 ($n_1 < n_c < n_2$), one can denote

$$\lambda = \frac{(s^2 - (in_c\Omega_0)^2)}{(-2in_c\Omega_0)},$$

$$A' = \frac{A}{n} = \frac{\Omega_0^2}{\Omega}\frac{I_0}{2q_{rf}V_{rf}\sin\phi_{s0}}.$$

With this denotation, eq.(19) becomes to be

$$\lambda\vec{D}^{nl} - (in\Omega_0)(\xi\hat{M}^n)\vec{D}^{nl} -$$

$$-A'\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\hat{B}^{|n||q|}(\frac{nq}{n_c})(\hat{E} - \xi\hat{M}^q)\vec{D}^{qj} = 0, \quad (20)$$

$$\text{for } \; n = n_c;$$

$$(\lambda\hat{E} + i\Omega_0\frac{(n^2 - n_c^2)}{2n_c}\hat{E} - i\Omega_0\frac{n^2}{n_c}\xi\hat{M}^n))\vec{D}^{nl} -$$

$$-A'\sum_{j=1}^{n_0}\frac{I_j}{I_0}\sum_m Z_m^{lj}(s)e^{-\frac{m^2\phi_0^2}{2}}\sum_{q>0}\hat{B}^{|n||q|}(\frac{nq}{n_c})(\hat{E} - \xi\hat{M}^q)\vec{D}^{qj} = 0, \quad (21)$$

$$\text{for } \; n = n_1, .., n_2, n \neq n_c.$$

One can see that near the n_c-th harmonic of the synchrotron frequency, when $|\lambda| \ll |\Omega_0|$, the equations with $n \neq n_c$ have an additional large diagonal term $|\frac{n^2-n_c^2}{2n_c}| \gg |\lambda|$, which subdues the influence of non-diagonal coupling terms depending on the impedance. Due to this fact, we believe that each multipole mode (n_c) should have the strongest coupling with the closest modes ($n_c \pm 1$, $n_c \pm 2$ and so on) and more distant modes can be dropped. This assumption is obvious for higher distant modes (with $n > n_c$), because the coupling terms for these multipole numbers contain higher powers of $\phi_0 \ll 1$. But for the lower multipole numbers ($n = n_c - 1, ..., 1$) one should check accurately the combined effect of increasing diagonal term, which decreases coupling, and lowering the power of ϕ_0, which increases coupling with decreasing n.

6 The code MBIM2

The method given in this paper is realized in the code MBIM2, analogous to the code MBIM1 [2] for short bunches.

The code allows to calculate the growth rates and synchrotron frequency shifts for eigen modes of unsymmetrical multibunch beam (and for the case of counterrotating bunches too) with long bunches. The user defines the number of expansion terms taken into account. The code implies the possibilities to take into account the spread of synchrotron frequencies along the bunch and to calculate simultaneously several neighbour types of multipole oscillations.

7 Conclusion

1. The alternative method for studying the coherent synchrotron oscillations was considered, in comparison with the method given in [2], which was developed in an approach of short bunches.

2. The present method is useful in the case of long bunches, and particularly for higher multipole synchrotronous oscillations, since the main contribution for their instability give the harmonics with numbers m, for which the condition of small bunch length is no more valid.

3. The method allows to take into account the spread of synchrotron frequencies along the bunch. In the absence of coherent motion, it gives the qualitative picture of stochastic motion with an infinite spectrum. For coherent motion, it gives the correction to the growth rates due to this spread.

4. The asymptotics of both methods coinside in the case of a short bunch, in the region of big growth rates or frequency shifts, already at a small number of expansion terms to be taken into account. In the region of growth rates comparable or less than the synchrotron frequencies spread, the results became to be the more close, the more number of expansion terms is taken into account.

5. Comparison of two methods in the region of validity of both methods shows the way of analytical extention of the previous solution ([2]) to the whole plane s, which corresponds to the plane Z'_{00}

(subsection 4.2) without a certain region along the real axis, where the continuous spectrum of solutions takes place, which corresponds to the nonperiodical motion in the presence of frequencies spread. Naturally, the previous method of searching for isolated poles does not describe this region.

6. Finally, one can propose the way how to combine these two methods in order to get the correct results at the lowest cost:

One should check the solution convergence at increasing N_r for a single bunch, without frequency spread. If there is an essential difference between the converged result and a short bunch approximation with $N_r = 1$, one should solve the whole problem for all bunches with the frequency spread and sufficient number N_r, defined for one bunch. In the other case, one can use the previous method with matrix order equal to the number of bunches and taking into account the frequency spread by recalculating eigen values ([2]).

Appendix 1

The Laguerre polinomials

Here, for convenience, the definition and the list of formulae for Laguerre polinomials are given (from [1]). Definition (for $\alpha > -1$):

[1],(22.1.1,22.1.2,22.2.12):

$$\int_0^\infty w(x)u_k(x)u_m(x)dx = \delta_{km}h_k, \tag{22}$$

$$w(x) = e^{-x}x^\alpha, u_k(x) = L_k^{(\alpha)}(x), h_k = h_k(\alpha) = \frac{\Gamma(\alpha+k+1)}{k!}.$$

For $\alpha = n$, the set of orthogonal functions with weight function e^{-x} is $f_k^{(|n|)}(x)$:

$$f_k^{|n|}(x) = \frac{1}{\sqrt{h_k}}x^{|n|/2}L_k^{(|n|)}(x), \quad \int_0^\infty e^{-x}f_k^{|n|}(x)f_m^{|n|}(x)dx = \delta_{km}. \tag{23}$$

[1],(22.3.9):

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} x^m \frac{(-1)^m}{m!}\left(\begin{array}{c} n+\alpha \\ n-m \end{array}\right). \tag{24}$$

[1],(22.7.12, 22.7.29-22.7.32):

$$\begin{array}{l} (n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x); \\ L_n^{(\alpha+1)}(x) = \frac{1}{x}[(x-n)L_n^{(\alpha)}(x) + (\alpha+n)L_{n-1}^{(\alpha)}(x)]; \\ L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x); \\ L_n^{(\alpha+1)}(x) = \frac{1}{x}[(n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)]; \\ L_n^{(\alpha-1)}(x) = \frac{1}{n+\alpha}[(n+1)L_{n+1}^{(\alpha)}(x) - (n+1-x)L_n^{(\alpha)}(x)]. \end{array} \tag{25}$$

[1],(22.8.6):

$$x\frac{d}{dx}L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{26}$$

[1],(22.9.16):

$$(xz)^{-\alpha/2}e^z J_\alpha(2(xz)^{1/2}) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\frac{z^n}{\Gamma(n+\alpha+1)}. \tag{27}$$

One can show that

$$\int xe^{-x} f_k^{|n|}(x) f_m^{|n|}(x) dx =$$

$$= \frac{1}{h_k} \int e^{-x} x^{|n|+1} L_k^{(|n|)}(x) L_m^{(|n|)}(x) dx = \frac{1}{\sqrt{h_k(|n|) h_m(|n|)}} \times$$

$$\times \int e^{-x} x^{|n|+1} (L_k^{(|n|+1)}(x) - L_{k-1}^{(|n|+1)}(x))(L_m^{(|n|+1)}(x) - L_{m-1}^{(|n|+1)}(x)) dx =$$

$$= (|n| + 2k + 1)\delta_{km} - \sqrt{k(|n| + k)}\delta_{k-1,m} - \sqrt{m(|n| + m)}\delta_{k+1,m}. \quad (28)$$

Appendix 2

Summing of the series

The matrix elements contain the series

$$S_N(\theta) = \sum_{m=-\infty}^{\infty} m^N e^{im\theta} e^{-m^2\phi_0^2/2} Z_m^+,$$

where $Z_m^+ = Z(-i(m\omega_0 + n\Omega))$ and $N = n_i + n_j + 2(k_1 + k_2) - 1$ is an odd number if we consider only one kind of multipole oscillations, separately from all others. If we consider a set of multipole oscillations with different multipole numbers n_i, N can be both odd and even.

Further, we will derive the approximate formulae for summing these series in the case of resonant impedance with resonant frequency $\omega_r = m_r\omega_0$, effective character impedance ρ and quality factor Q:

$$Z(s) = \frac{\rho s}{(s - s_1)(s - s_2)},$$

$$s = \frac{-i\omega}{\omega_r}, \; s_{1,2} = \pm i\nu_2 - \nu_1, \; \nu_1 = \frac{1}{2Q}, \; \nu_2 = \sqrt{1 - \nu_1^2}.$$

For the series given above the argument of the impedance is $s = s(m) = -i\frac{m}{m_r} + \nu'$, $\nu' = -i\frac{n\nu_z}{m_r}$, $\nu_z = \frac{\Omega}{\omega_0}$. In the case of transformation to the symmetric modes, we have $q\sum_{p=-\infty}^{\infty}$ with $m = pq + k$ instead of $\sum_m$. In this case $s = s(p) = -i\frac{pq+k}{m_r} + \nu'$, $k = 0, 1, ..., q - 1$.

Let us denote also $s' = s - \nu' = -i\frac{m}{m_r}$ - proportional to m and $m_{1,2} = (s_{1,2} - \nu')im_r$.

With all these denotations, we can write $Z(s) = Z(s(m))$ (included into the series given above) as

$$Z(s) = \frac{\rho s}{(s-s_1)(s-s_2)} = \frac{\rho s}{(s_1-s_2)}\left(\frac{1}{(s-s_1)} - \frac{1}{(s-s_2)}\right) =$$

$$= \frac{\rho}{(s_1-s_2)}(1+\frac{\nu'}{s'})\sum_{k=0}^{\infty}\left(\left(\frac{s_1-\nu'}{s'}\right)^k - \left(\frac{s_2-\nu'}{s'}\right)^k\right) =$$

$$\frac{\rho}{(s_1-s_2)}\left\{\sum_{k=0}^{N} + \sum_{k=N+1}^{\infty} + \frac{\nu'}{s'}(\sum_{k=0}^{N-1} + \sum_{k=N}^{\infty})\right\} = Z_1(s) + Z_2(s),$$

$$Z_1(s) = \frac{\rho}{(s_1-s_2)}\left\{\sum_{k=0}^{N} + \frac{\nu'}{s'}\sum_{k=0}^{N-1}\right\} =$$

$$= \frac{\rho}{(s_1-s_2)}\sum_{k=1}^{N}\frac{1}{(s')^k}(s_1(s_1-\nu')^{k-1} - s_2(s_2-\nu')^{k-1});$$

$$Z_2(s) = \frac{\rho}{(s_1-s_2)}\left\{\sum_{k=N+1}^{\infty} + \frac{\nu'}{s'}\sum_{k=N}^{\infty}\right\} =$$

$$= \frac{\rho}{(s_1-s_2)}\left(\frac{s_1}{s'+\nu'-s_1}\left(\frac{s_1-\nu'}{s'}\right)^N - \frac{s_2}{s'+\nu'-s_2}\left(\frac{s_2-\nu'}{s'}\right)^N\right).$$

The series with the second addentum $Z_2(s)$ can be sumed up via the formulae of Watson-Sommerfeld (putting $\exp(-m_r^2\phi_0^2/2)$ instead of $\exp(-m^2\phi_0^2/2)$, for sufficiently sharp resonance), giving the same expressions as previously for $N = 1$, but with factors $m_{1,2}^N$ instead of $m_{1,2}$ (see [2], App.1):

$$S_2 = -\frac{i\pi\rho m_r}{2}e^{-m_r^2\phi_0^2/2}\cdot$$

$$\cdot\sum_{1,2}(1 \pm i\frac{\nu_1}{\nu_2})m_{1,2}^N e^{im_{1,2}\theta}(ctg(\pi m_{1,2}) - i\cdot sign(\theta)) =$$

$$= -\frac{\pi\rho m_r}{2\nu_2} e^{-m_r^2\phi_0^2/2} \cdot \sum_{1,2} (\pm s_{1,2}) m_{1,2}^N e^{im_{1,2}\theta} (ctg(\pi m_{1,2}) - i \cdot sign(\theta)),$$

$$sign(\theta) = \left\{ \begin{array}{ll} 1, & 0 \le \theta < 2\pi, \\ -1, & -2\pi < \theta < 0. \end{array} \right.$$

Now, let us transform the first part of the series containing $Z_1(s)$:

$$S_1 = \sum_m m^N e^{im\theta} e^{-m^2\phi_0^2/2} \frac{\rho}{(s_1 - s_2)} \times$$

$$\times \sum_{k=1}^{N} \frac{1}{(s')^k} (s_1(s_1 - \nu')^{k-1} - s_2(s_2 - \nu')^{k-1}) =$$

$$= \frac{\rho m_r}{2\nu_2} \sum_{k=0}^{N-1} (s_1 m_1^{N-k-1} - s_2 m_2^{N-k-1}) \tilde{S}_k,$$

where

$$\tilde{S}_k = \tilde{S}_k(\theta, \phi_0) = \sum_m m^k e^{im\theta} e^{-m^2\phi_0^2/2}.$$

For even $k = 2l$

$$\tilde{S}_{2l}(\theta, \phi_0) = \left(\frac{\partial}{\partial(-\frac{\phi_0^2}{2})} \right)^l \tilde{S}_0(\theta, \phi_0);$$

for odd $k = 2l + 1$

$$\tilde{S}_{2l+1}(\theta, \phi_0) = \left(\frac{\partial}{\partial(-\frac{\phi_0^2}{2})} \right)^l \tilde{S}_1(\theta, \phi_0) = \left(\frac{\partial}{\partial(i\theta)} \right) \left(\frac{\partial}{\partial(-\frac{\phi_0^2}{2})} \right)^l \tilde{S}_0(\theta, \phi_0),$$

$$\tilde{S}_{2l+1}(\theta, \phi_0) = \frac{\partial}{\partial(i\theta)} \tilde{S}_{2l}(\theta, \phi_0),$$

where

$$\tilde{S}_0(\theta, \phi_0) = \sum_m e^{im\theta} e^{(-m^2\phi_0^2/2)}.$$

The last series can be summed up in an approach $\phi_0 \ll \pi$, which is valid in the most cases of bunched beams.

In this approach, neglecting terms of order $\exp -\pi^2/2\phi_0^2$, we have

$$\tilde{S}_0(\theta,\phi_0) = \frac{\sqrt{2\pi}}{\phi_0} e^{-\theta^2/2\phi_0^2},$$

$$\tilde{S}_{2l}(\theta,\phi_0) \approx \begin{cases} \frac{\sqrt{2\pi}}{\phi_0}\frac{(2l-1)!!}{(\phi_0^2)^l}, & \text{for } \theta = 0, \ l > 0 \\ \frac{\sqrt{2\pi}}{\phi_0}\left(\frac{-\theta^2}{\phi_0^4}\right)^l e^{-\theta^2/2\phi_0^2}, & \text{for } \theta \gg \phi_0. \end{cases} ;$$

$$\tilde{S}_{2l+1}(\theta,\phi_0) \approx (\frac{i\theta}{\phi_0^2})\tilde{S}_{2l}(\theta,\phi_0),$$

$$\text{for } \theta \gg \phi_0 \text{ or } \theta = 0.$$

For convenience, denote $S'_k = \tilde{S}_k \phi_0^k$. Hence, we get:

$$S'_k(\theta,\phi_0) =\approx \begin{cases} \frac{\sqrt{2\pi}}{\phi_0} e^{-\theta^2/2\phi_0^2}, & \text{for } k = 0; \\ \frac{\sqrt{2\pi}}{\phi_0}(k-1)!!, & \text{for } \theta = 0, \ k \ - \ \text{even}; \\ \frac{\sqrt{2\pi}}{\phi_0}\left(\frac{-\theta^2}{\phi_0^2}\right)^{k/2} e^{-\theta^2/2\phi_0^2}, & \text{for } \theta \gg \phi_0, \ k \ - \ \text{even}; \\ \frac{i\theta}{\phi_0}S'_{k-1}, & \text{for } k \ - \ \text{odd}. \end{cases}$$

Note that $B^{nq}_{kk_1}$ contains a factor $(\phi_0/2)^{N-1}$, where $N = n + q + 2(k + k_1) - 1$ (see above). Hence, one can write down

$$S_1(\frac{\phi_0}{2})^{N-1} = \frac{\rho m_r}{2\nu_2}\sum_{k=0}^{N-1}(s_1 m_1^{N-k-1} - s_2 m_2^{N-k-1})\frac{S'_k}{\phi_0^k}(\frac{\phi_0}{2})^{N-1} =$$

$$= \frac{\rho m_r}{2\nu_2 \cdot 2^{N-1}}\sum_{k=0}^{N-1}(s_1(m_1\phi_0)^{N-k-1} - s_2(m_2\phi_0)^{N-k-1})S'_k\,.$$

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Contents

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