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DYNAMICS OF SAWTOOTH MAP:
1. NEW NUMERICAL RESULTS

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1. New numerical results

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Abstract

Some results of numerical study of the canonical map with a sawtooth force are given and discovered new unexpected dynamical effects are described. In particular, it is shown that if the values of the system parameter K belong to the countable set determined by Ovsyannikov's theorem, separatrices of primary resonances are not splitted and chaotic layers are not formed. One more set of values of the parameter related to the other family of nondestructed separatrices of primary resonances was found. The mechanism explaining the stability of the primary resonance separatrix in the critical regime is found and described. First secondary resonances were studies and for them were found the K-values at which their separatrices are not splitted also. These facts are important since the presence of the nondestructed separatrix of the any order resonance eliminates the possibility of a global diffusion in a phase space. New problems and open questions occurred in this connections whose solution can facilitate the further development of the nonlinear Hamiltonian systems theory are presented.

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1 Introduction

Two-dimensional maps of the form

$$\overline{p} = p + K \cdot f(x)\,, \quad \overline{x} = x + \overline{p} \quad (mod\ 1) \tag{1}$$

with the only parameter K is long and broadly used in the nonlinear physics as quite convenient and very informative models [1,2,3]. For example, function $f(x) = \sin(2\pi x)$ corresponds to the so-called standard map, to the study of which many papers are devoted and which is used in a great number of studies. From the very beginning, the creators of the KAM theory noted different dynamical behavior of the systems (1) for analytical and smooth dependences $f(x)$. The main subject here is the question of the smoothness degree (the number of l continuous derivatives of the force $f(x)$) at which the global chaos takes place not for any arbitrarily small parameter of the system but only higher of some threshold value $K > K_g$. The studies of Moser and Rűssman have shown that such a threshold always exists at $l > 3$ [4]. However, there is no proof of the inverse statement that at $l \leq 3$ there is no threshold and the diffusion in a phase space is unlimited.

Consider the system with the sawtooth force

$$f(x) = \begin{cases} 4\,x, & \text{if } x \leq 0.25\,, \\ 4\,(0.5 - x), & \text{if } |0.5 - x| \leq 0.25\,, \\ 4\,(x - 1.0), & \text{if } |x - 1.0| \leq 0.25\,. \end{cases} \tag{2}$$

This function is asymmetric $f(-x) = -f(x)$, its period is equal to unity and its smoothness degree is $l = 0$.

This our work was induced by Ovsyannikov's theorem [5] that for map (1),(2) there exist separatrices of primary resonances at exactly determined countable set of values of the system parameter K [5]. Unfortunately, this important result is not yet published (in the later work on this subject [6]

Appendix nearly completely text of the communication [5].

By the example of a pendulum we remind that the separatrix of a single nonlinear resonance is a special trajectory separating the phase oscillations (inside the resonance) from its rotation (out of the resonance). In fact, these are two spatially coincided branches corresponding to the back and forth course of time, respectively. Each branch is a continuous trajectory with an infinite period of motion, which outcomes the position of unstable equilibrium (saddle) and then aproaches it asymptotically. In the presence in the system of other (at least one) nonlinear resonances, the separatrix is splitted into two intersecting branches outcoming from the saddle in opposite directions but do not return to it. Free ends of the branches produce an infinite number of loops of the infinitely increasing length which fill the narrow region along the unperturbed separatrix forming the so-called chaotic layer [1,2,3]. The overlapping of chaotic layers of all the system resonances just means the onset of the global chaos.

The central point of the modern nonlinear Hamiltonian systems theory could be considered the statement that the splitting of the resonance separatrix and formation on its place of the chaotic layer in the typical (i.e. nonintegrable) Hamiltonian system occur at almost any perturbation. It is also assumed that namely separatrices are first destroyed since they have zeroth frequencies of motion and the interaction of nonlinear resonances in their vicinity are always essential [1,2,3]. As the perturbation increases, invariant curves with irrational winding numbers vanish last (for the standard map, this is the "golden" number $(\sqrt{5}-1)/2$ maximally remoted from all rational numbers [7]).

For the system (1), (2) all looks otherwise and recently obtained results will be given below.

2 Critical numbers of primary resonances

The main subject of further consideration will be resonances of the system (1),(2) and their separatrices. The periodic orbit and corresponding resonance are adopted to denote by the ratio of integer numbers $P : Q$, where Q is the number of iterations of the map to P periods of the orbit [2]. Resonances with $Q = 1$ are called primary and the remaining resonances with $Q > 1$ are called secondary.

One of the most unusual and estonishing features of the studied dynamics as will be shown below is the presence of "critical" values of the parameter chaotic layers are not formed. The search for such regimes is quite a fine problem and its accurate solution requires some special computational techniques. To this end, in the present paper we used the earlier developed technique of measuring the intersection angle of separatrix branches of primary resonances at the central homoclinic point, which for the system (1), (2) lies always in the symmetry line $x = 0.5$ (see [8] with the description of the details and measurement results for the standard map). The justification of such a choice is discussed below.

Fig.1 shows the obtained numerically for the system (1), (2) the dependence $\alpha_1(K)$, which turned to be sign-variable and oscillating. It is important to stress that it differs qualitatively from the well studied by now the similar dependence for the standard map (the latter is the sign-constant and strictly monotonical, for details see [8]). As will be shown below, this difference is related with the essentially different dynamical behavior of these two systems.

Points $\alpha_1(K) = 0$ in the Figure indicates the absence of splitting of separatrices and corresponding values of the parameter K we will call critical numbers. It is seen that these numbers can be of two types: for some of

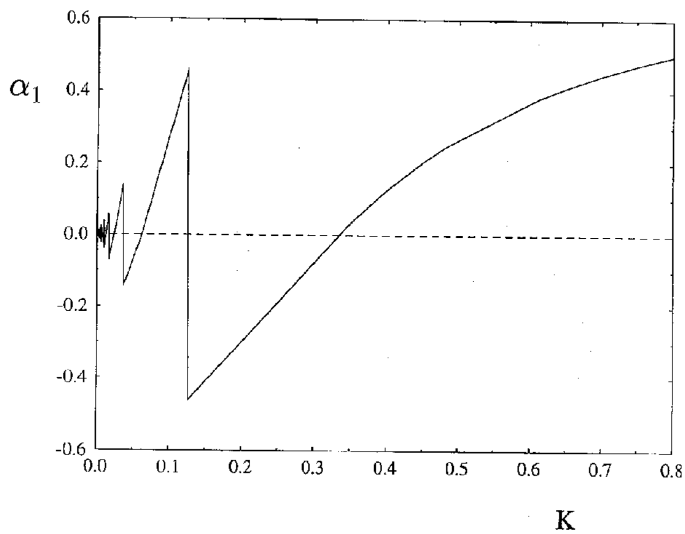

Figure 1: Splitting angle α_1 of separatrix branches of the system (1), (2) primary resonances as a function of parameter K.

for others in a jump. Since at the monotonical increase in the parameter value the elements of these sets are alternate (see also Fig.3 below), we can introduce for them the through enumeration $K_{1,m}$, $m = 1, 2, 3...$ (the first index indicates the relation to the primary resonance) and then all the odd numbers will be referred to the smooth change of an angle and the even numbers with the jump.

A thorough analysis showed that even critical numbers coincide exactly with the elements of the countable set given in Ovsyannikov's theorem and determined by the solutions of the transcendent equation (1.4) at integer values of the coefficient k (see Appendix). It turned out that found above and not included into Ovsyannikov's theorem odd critical values are also determined by the solutions of the same equation (1.4) but for halfinteger values of the coefficient, therefore for any critical number we have

$$K_{1,m} = \sin^2(\beta_m/2), \quad m = 1, 2, 3, \dots, \tag{3}$$

where β_m is the least positive root of the equation

$$\sqrt{2}\sin(m\beta_m/2) = \cos(\beta_m/2)\,. \tag{4}$$

The latter relations enable one, in particular, to find out exact values for two first critical numbers $K_{1,1} = 1/3$ and $K_{1,2} = 1/8$. Let us emphasize again that these sets are referred only to primary $Q = 1$ resonances of the map (1), (2).

The nature of jump-like changes of the angle at even critical numbers can be interpreted in the following way. An important part of the Ovsyannikov theorem is the presence of exact formulae (see (1.5) in Appendix) by which the separatrix can actually be constructed for any number $K_{1,m}$, $m = 2, 4, 6, \dots$ All these separatrices turned to be broken lines with one the break point coincides always with the central homoclinic point of primary resonances in the line $x = 0.5$. Fig.2 shows the picture of intersecting branches of the primary resonance separatrix in the vicinity of this point at the system parameter value a bit less than the critical value $K_{1,2} = 1/8$. It is seen that the transition to the horizontal section in the forward course of time is phase shifted to the right and in the backward course to the left. If one constructs such a picture for K a bit larger than $K_{1,2}$, the branches exchanged their roles and the angle conserving its value changes its sign to the opposite sign.

From this picture it also follows that the branch intersection angle is quite close to the break angle value of the unsplitted separatrix (the upper line in Fig.2) in the central homoclinic point which can be exactly calculated.

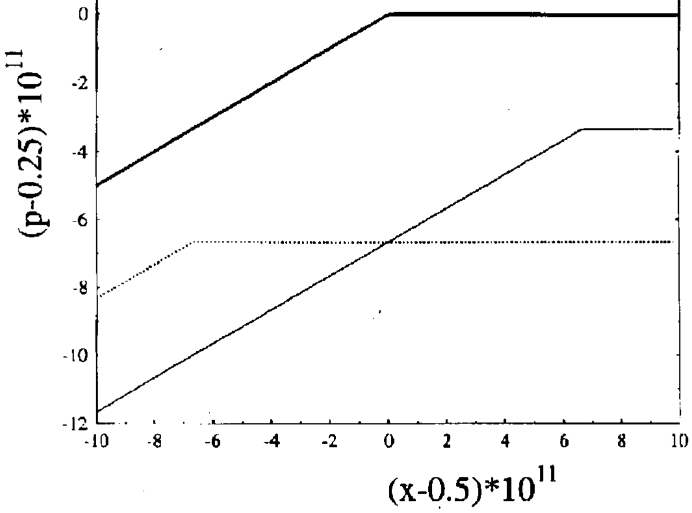

Figure 2: The upper broken line is the section of unsplitted separatrix of the primary resonance at $K = K_{1,2} = 0.125$. Below are the branches of the splitted separatrix at $K = 0.125 - 5 \times 10^{-11}$. The solid line corresponds to the forward course of time and the dashed line to the backward course. The branches intersection angle is $\alpha_1 = 0.464$.

From formulae (1.5) in Appendix, it follows the angle value asymptotical dependence on the map parameter K in the even critical points:

$$\alpha_{1,m} = 4\,K_{1,m}, \quad m \gg 1\,. \tag{5}$$

Fig. 3 shows the function $\alpha_1(K)$ at $K \leq 0.001$ whose envelope follows well the theoretical law (5). To our opinion, this fact is an evidence of the high quality of the calculation technique we used. As already mentioned in literature, among of all the chaos attributes in many cases only the intersection angle of separatrix branches can be found as accurately as required thus facilitating the revealing of the finest details of the interaction of nonlinear resonance and the formation of the chaotic layer [8]. The technique can be substantially reinforced by the use of the user specified arbitrary accuracy of calculations (in [8], for example, the mantissa of the decimal representation of the real number had 300 significant digits).

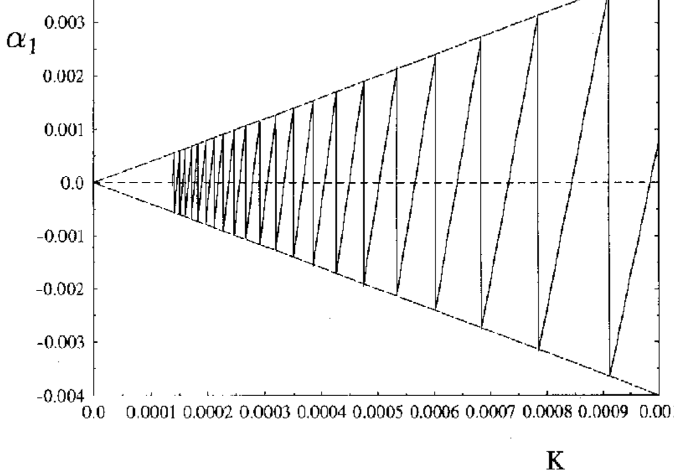

Figure 3: Illustration of the asymptotical theoretical dependence (5) of the break angle value of the primary resonance separatrix at the central homoclinic point as a function of the critical number value (see text). Inclined dashed lines are constructed by equations $\alpha_1 = \pm 4K$.

The found above number $K_{1,1} = 1/3$ is not only the largest of all the critical numbers. It turned out that it is at the boundary of two regions with distinct and qualitatively different dynamical behavior of the system (1), (2). At $K \leq K_{1,1}$ takes place far from trivial dynamics whose discussion will be in the remaining part of the work. On the contrary, at $K > K_{1,1}$ the system behavior is quite simple: with the growth of the parameter K value, secondary resonances one by one loose its stability and corresponding islands of stability "submerge" in the chaotic sea. This process is distinctly traced by the transition of eigenvalues of the linearized motion matrix from the complex-conjugated values (a stable point of the elliptical type) to the real values via the equality $\lambda_1 = \lambda_2 = -1$ (the hyperbolic point with reflection), which is the evidence of the shift from the regular motion to the chaotic motion [2]. At $K \approx 0.39$ resonances Q=3 vanish, half-primary resonances $Q = 2$ vanish at $K = 0.5$, and finally, primary resonances vanish at $K = 1.0$. It may turn out that for $K > 1$ the regular component of motion in the system (1), (2) is completely absent. This property if approved is also the feature of the sawtooth perturbation. In the standard map, for example, regular component is always higher than zero (new interesting results on the subject were recently obtained in the work [9]).

The established above fact of the existence of a countable set of values of the parameter K at which separatrices of the primary resonances are not destroyed in spite of the perturbing influence of many other resonances is of course of interest in itself and, to our opinion, it demands its additional study. But of no less interest are its dynamical consequences and we proceed to their description.

3 Dynamics of critical regimes

Fig.4 in small window shows the region occupied by one chaotic trajectory at the system parameter value equal to the first even critical number $K = K_{1,2} = 1/8$. The lower and upper boundaries of this region are quite close (this definition will be clarified) to separatrices of primary resonances calculated by the exact Ovsyannikov's formulae. Here all contradicts to the ideas of the "usual" dynamics.

It is known that there are infinitely many "untypical" Hamiltonian systems whose separatrices of all the resonances are not splitted. These are the so-called fully integrable systems whose dynamical behavior has no any chaos [2,3]. A striking feature of the situation given in Fig.4 is the coexistence of unsplitted separatrices of two adjacent primary resonances with the region of the powerful chaos, where all the invariant curves with the most stable irrational winding numbers are destroyed and chaotic layers of all secondary resonances are overlapped. We have checked also the fact that the chaotic trajectory started inside a primary resonance remains there during the entire counting time (10^{10} iterations) in full agreement with the Fig.3 and definition of the separatrix.

This is apparently the main dynamical effect: at critical numbers the primary resonances separatrices are not only destroyed with no formation of the chaotic layer but they form a stable invariant manyfold which does not allow other trajectories to be intersected. Having the full phase extension they, as "dams", divide the phase space into sections isolated each from other. This circumstance is especially important for applications since it prohibits the momentum global diffusion and eliminates, for example, an unlimited rise of particle energy. As far as we know, anything similar was never observed !

It is relevant to note that far from critical numbers there is apparently no unpenetrable barriers and the system behaves as "usual". For example,

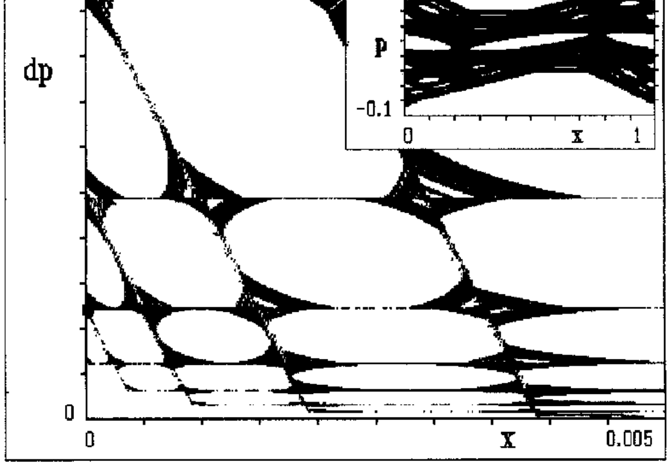

Figure 4: The system parameter is equal to the second critical number $K = K_{1,2} = 1/8$. Small window: the region occupied by one chaotic trajectory with initial coordinates $x = 0$, $p = 0.37$. The number of the map iterations is 10^{10}. Large window: magnified small region of the "gap" between this trajectory and the lower separatrix of the primary resonance (see text). Here $dp = p - p_s$, where $p(x)$ is the momentum of a trajectory and $p_s(x)$ is the momentum on the separatrix.

at $K = 0.3$, the chaotic trajectory started at the initial conditions $x = 0$, $p = 0.123$ during 10^6 iterations was occurred in the regions of seven adjacent primary resonances.

On the base of all mentioned above one has to recognize that the unusual dynamics is not related to the circumstance whether the separatrix is splitted or not but rather to that whether it is "transparent" for other trajectories or not. In the recent paper [6], there is a theorem (let us call it the second Ovsyannikov's theorem) where introduced the new and different from the generally accepted definition of the separatrix. The new separatrix is not splitted at any value of the system parameter $K < 1$. But, as our studies have shown, it is penetrable for other trajectories for almost all the values K except the critical values (3), (4). But for the critical K separatrices of both kind (according to conventional viewpoint and to second Ovsyannikov's theorem) coincide exactly with the objects of the first Ovsyannikov's theorem

The chaotic trajectory in Fig.4, as said above, approaches quite close to the separatrices. For obtaining the quantitative estimate of this closeness, the minimum distance in momentum dp_{min} between this trajectory and of primary resonances separatrices (calculated by exact formulae (1.5)) was fixed. As it turned out, between the trajectory and the lower separatrix there is a "gap" whose minimum width is $dp_{min} \approx 3 \cdot 10^{-6}$. A very magnified small part of the gap is shown in large widow of Fig.4, from which it is seen that the gap is filled in with secondary resonances of relatively high orders. The dynamics of this region will be discussed later (see Section 5) after considering the properties of separatrices and critical numbers of the secondary resonances.

With the deviation of the system parameter K from the critical value $K_{1,2}$ to the side of an increase, the separatrix starts to allow other trajectories to penetrate but the mean time $< T_c >$ (the number of iterations) of resonance passage depends on the detuning value $\Delta K = K - K_{1,2} > 0$. For finding out this dependence the following measurements were performed. At a fixed value of the parameter $K = K_{1,2} + \Delta K$ in the region between two adjacent resonances (see small window of Fig.4) some random chaotic trajectories are introduced and time T_c is fixed of their first occurrence in the regions either below the lower or above the upper resonances. Here it is necessary to make the general note which is related to our work on the whole. It is very important to choose initial conditions in such a way to have jrajectories chaotic for sure since only these trajectories can abandon "their" resonances and shift to others. This circumstance was checked in all the cases by the value and behavior of the Lyapunov exponent [2].

Results of these measurements are given in Fig.5 and their treatment by the least-squares method enables us to write down an empirical formula for the passage mean time of primary resonance:

$$< T_c >= 135 \left(\frac{K_{1,2}}{K - K_{1,2}} \right)^{1.193} (1 \pm 0.09), \quad K > K_{1,2}. \tag{6}$$

It is seen that with an approach to the critical regime the passage time grows unlimitly. This circumstance enables, in principle, to use the construction of dependencies of the kind (6) for the search for critical numbers but such method is much less exact and much more cumbersome than the measurement of the separatrix splitting angle.

It would be natural to see what is happening with the deviation of the system parameter K from the critical value $K_{1,2}$ to the side of an decrease. It was found out, however, that the dynamical situation occurred is qualitatively

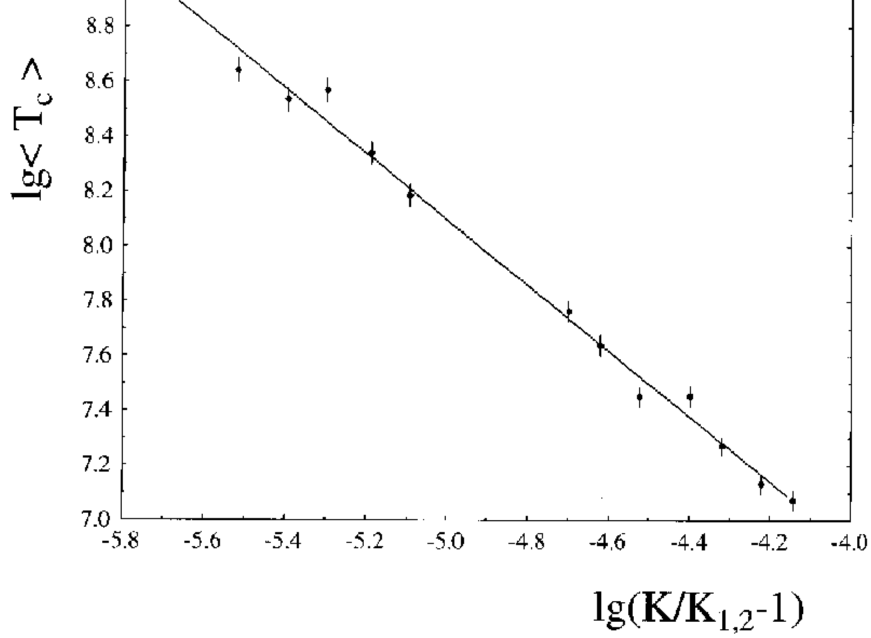

Figure 5: Time $< T_c >$ required for passing the primary resonance (mean by 100 random chaotic trajectories) as a function of the relative deviation of the system parameter K from the critical value $K_{1,2} = 1/8$ at $K > K_{1,2}$. Logarithms here are decimal.

different. For its understanding, it is necessary to learn the characteristics of secondary resonances and therefore the discussions of this dynamics will also be done in Section 5.

The situation with the first odd critical number looks even more estonishing. Fig.6 constructed for the parameter value $K = K_{1,1} = 1/3$ shows separatrices of the adjacent primary resonances and three chaotic trajectories starting inside secondary resonances 1:4, 1:2, and 3:4, respectively. Similar calculations were carried out for resonances of other orders but the dynamical picture always turned to be qualitatively the same: the chaotic trajectory for the whole interval of calculation does not escape the region of the secondary resonance inside of which it is started. We have shown in this Figure only three trajectories just to clarify the fact. The picture with many resonancs

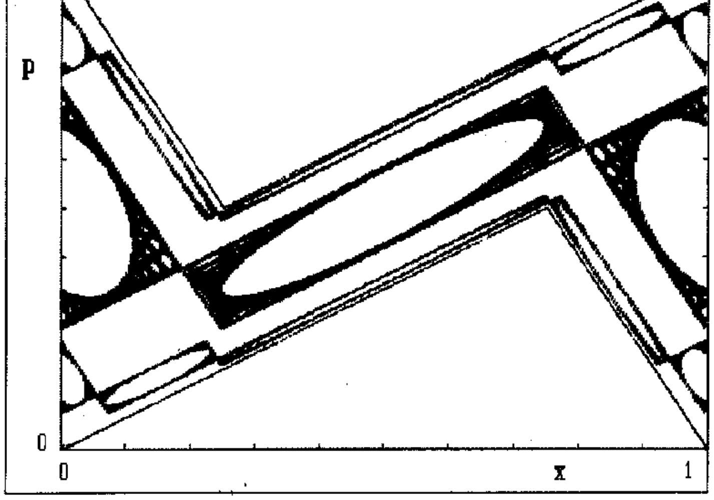

Figure 6: The system parameter is equal to the first critical number $K = K_{1,1} = 1/3$. The lower and upper broken lines are the separatrices of adjacent primary resonances. Three trajectories located between them with initial coordinates providing their belonging to the chaotic components of motion inside resonances (from below to up) 1:4, 1:2, and 3:4, respectively. The number of the map iterations is 10^{11} for each trajectory.

looks very attractive on the color display screen, where each resonance is presented by its own color and these colors are not mixed.

Here, similarly to that given in Fig.4, all the shown region is "closed" between separatrices of adjacent primary resonances. However, there is an impression that each individual secondary resonance is a stable invariant manyfold (it seems to more correct to say: the stable invariant resonance structure) with the boundary through which the trajectories do not penetrate. The question is relevant "What is this boundary ?". By the laws of the "usual" dynamics this may be a stable invariant curve with irrational winding number, which isolates adjacent resonances from each other [1,2,3]. But after the discussion of the situation given in Fig.4 we can rightfully assume that this boundary might also be the unsplitted separatrix of the secondary resonance itself. The answer to this and other questions demands the detail analysis of secondary resonances to be done in the nest Section.

For a study of secondary $Q > 1$ resonances we decided to use the same techniques, which was used above for primary resonances, namely the measurement of the splitting angle of separatrix α_Q as a function of the system parameter K. Note that with an increase in Q technical difficulties grow therefore, by now the reliable data are obtained only for the first four secondary resonances $Q = 2, 3, 4, 5$.

First of all, intersection angles of separatrix branches were measured for the critical value of parameter $K = K_{1,2} = 1/8$ and they turned to be rather large: $\alpha_2 \approx -0.036$, $\alpha_3 \approx -0.36$, $\alpha_4 \approx -0.45$, $\alpha_5 \approx -1.01$. This fact evidences the presence of wide chaotic layers in all the studied resonances, which is entirely agree with the picture shown in small window of Fig.4.

The next step was the calculation of critical numbers for these resonances whose results are given in Table 1. The data of the first column (referred to the primary resonances) calculated by formulae (3), (4), all the ramaining data were obtained numerically. It turned out that each secondary resonance has its own (apparently, countable) set of critical numbers $K_{Q,m}$, $m = 1, 2, 3, ...$ which, except for the first element, does not coincide with other similar sets. And also, as in the case of primary resonances, odd critical numbers are related to the smooth variations of the intersection angle of separatrix branches and even numbers to the jump.

Table 1. First critical numbers of resonances

m	$Q = 1$	$Q = 2$	$Q = 3$	$Q = 4$	$Q = 5$
1	$1/3$	$3.3333333 \cdot 10^{-1}$	$3.3333333 \cdot 10^{-1}$	$3.3333333 \cdot 10^{-1}$	$3.3333333 \cdot 10^{-1}$
2	$1/8$	$2.2949699 \cdot 10^{-1}$	$2.9787198 \cdot 10^{-1}$	$3.2189964 \cdot 10^{-1}$	$3.2960547 \cdot 10^{-1}$
3	$6.1916956 \cdot 10^{-2}$	$1.7387100 \cdot 10^{-1}$	$2.8038338 \cdot 10^{-1}$	$3.1678073 \cdot 10^{-1}$	$3.2801245 \cdot 10^{-1}$
4	$3.6340580 \cdot 10^{-2}$	$1.3985656 \cdot 10^{-1}$	$2.7079692 \cdot 10^{-1}$	$3.1415833 \cdot 10^{-1}$	$3.2721290 \cdot 10^{-1}$
5	$2.3743290 \cdot 10^{-2}$	$1.1700662 \cdot 10^{-1}$	$2.6505200 \cdot 10^{-1}$	$3.1265736 \cdot 10^{-1}$	$3.2676630 \cdot 10^{-1}$
6	$1.6679196 \cdot 10^{-2}$	$1.0061849 \cdot 10^{-1}$	$2.6136469 \cdot 10^{-1}$	$3.1172417 \cdot 10^{-1}$	$3.2648587 \cdot 10^{-1}$
7	$1.2340650 \cdot 10^{-2}$	$8.8293529 \cdot 10^{-2}$	$2.5886807 \cdot 10^{-1}$	$3.1110653 \cdot 10^{-1}$	$3.2630354 \cdot 10^{-1}$
8	$9.4919663 \cdot 10^{-3}$	$7.8686519 \cdot 10^{-2}$	$2.5710423 \cdot 10^{-1}$	$3.1067741 \cdot 10^{-1}$	$3.2617739 \cdot 10^{-1}$
9	$7.5237127 \cdot 10^{-3}$	$7.0986237 \cdot 10^{-2}$	$2.5581434 \cdot 10^{-1}$	$3.1036754 \cdot 10^{-1}$	$3.2608659 \cdot 10^{-1}$
10	$6.1081453 \cdot 10^{-3}$	$6.4674926 \cdot 10^{-2}$	$2.5350778 \cdot 10^{-1}$	$3.1013667 \cdot 10^{-1}$	$3.2601910 \cdot 10^{-1}$

As is seen from the Table 1, the first critical numbers $K_{Q,1}$ of the secondary resonances coincide with the first critical number of the primary reswas calculated for all the resonances with an accuracy of 25 true decimal digits and all of them turned to be triplets. Actually, any calculation results not supported by the theory can always be subjected to a doubt because of the insufficient duration or accuracy. But the results obtained are, to our opinion, convinceble evidences of the fact that the unpenetrable for other trajectories boundaries of the stable structures shown in Fig.6 are their own separatrices. Whether the equality $K_{Q,1} = 1/3$ is true for any Q (this would mean the unsplitting of all the separatrices) will be found by future studies.

In the course of described measurements at $K_{1,1} = 1/3$ it was noticed that the winding number of the chaotic trajectory inside any resonance during the long calculation is quite close to the value P/Q. This fact can be a base of one more method of critical numbers localization but it is also (as the construction of dependences of the kind (6)) is less exact and more cumbersome than the measurement of the splitting angle of separatrices.

Fig.7 shows the "work" of critical regimes of the secondary resonances. Here, the system parameter $K = K_{2,4}$ is equal to the second critical number

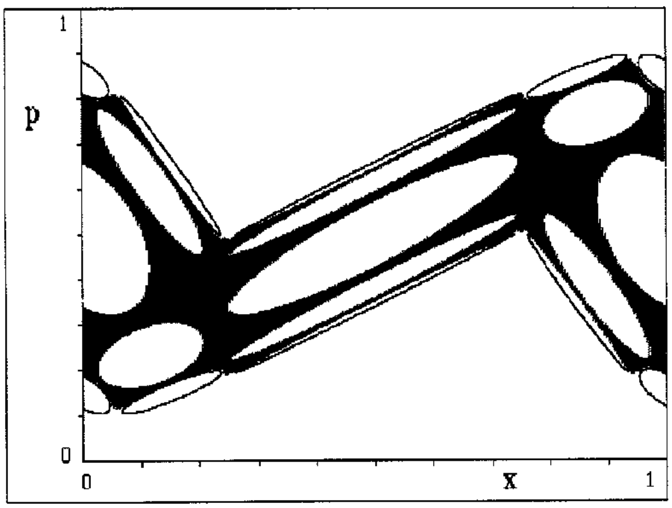

Figure 7: The system parameter is equal to the second critical number $K = K_{4,2} = 0.321899641461...$ of secondary resonances 1:4 and 3:4 which are presented by islands of stability. One chaotic trajectory "closed" between unsplitted separatrices (not shown in the Figure) of these resonances. The number of the map iterations is 10^{11}.

trajectory started between them.

It is necessary to discuss one more important feature of the system (1), (2), which can conventionally called so: the retuning of the secondary resonances. The map (1), (2) similarly to the standard map, has some kind of symmetry and can be presented as product of two involutions [2,8]. As known, this facilitates substantially the search for fixed points since it is necessary to examine only four lines of symmetry enumerated for convenience in the following order: 1) $x = 0$; 2) $x = 0.5$; 3) $x = p/2$ and 4) $x = (1 + p)/2$ [10].

For the standard map, the central and saddle points of any resonance in the line of symmetry do not escape it with the change of the system parameter. In this respect, all was turned quite different for the system (1),(2) under study. For example, for the resonance 1:3 at $K = K_{1,1} = 1/3$ one of the stable point is located in the second line of symmetry $x = 0.5$ whereas in the first line $x = 0$ is located the saddle. At $K = K_{1,2} = 1/8$ all turned to be just the other way around. The dynamics of this transition is quite specific and it looks like that. With an decrease in the parameter K from $1/3$, the area of the related to the resonance 1:3 stable region decreases monotonically and at $K = 0.250$ it vanishes completely (the value of the Green's residue calculated at this point is equal to zero [7]). With an additional decrease in the parameter K it occurs again and grows monotonically but the center turned to be already in the first line and the saddle is located in the second line.

Positions of resonances $Q = 3, 4, 5, 6$ were traced for the first six critical numbers from $K_{1,1}$ to $K_{1,6}$ (see Table 1) but we still failed to find out here some general law. At $K_{1,1}$ one of the stable points of all these resonances is in the second line of symmetry. At $K_{1,2}$ the center of the resonance 1:3 shifts to the first line and does not escape it. At $K_{1,2}$ the resonance 1:4 shifts to the third line and remains there. For the resonance 1:5, the picture turned to be a little complex: at $K_{1,2}$ its center is moved to the first line, at $K_{1,3}$ it returns again to the second line and remains there. The center of resonance 1:6 does not escape its initial (second) line of symmetry.

One of the most important stages in a study of the system should be the search for and explanation of the mechanism responsible for the oscillatory and sign-variable character of the main dependence $\alpha_1(K)$ (Fig.1) It may turn out that the above described "movability" of secondary resonances is related directly to the mechanism.

at $K_{1,2}$

Upon the acquaintance with the critical numbers and properties of separatrices of secondary resonances we can return to the subjects formulated but not considered in Section 3. Let us start with the discussion of the dynamical situation in a "gap" between the chaotic trajectory and separatrix of the lower primary resonance (for the brevity let us call it the main separatrix). As is seen from large window of Fig.4, this region is populated by resonances of relatively high orders 1:Q (the lowest still recognizable resonance in this picture is 1:23).

As the first step, critical numbers of some of them were found. Central homoclinic points at odd values $Q = 15, 17, 19...$ are in the first line of symmetry $x = 0$ that enables the use of our technique of measurement of the separatrix intersection angle. Of the main interest are numbers closest to the value $K_{1,2} = 1/8$ for which Fig.4 is constructed. Results are given in Table 2, where the difference between $K_{1,2} = 1/8$ and critical number K_Q of the resonance $1 : Q$ is indicated. From these data it is clearly seen that with an increase in the resonance Q order, its winding number $1/Q$ and its critical number K_Q tend to the winding number (zero) and critical number $K_{1,2}$ of the main separatrix. All these resonances have the full phase extension and any chaotic trajectory tending to approach the main separatrix should intersect them, which takes some certain time.

Table 2. Critical numbers of the "gap" secondary resonances.

Q	13	15	17	19	21	23
$K_{1,2} - K_Q$	$6.71 \cdot 10^{-8}$	$4.19 \cdot 10^{-9}$	$2.62 \cdot 10^{-10}$	$1.63 \cdot 10^{-11}$	$1.02 \cdot 10^{-12}$	$6.39 \cdot 10^{-14}$

Formula (6) from the Section 4 provides the estimate for the mean passage time of the primary resonance, which grows infinitely with the approach of the parameter K to the critical number $K_{1,2}$ from above. Table 2 helps the understanding the reason of this: all the secondary resonances of the "gap" approach (from above too) to their critical numbers. If we assume that there is some similar to the dependence (6) for these resonances, the time required for their passage should also grow unlimitedly at $Q \to \infty$ and $(K_{1,2} - K_Q) \to 0$.

The direct measurement of the trajectory penetration time from the "chaotic sea" inside the secondary resonance of high order is technically for the escape from the resonance region of the trajectory started inside it. For each resonance with $Q = 13, 15, 17, 19, 21$ about 500 chaotic trajectories were calculated and the treatment of the results by the least squares method gave the dependence

$$< T_e >= 80\, e^{0.705\,Q}(1 \pm 0.07)\,, \qquad (7)$$

which proves the above made assumption about infinite escape time with $Q \to \infty$.

These data enable us to understand the mechanism providing the stability of the main separatrix in the critical regime $K = K_{1,2}$. The region adjacent to the separatrix is filled with high order resonances $1 : Q$ which are located by layers the closer to the ceparatrix, the higher is the value Q (see Fig.4). Since each such resonance has the full phase extension, in order to reach the lower layer it is necessary to intersect the upper layer. Critical numbers of resonances with the approach to the main separatrix and the growth of the value Q approach the value $K = K_{1,2}$ (see Table 2). As a consequence, eigen separatrices of these resonances become less and less "transparent" for the chaotic trajectory and its escape time from them, according to (7), grows infinitely.

Let us turn now to the second question put in section 3 and consider what is happening at the deviation of the system parameter K from the critical value $K = K_{1,2}$ to its decrease. From Table 2 it is clear that, in this case, the gap resonances will be one by another successively put into their critical regimes and do not allow the trajectory to approach the main separatrix at a distance (in momentum) closer than some certain dp_{min}. So, at $K = 0.12499999581..$ (the critical value for the resonance 1:15) the minimum distance between the trajectory and the separatrix turned to be $dp_{min} \approx 4.6 \cdot 10^{-5}$, at $K = 0.12499993294..$ (the critical value for the resonance 1:13) $dp_{min} \approx 1.8 \cdot 10^{-4}$ with the number of map iterations equals 10^{11} in every case.

This consideration can be summarized in the following somewhat figurative form. The higher order resonance form in front of the main separatrix some kind of the "obstacle" resisting against an approach of the chaotic trajectory. In the supercritical regime $K > K_{1,2}$, the trajectory manages to intersect the primary resonance spending the more time the closer K to $K_{1,2}$. At $K = K_{1,2}$ in the process of very long evolution the trajectory can approach the main separatrix arbitrary close but never reaches it. In the subcritical regime $K < K_{1,2}$, the trajectory cannot intersect the secondary a distance less than some certain distance.

Whether the described scenario is typical and whether it is valid for not only primary but for secondary separatrices is one of the open questions. It is not yet clear also what is the onset of the supercritical and what is the final of the subcritical regimes and where (by the parameter K) are their boundaries?

6 Conclusion

A detail study of the map (1), (2) is only started and it is too early to draw some conclusions. But if we assume that the above described effects are proved and generalized, the following far from simple dynamical picture takes place.

Each resonance from the countable set of all the system resonances is related to its own set of critical values of the parameter K at which the intersection angle α_Q of separatrix branches passes through zero, which evidences the absence of splitting and chaotic layer. The separatrix of such a resonance (independent of its order Q) has the full phase extension and is an unpenetrable barrier for other trajectories thus eliminating the possibility of the momentum global diffusion.

In this connection, note that in Ref.[11] in the course of the numerical study of the map (1), (2) with the value of the parameter K=0.28625 (we draw your attention to its closeness to the critical number $K_{3,3}$, see Table 1) the limitation of diffusion was noticed but not yet explained. This circumstance can probably be understood in the course of further studies taking into account new information.

To our opinion, the system (1), (2) deserves its further study and we would like to note some most important problems.

Above mentioned facts evidence that the change of sinus by sawtooth in the standard map, generates the qualitatively different new dynamics, which is beyond the frame of now existing ideas. Recall that in the "usual" dynamics, with an increase of the parameter K the transition from the chaotic layer to the chaotic sea is studied well enough and explained by the destruction of the invariant curves with irrational winding numbers and by the formation of the so-called Cantorus [12]. For the system (1), (2), this picture is apparently not valid since the main dynamical effects turned to be related to the resonances whose winding numbers are strictly rational. With a distance from the critical value of the parameter the separatrix of any resonance is the process is apparently one of the main and intriguing prolems.

Another problem is related with the search for the mechanism respondible for the oscillatory and sign-varying character of the dependence $\alpha_Q(K)$ (see Fig.1 for $Q = 1$) which mainly determines an unusual dynamics of the system.

Of undoubted interest is also a study of the global diffusion at $K \leq 1/3$ taking into account of the fact that this diffusion is limited at any critical number of any resonance. The solution of this problem seems to be dependent of that what is the structure of the set of all the critical numbers.

One can hope that in the course of the search for answers to the posed and many other questions the information can be obtained, which enables to refine of some ideas of the modern nonlinear Hamiltonian systems theory.

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Appendix

L.V.Ovsyannikov's theorem on separatrices of sawtooth map [5]

Consider the equation for the function $x(t)$ defined through the entire line $\mathcal{R}(-\infty < t < +\infty)$

$$x(t+h) + x(t-h) - 2x(t) = h^2 f(x(t)) \,, \tag{1.1}$$

where $h > 0$ is the fixed constant; the function $f(x)$ is odd with period equals 4 and it is specified for $0 \leq x \leq 2$ by the formula

$$f(x) = \begin{cases} -x, & (0 \leq x \leq 1)\,, \\ x - 2, & (1 \leq x \leq 2)\,. \end{cases} \tag{1.2}$$

The solution $x(t)$ of Eq.(1.1) with the function (1.2) is called the separatrix if $x(t)$ is monotonical and continuous through the entire real axis $\mathcal{R}$, $x(0) = 0$ and $x(t) \to 2$ at $t \to +\infty$, $x(t) \to -2$ at $t \to -\infty$.

Theorem. There exists such a sequence $\{h_k\}$, $k = 1, 2, ...$, that $h_k \to 0$ at $k \to \infty$ and for each $h = h_k$ there is the separatrix $x = x^k(t)$ as the solution of Eq.(1.1) with the function (1.2). The sequence $\{x^k(t)\}$ at $k \to \infty$

equation $x''(t) = f(x)$

$$x(t) = \begin{cases} \sqrt{2}\sin t, & (0 \le t \le \pi/4)\,, \\ 2 - e^{\pi/4 - t}, & (\pi/4 \le t \le \infty)\,. \end{cases} \tag{1.3}$$

Construction. Let $h_k = 2\sin(\alpha_k/2)$, where α_k is the least positive root of the equation

$$\sqrt{2}\sin(k\alpha) = \cos(\alpha/2)\,. \tag{1.4}$$

Then for $nh_k \le t \le (n+1)h_k$ the separatrix is given by formulae

$$x^k(t) = X_n(n + 1 - t/h_k) + X_{n+1}(t/h_k - n) \;\; (n = 0, 1, 2...) \tag{1.5}$$

with constants

$$X_n = \frac{\sin(n\alpha_k)}{\sin(k\alpha_k)} \quad (n \le k)\,, \qquad X_n = 2 - \left(\frac{\sqrt{h_k^2 + 4} - h_k}{2}\right)^{2(n-k)} \quad (n \ge k)$$

Note that since the period and position of the force (1.2) in this theorem are different from those adopted in the text (2), for the transition from the Ovsyannikov's parameter h in Eq. (1.1) to the parameter K of the map (1) and vice versa, we need to use the relation

$$K = h^2/4\,.$$

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