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KINETICS OF 3-D IONIZATION COOLING OF MUONS

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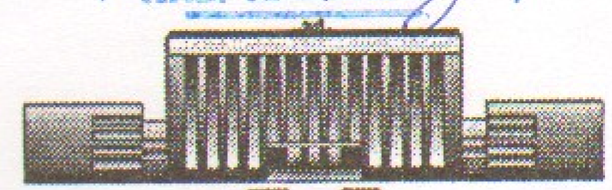

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Kinetics of 3-d ionization cooling of muons[1]

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Abstract

Considered is the ionization cooling of muon beam in all three – the longitudinal and two transverse – directions simultaneously in a scheme, based on bent lithium lenses. An analysis of 3-dimensional cooling is performed with the use of kinetic equation method. Results of solution for a concrete beam line are presented together with computer simulation results. To create the dipole constituent of magnetic field in current-carrying rod a special configuration of the rod is proposed.

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1 Introduction

As it was already shown [1,2,3], the ionization cooling is efficient for the transverse beam emittances and not at all for the longitudinal one. The natural longitudinal decrement is equal to a derivative of the mean rate of ionization loss with respect to the particle energy. In a region of logarithmic grow of loss rate the magnitude of this derivative does not exceed 7% of transverse decrement, whereas at momentum below $\sim$ 400 MeV/c, where it is negative, its absolute value grows fast with energy decrease and becomes of the order of $\delta_\perp$ at $pc \sim 100$ MeV, thus resulting in a strong heating of longitudinal emittance instead of cooling.

Meanwhile, to arrive in a limit at minimum 6-dimensional emittance of cooled beam, the summary decrement $\delta_\Sigma = 2\delta_\perp + \delta_\parallel$ is to be about equally divided between all three directions [2].

Besides that, with no efficient cooling of longitudinal emittance, the energy spread in a beam is growing, and chromatic aberration in a lens, matching the amplitude functions of low-beta focus at slowing target and that of rather long-wave accelerator unit, leads to sufficient increase in transverse emittance, which puts a lower limit to its minimum value, achievable without an efficient cooling in longitudinal direction.

Thus, the violent redistribution of summary decrement between the transverse and longitudinal degrees of freedom has to be fulfilled in a scheme for ionization cooling by way of creation of transverse gradient of energy loss rate, correlated with dispersion function.

2 Kinetic equation and three-dimensional cooling analysis

Kinetic equation for ionization cooling of muons [1] in three-dimensional case is convenient to be written in cylindrical coordinates R, φ, z with φ taken as independent variable:

$$\frac{1}{R_0}\left(\frac{\partial P}{\partial \varphi} + \frac{R}{v}\frac{\partial P}{\partial t} + \frac{\partial P}{\partial \mathbf{r}}\frac{d\mathbf{r}}{d\varphi} + \frac{\partial P}{\partial \theta}\frac{d\theta}{d\varphi}\right) + e\varepsilon\left(\frac{\partial P}{\partial E} - \frac{1}{pv}\frac{\partial \theta P}{\partial \theta}\right) = \tag{1}$$

$$\frac{R}{R_0}\left\{\frac{\pi Z^2 e^4 n L_c}{(pv)^2}\Delta_\theta P - \frac{\partial}{\partial E}\left[P\left(\overline{\frac{dE}{ds}}\right)_{ion}\right] + \frac{1}{2!}\frac{\partial^2}{\partial E^2}\left[P\left(\overline{\frac{d\Delta E^{\,2}}{ds}}\right)_{ion}\right]\right\}.$$

Here R_0 stands for a beam line curvature radius, while $r = R - R_0$ – for the radial coordinate of particle with respect to the beam axis, $\mathbf{r}$ and θ are two-dimensional vectors of transverse coordinate (r, z) and angle (θ_r, θ_z), and $e\varepsilon$ denotes an acceleration rate.

To extract the free radial motion we make a change of variables $\rho = r - \psi\frac{\Delta E}{pv}$ and $\vartheta_r = \theta_r - \psi'\frac{\Delta E}{pv}$, where ψ is the coordinate dispersion function while ψ' is the angular one.[2] By that the derivatives $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta_r}$ in above equation just turn into $\frac{\partial}{\partial \rho}$ and $\frac{\partial}{\partial \vartheta_r}$, while derivative $\frac{\partial}{\partial E}$ transforms into a sum $\frac{\partial}{\partial E} - \frac{\psi}{pv}\frac{\partial}{\partial \rho} - \frac{\psi'}{pv}\frac{\partial}{\partial \vartheta_r}$.

Being integrated – after change of variables – over spreads of energy and longitudinal particle coordinate $\Delta s = v\Delta t$, the equation (1) transforms into one, describing the free transverse motion only, where from the differential equations for the mean square characteristics of free radial motion – $\langle\rho^2\rangle$,$\langle\rho\vartheta_r\rangle$ and $\langle\vartheta_r^2\rangle$ are got in a form:

$$\frac{\partial\langle\rho^2\rangle}{\partial s} - 2\langle\rho\vartheta_r\rangle - 2\delta_0\psi\left(\eta + \frac{1}{R_0}\right)\langle\rho^2\rangle = \frac{\sigma_{st}^2\psi^2}{p^2v^2},$$

$$\frac{\partial\langle\rho\vartheta_r\rangle}{\partial s} + \delta_0\left[1 - \psi\left(\eta + \frac{1}{R_0}\right)\right]\langle\rho\vartheta_r\rangle - \langle\vartheta_r^2\rangle + k_r\langle\rho^2\rangle = 0, \tag{2}$$

$$\frac{\partial\langle\vartheta_r^2\rangle}{\partial s} + 2\delta_0\langle\vartheta_r^2\rangle + 2k_r\langle\rho\vartheta_r\rangle = \frac{E_k^2/X_0}{2(pv)^2}.$$

[2] To be precise, the change of variables is to be $\rho = r - \psi\frac{\Delta E}{pv} - \chi\Delta s$ and $\vartheta_r = \theta_r - \psi'\frac{\Delta E}{pv} - \chi'\Delta s$ to make an account for possible correlation between the radial coordinate and angle and the longitudinal coordinate, which we neglect.

Here s substitutes for φR_0, $k_r = k + \frac{1}{R_0^2}$ with $k = \frac{e}{pc}\frac{dH}{dr}$; δ_0 stands for the natural transverse decrement, $\delta_0 = \frac{\xi}{pv}$, where ξ denotes the mean rate of energy loss, $\xi = -\left(\overline{\frac{dE}{ds}}\right)_{ion}$, while σ_{st}^2 stands for its mean square straggling, $\sigma_{st}^2 = \left(\overline{\frac{d\Delta E^{\,2}}{ds}}\right)_{ion}$. The radial gradient of ξ is presented by η, $\eta = \frac{1}{\xi}\frac{\partial\xi}{\partial r}$. Average value of ξ is taken equal to the acceleration rate, $\overline{\xi} = \xi_0 = e\varepsilon$.

In the right hand side of the third equation stands the mean square rate of multiple Coulomb scattering in radial direction, expressed through the characteristic energy E_k (~ 20 MeV) and radiation length X_0.

The equation for radial emittance, got with the use of system (2), reads:

$$\frac{\partial\epsilon_r}{\partial s} + \delta_0\left(1 - \psi\eta - \frac{\psi}{R_0}\right)\epsilon_r = \frac{1}{2(pv)^2}\left(\frac{E_k^2\beta_r}{2X_0} + \frac{\psi^2\sigma_{st}^2}{\beta_r}\right) \tag{3}$$

with β_r standing for radial betatron function.

The radial decrement is defined from this equation as:

$$\delta_r = \delta_0\left[1 - \psi\left(\eta + \frac{1}{R_0}\right)\right]$$

and equilibrium emittance as:

$$\epsilon_{r,eq} = \frac{1}{2\delta_r p^2v^2}\left(\frac{E_k^2\beta_r}{2X_0} + \frac{\psi^2\sigma_{st}^2}{\beta_r}\right),$$

which, after substitutions made: $E_k^2/X_0 = 4\pi Z(Z+1)e^4 n L_c$, $\delta_0 = \frac{\xi_0}{pv}$, $\xi_0 = \frac{2\pi Z e^4 n}{mv^2}L_i$ and $\sigma_{st}^2 = 2\pi Z e^4 n(\gamma^2+1)$, reads:

$$\epsilon_{r,eq} = \frac{\beta_r mc^2 L_c(Z+1)}{2EL_i\left[1 - \psi\left(\eta + \frac{1}{R_0}\right)\right]}\left(1 + \frac{\psi^2}{\beta_r^2}\frac{\gamma^2+1}{(Z+1)L_c}\right). \tag{4}$$

Here L_c denotes the logarithmic factor in the mean square angle of multiple Coulomb scattering, calculated for an effective depth of slowing medium $s_{\text{eff}} = \frac{1}{2\delta_0}$, while L_i is the logarithmic factor in the mean rate of ionization loss of energy, $L_i = \ln\frac{2mc^2\beta^2\gamma^2T_{max}}{I^2} - 2\beta^2 - \delta$.

As it is seen from (4), the equilibrium radial emittance gets an increase, caused not only by the decrement reduction, but also by an influence of the energy loss straggling. Amplified by square ratio of dispersion function to betatron one, this becomes significant for free radial motion and puts a limit for dispersion function magnitude as follows:

$$\psi < \beta_r \sqrt{\frac{(Z+1)L_c}{\gamma^2+1}}.$$

This means, for instance, $\psi < 3\beta_r$ at particle momentum $\sim$ 200 MeV/c. With high value of field gradient ($kR_0^2 >> 1$) the dispersion function is found as $\psi \cong r_0 \frac{H_0}{H_m}$, where H_0 denotes the bending field, while H_m – the maximum focusing one ($H_m = r_0 \frac{dH}{dr}$), and r_0 is the aperture radius. When $H_m \sim H_0$ the above condition is easily satisfied.

To consider the longitudinal motion we integrate the equation (1) over free transverse coordinates ρ and z and over angles ϑ_r and θ_z. Having made an account for velocity spread $\Delta v = v - v_0 = \frac{1}{\gamma^2}\frac{\Delta E}{pv}$ and then omitting index "0" in R_0 and v_0, one gets:

$$\frac{\partial P}{\partial s} + \left[1 + \frac{\Delta E}{pv}\left(\frac{\psi}{R} - \frac{1}{\gamma^2}\right)\right]\frac{1}{v}\frac{\partial P}{\partial t} + e\varepsilon\frac{\partial P}{\partial E} =$$
$$\left(1 + \frac{\psi\Delta E}{pvR}\right)\left[\frac{\partial}{\partial E}(\xi P) + \frac{1}{2!}\frac{\partial^2}{\partial E^{\,2}}(\sigma_{st}^2 P)\right] + \frac{\psi}{pvR}\left[\xi P + \frac{\partial}{\partial E}(\sigma_{st}^2 P)\right]. \quad (5)$$

In presence of radial gradient of ξ it has a form: $\xi = \xi(E)(1+\eta r)$, which after integration over ρ transforms into $\xi \cong (\xi_0 + \xi'\Delta E)\left(1 + \eta\psi\frac{\Delta E}{pv}\right)$, where an account is made as well for dependence of mean rate of energy loss on particle energy.

Let us also make an account for acceleration rate dependence on deviation of particle longitudinal coordinate from the equilibrium one $\varepsilon \cong \varepsilon_0 + \varepsilon_0'\Delta s$.

The equations for mean square characteristics of longitudinal motion $\overline{\Delta E^2}$, $\overline{\Delta E\Delta s}$ and $\overline{\Delta s^2}$, got from Eq.(5), read:

$$\frac{\partial\overline{\Delta E^2}}{\partial s} + 2\overline{\Delta E^2}\left[\xi' + \xi_0\frac{\psi}{pv}\left(\eta + \frac{1}{R}\right)\right] + 2e\varepsilon_0'\overline{\Delta E\Delta s} = \sigma_{st}^2,$$
$$\frac{\partial\overline{\Delta E\Delta s}}{\partial s} + \overline{\Delta E\Delta s}\left[\xi' + \xi_0\frac{\psi}{pv}\left(\eta + \frac{1}{R}\right)\right] - \frac{\overline{\Delta E^2}}{pv}\left(\frac{\psi}{R} - \frac{1}{\gamma^2}\right) + e\varepsilon_0'\overline{\Delta s^2} = 0, \quad (6)$$
$$\frac{\partial\overline{\Delta s^2}}{\partial s} - 2\frac{\overline{\Delta E\Delta s}}{pv}\left(\frac{\psi}{R} - \frac{1}{\gamma^2}\right) = 0.$$

From this system an equation for the longitudinal emittance

$\epsilon_L = \sqrt{\overline{\Delta E^2}\,\overline{\Delta s^2} - \overline{\Delta E\Delta s}^{\,2}}$ is found in a form:

$$\frac{\partial\epsilon_L^2}{\partial s} + 2\left[\xi' + \xi_0\frac{\psi}{pv}\left(\eta + \frac{1}{R}\right)\right]\epsilon_L^2 = \sigma_{st}^2\overline{\Delta s^2},$$

where from the equilibrium emittance is defined as

$$\epsilon_{L,eq} = \frac{\sigma_{st}^2}{2pv\delta_L}\beta_L \quad (7)$$

with δ_L being the longitudinal decrement,

$$\delta_L = \xi' + \xi_0\frac{\psi}{pv}\left(\eta + \frac{1}{R}\right), \quad (8)$$

and β_L – an amplitude function for longitudinal motion, defined with expression: $\frac{\beta_L\epsilon_L}{pv} = \overline{\Delta s^2}_{eq}$.

In rather hypothetical approach of homogeneously distributed acceleration rate this function is $\beta_L = \sqrt{\frac{pv}{e\varepsilon_0'}\left(\frac{1}{\gamma^2} - \frac{\psi}{R}\right)}$, while in a real case, where the linear accelerator sections succeed the energy degraders (with small enough pulsation of mean energy) the amplitude function is found as:

$$\beta_L \cong \sqrt{\frac{pv\bar{\lambda}}{\gamma^2}\left[\frac{1}{e\varepsilon_0} + \frac{1}{\xi_0}\left(1 - \gamma^2\frac{\psi}{R}\right)\right]}. \quad (9)$$

Symbol $\bar{\lambda}$ here stands for the ratio $\varepsilon_0/\varepsilon_0'$, i.e. $\bar{\lambda} = \frac{\lambda}{2\pi}\,tg\varphi_a$, with λ being a wave length of accelerating voltage and φ_a – an acceleration phase ($\varepsilon_0 = \varepsilon_0^{max}\sin\varphi_a$).

The equilibrium mean square energy spread, also for small enough pulsation of mean energy, is found as

$$\overline{\Delta E^2}_{eq} \cong \frac{\sigma_{st}^2}{2\delta_L}. \quad (10)$$

The longitudinal decrement (8) is composed of derivative ξ' of the mean rate of ionization loss with respect to the particle energy and of fraction $\Delta\delta = \delta_0\psi\left(\eta + \frac{1}{R}\right)$, transferred from the radial direction. This fraction is not equal zero even by $\eta = 0$, i.e. with no gradient of energy loss rate, however, the ratio $\frac{\psi}{R} \sim \frac{1}{\nu^2}$, defining here the value of $\Delta\delta$, is to be much less than unity to provide with a small value of transverse amplitude function $\beta_\perp \sim R/\nu$.

In a real beam line, where the dispersion function is not constant trough, the free radial motion and the longitudinal one can not be easily separated from each other. Their mutuel behavior is defined with a system of equations, got from (1):

$$
\begin{aligned}
&\frac{\partial\langle r^2\rangle}{\partial s} - 2\langle r\theta_r\rangle = 0,\\
&\frac{\partial\langle r\theta_r\rangle}{\partial s} + \frac{e\epsilon_0}{pv}\langle r\theta_r\rangle - \langle\theta_r^2\rangle + k_r\langle r^2\rangle - \frac{\langle r\Delta E\rangle}{pvR} = 0,\\
&\frac{\partial\langle\theta_r^2\rangle}{\partial s} + 2\frac{e\epsilon_0}{pv}\langle\theta_r^2\rangle + 2k_r\langle r\theta_r\rangle = \frac{E_k^2/X_0}{2(pv)^2},\\
&\frac{\partial\langle\Delta E^2\rangle}{\partial s} + 2e\epsilon'\langle\Delta E\Delta s\rangle = -2\xi_0\left(\eta+\frac{1}{R}\right)\langle r\Delta E\rangle - 2\xi'\langle\Delta E^2\rangle + \sigma_{st}^2,\\
&\frac{\partial\langle\Delta s^2\rangle}{\partial s} - 2\frac{\langle\Delta S^2\rangle}{R} + 2\frac{\langle\Delta E\Delta s\rangle}{pv\gamma^2} + e\epsilon'\langle\Delta s^2\rangle = 0, \qquad (11)\\
&\frac{\partial\langle\Delta E\Delta s\rangle}{\partial s} - \frac{\langle r\Delta E\rangle}{R} + \frac{\langle\Delta E^2\rangle}{pv\gamma^2} + e\epsilon'\langle\Delta s^2\rangle =\\
&-\xi_0\left(\eta+\frac{1}{R}\right)\langle r\Delta s\rangle - \xi'\langle\Delta E\Delta s\rangle,\\
&\frac{\partial\langle r\Delta E\rangle}{\partial s} + e\epsilon'\langle r\Delta s\rangle - \langle\theta_r\Delta E\rangle = -\xi_0\left(\eta+\frac{1}{R}\right)\langle r^2\rangle - \xi'\langle r\Delta E\rangle,\\
&\frac{\partial\langle\theta_r\Delta E\rangle}{\partial s} + e\epsilon'\langle\theta_r\Delta s\rangle + \frac{e\epsilon_0}{pv}\langle\theta_r\Delta E\rangle + k_r\langle r\Delta E\rangle - \frac{\langle\Delta E^2\rangle}{pvR} =\\
&-\xi_0\left(\eta+\frac{1}{R}\right)\langle r\theta_r\rangle - \xi'\langle\theta_r\Delta E\rangle,\\
&\frac{\partial\langle r\Delta s\rangle}{\partial s} - \frac{\langle r^2\rangle}{R} + \frac{\langle r\Delta E\rangle}{pv\gamma^2} - \langle\theta_r\Delta s\rangle = 0,\\
&\frac{\partial\langle\theta_r\Delta s\rangle}{\partial s} + \frac{e\epsilon_0}{pv}\langle\theta_r\Delta s\rangle - \frac{\langle r\theta_r\rangle}{R} + \frac{\langle\theta_r\Delta E\rangle}{pv\gamma^2} + k_r\langle r\Delta s\rangle - \frac{\langle\Delta E\Delta s\rangle}{pvR} = 0.
\end{aligned}
$$

This is to be completed with an equation for the beam energy variation:

$$\frac{\partial E}{\partial s} - e\epsilon_0 = -\xi_0.$$

The dispersion function and its derivative are found using the variables of above system as: $\psi = pv\langle r\Delta E\rangle/\langle\Delta E^2\rangle$ and $\psi' = pv\langle\theta_r\Delta E\rangle/\langle\Delta E^2\rangle$.

The axial motion is described by equations, similar to the first three of above system with R put infinitely large. Besides that, when it is not inde-

pendent of radial motion, several more equations are to be added, describing the correlations between them.

Out of above consideration there are left the single Coulomb scattering by a large angle and the asymmetry of energy loss distribution around the mean value, described by the Landau curve, while the first process is taken into account in the Moliere distribution of scattering angle. The difference of these two distributions from the Gaussian one is mostly observed at large deviation of scattering angle or energy loss from the mean values, which takes place rather seldom. Due to that these processes only slightly effect the mean square parameters of a beam, but manifest themselves in the main in a loss of particles by a finite acceptance of cooling channel. This is to be studied with computer simulation.

3 The ultimate 6-dimensional emittance

Having obtained the analytic expressions for all equilibrium emittances, we can now evaluate the ultimate value of 6-dimensional equilibrium emittance $\epsilon_{eq}^{(6)} = \epsilon_{r,eq}\epsilon_{z,eq}\epsilon_{L,eq}$, achievable with the use of current-carrying rod focus, and find the optimum correlation between the parameters, determining this value.

An optimization is applied, first of all, to the rod radius r_0 and product $\psi\eta^*$ ($\eta^* = \eta + \frac{1}{R}$). With fixed magnitude of magnetic field at the rod surface, restricted by a value of the order of 10 – 20 T [4], an achievable field gradient is defined by the minimum available value of r_0, restricted from below by an evident relation $r_0^2 > \langle r^2\rangle = \epsilon_{r,eq}\beta_r + \epsilon_{z,eq}\beta_z$. If to denote with R_t a permissible ratio of r_0 to the r.m.s. transverse coordinate of particles, $R_t = \frac{r_0}{\sqrt{\langle r^2\rangle}} \sim \frac{r_0}{\sigma_\perp\sqrt{2}}$, the ultimate value of r_0 is got, using (4), in a form:

$$r_{0_{ult}} = R_t^2\beta\frac{mc^2}{eH_m\left(1-\frac{\psi\eta^*}{2}\right)}\frac{(Z+1)L_c}{L_i}.$$

Here m stands for the electron mass and β – for the particle velocity in c units. We have also supposed, that the decrease in transverse decrements in favor of the longitudinal one is equally divided between both of them, and that the dispersion function meets the condition of its smallness (see (4)).

An expression for ultimate value of transverse amplitude function by that reads:

$$\beta_{\perp,ult} = \frac{R_t}{eH_m}\sqrt{\frac{mc^2pv(Z+1)L_c}{\left(1-\frac{\psi\eta^*}{2}\right)L_i}}.$$

With this expression used for transverse emittances definition, the equilibrium 6-dimensional emittance appears to be inversely proportional to a product $\left(1-\frac{\psi\eta^*}{2}\right)^3\left(\psi\eta^*+\frac{\xi'}{\delta_0}\right)$. It is to be maximized to minimize the 6-emittance, which defines the optimum value of $\psi\eta^*$ as:

$$(\psi\eta^*)_{opt} = \frac{1}{2} - \frac{3}{4}\frac{\xi'}{\delta_0}$$

and the decrements as:

$$\delta_r = \delta_z = \frac{3}{8}\delta_\Sigma, \ \delta_L = \frac{1}{4}\delta_\Sigma,$$

where $\delta_\Sigma = 2\delta_0 + \xi'$ denotes the summary decrement.

The optimum longitudinal decrement here occurs to be by 1.5 times less than the transverse ones, whereas in the case, where the ultimate value of betatron function is independent of the transverse emittance magnitude, the optimum values of decrements are equal each other $\delta_r = \delta_z = \delta_L = \frac{1}{3}\delta_\Sigma$.

With all substitutions made the ultimate value of normalized 6-dimensional emittance, defined as $\epsilon^{(6)}_{ult,N} = \frac{\beta^3\gamma^3}{pv}\epsilon_{r,ult}\epsilon_{z,ult}\epsilon_{L,ult}$, is found equal:

$$\epsilon^{(6)}_{ult,N} = \frac{16}{27}\left(\frac{R_t}{eH_m}\right)^2\frac{m^4c^4(Z+1)^3}{M^2L_i}\left(\frac{L_c}{L_i}\right)^3\times$$
$$\frac{\beta^6(\gamma^2+1)}{\left(1+\frac{1}{2}\frac{\xi'}{\delta_0}\right)^4}\sqrt{\bar{\lambda}E\left[\frac{1}{e\varepsilon_0}+\frac{1}{\xi_0}\left(1-\gamma^2\frac{\psi}{R}\right)\right]}. \quad (12)$$

At low enough particle momenta the sum $1+\frac{1}{2}\frac{\xi'}{\delta_0}$ is equal to $\beta^2\left(1+\frac{2}{L_i}\right)$. This compensates a high degree dependence on particle velocity in (12), so that the energy dependent factor in reality appears to be $\sim\frac{(\gamma^2+1)\sqrt{\gamma}}{\beta^2}$. Near its minimum, at $pc \cong 100 MeV/c$, the value of $\epsilon^{(6)}_{eq,N}$ is estimated as

$$\epsilon^{(6)}_{ult,N} \cong 1.4\cdot 10^{-5}\left(\frac{R_t}{H_m}\right)^2\sqrt{\frac{\bar{\lambda}}{e\varepsilon_{\rm eff}}}\ \mathrm{cm}^3$$

with H_m to be taken in *Tesla*, $\bar{\lambda}$ – in *cm* and $e\varepsilon_{\rm eff}$ – in MeV/cm. Denoted with $\frac{1}{e\varepsilon_{\rm eff}}$ here is an expression from square brackets in (12),

$$\frac{1}{e\varepsilon_{\rm eff}} = \frac{1}{e\varepsilon_0} + \frac{1}{\xi_0}\left(1-\gamma^2\frac{\psi}{R}\right).$$

Second term in the sum may be neglected in most practical cases. Its contribution to $\epsilon^{(6)}_{ult,N}$ at $pc \sim 100$ MeV is about equal to 20% when the acceleration rate is ~ 1 MeV/cm.

The numerical result strongly depends on a considered reliable ratio of rod radius to the r.m.s. size of a beam. For $R_t \cong 2.5$ ($r_0 \cong 3.5\sigma_\perp$) by $H_m \cong 10$, $\bar{\lambda} \cong 1.6$ ($\lambda \cong 10$ *cm*, tg $\varphi_a \cong 1$) and $e\varepsilon_0 \cong 1$, the ultimate value of 6-dimensional normalized emittance $\epsilon^{(6)}_{ult,N}$ is got equal to $\sim 1.2\cdot 10^{-6}$ cm^3.

At particle momentum 200 MeV/c and with more moderate accelerator parameters – $\bar{\lambda} \cong 5$ cm and $e\varepsilon_{\rm eff} \cong 0.5$ MeV/cm – this value is: $\epsilon^{(6)}_{ult,N} \sim 4.0\cdot 10^{-6}$ cm^3

Now we are to define, how close can we approach to above values.

4 Beam line for 3-d cooling

The first problem for such a beam line consists in organizing the proper exchange of cooling rates in a real scheme. A creation of transverse gradient of electron density in conductor without a disturbance of conductivity homogeneity looks rather problematic, while with no gradient of n_e the value of $\psi\eta^* = \frac{\psi}{R} \cong \frac{1}{\nu^2}$ can satisfy the condition of efficient transfer of decrements only by a small value of ν, inconsistent with small betatron function. Really, with $\nu^2 = 1 + kR^2 \cong \frac{H_m}{H_0}\frac{R}{r_0}$, the value of $\psi\eta^*$ is of the order of 0.1 by $H_0 \sim H_m$ and R being by the order larger than r_0. This is far from optimum even by $\xi' = 0$ and does not compensate the negative contribution to longitudinal decrement from ξ' at momentum below ~ 300 MeV/c.

A solution may be found in insertion of wedges with higher electron density between the bent sections of current-carrying rod. If a length of bent section is l, and a wedge length (by the bottom) – Δl, this is equivalent to creation of an effective density gradient, whose relative value is found as follows:

$$\eta_{\rm eff} = \left(\frac{1}{n_e}\frac{dn_e}{dr}\right)_{\rm eff} \cong \frac{1}{r_0}\,\frac{n_{e,w}\Delta l}{2ln_{e,li}+\Delta ln_{e,w}}.$$

A choice of material for wedges is restricted by a necessity of low nuclear number, because an effective value of Z sufficiently depends on Z_w:

$$Z_{\text{eff}} \cong Z_{li} + \eta_{\text{eff}} r_0 \left(Z_w - Z_{li} \right) \ .$$

At optimum $\psi\eta_{\text{eff}}$ by $H_0 \sim H_m$ and $p < 300 MeV/c$ this means $Z_{\text{eff}} > \frac{1}{2}(Z_w + Z_{li})$. Among pure elements the beryllium and carbon seem to be practically the only candidates.

Insertion of wedges violates the homogeneity of current-carrying rod focus. This restricts the stability region as $j\pi \le \frac{\nu l}{R} \le (j+1)\pi - 2\text{arctg}\frac{\nu \Delta l}{2R}$ and creates the betatron functions modulation, resulting in an increase of the maximum beam size. The rod radius is to be also increased, which means the field gradient reduction and the equilibrium emittance enlargement. This puts a limitation to the ratio $\Delta l/l$, whose value practically can not be taken sufficiently more than ~ 0.2, and thus the product $\psi\eta_{\text{eff}}$ still remains far from optimum.

The next step consists in amplifying the effect of wedges in a way of increase of dispersion function value at them. This is achieved when between two subsequent wedges the beam gets a parallel shift by means of two subsequent deflections trough the same angle ϕ_0 in opposite directions (see fig. 1).

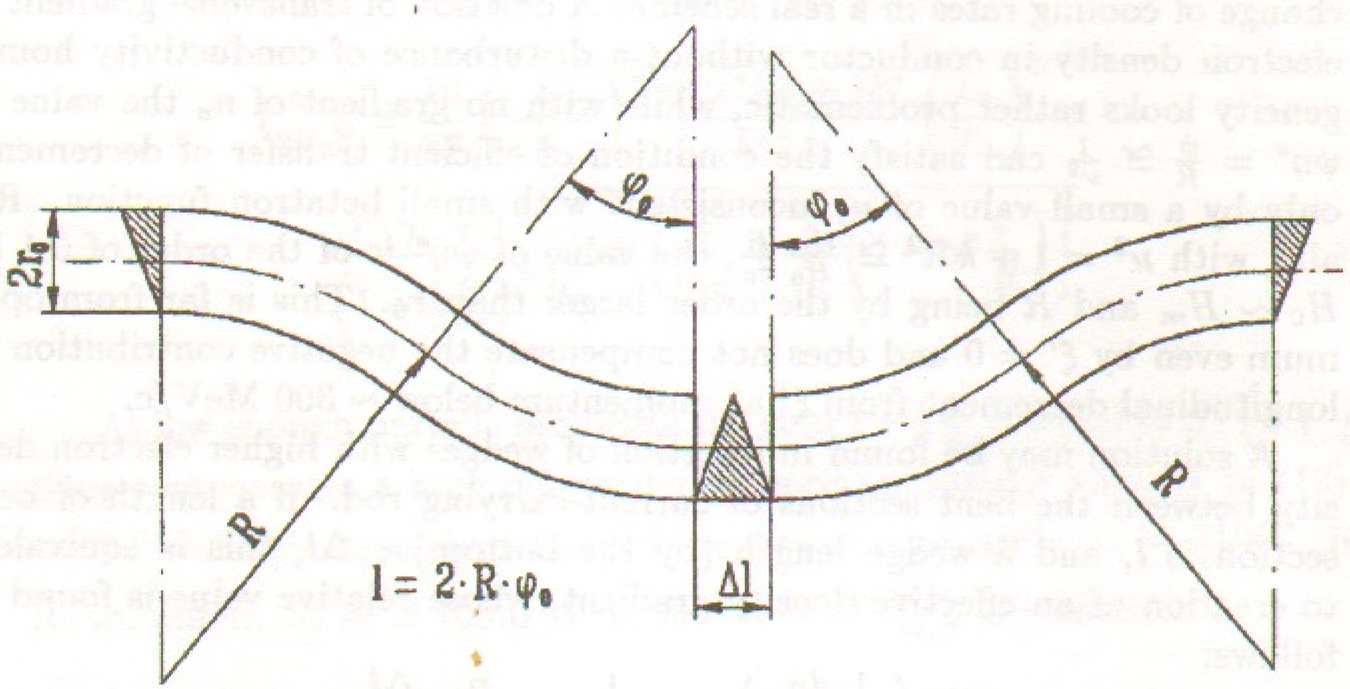

Figure 1:

In this case the dispersion function at wedges has its maximums by absolute value, equal to $\psi_w = \frac{R}{\nu^2} \mid (1 - \cos\nu\phi_0)/\cos\nu\phi_0 \mid$. When phase advance $\nu\phi_0$ is chosen between $\pi/2$ and π, the value of ψ_w appears to be several times larger than the average value in rods $\mid \psi \mid = R/\nu^2$.

With wedges made of beryllium and $\nu\phi_0 = \frac{2}{3}\pi$, the value of $\Delta l/l$ for optimum $\psi\eta_{\text{eff}}$ by $H_0 \sim H_m$ is estimated equal to ~ 0.14 at beam momentum 200 MeV/c. When $H_0 < H_m$ the value of $\Delta l/l$ for optimum $\psi\eta_{\text{eff}}$ is enlarged in proportionality with H_m/H_0.

The bending field H_0 in figure 1 is supposed to be the external one, produced by a special dipole. However, it can be produced in a rod itself, and two possibilities are presented below how to fulfill this.

The first, the evident one, consists in a use of a rod of enlarged cross section with the central trajectory of particle beam being shifted from the central line of the rod in direction out of the bend center (see fig. 2). The dipole field is proportional to a shift magnitude, $H_0 = \Delta r \frac{dH}{dr}$. To get it equal to the maximum focusing field H_m the shift is to be equal to a half of the rod radius, which in this case has to be two times larger than r_0 in figure 1. Simultaneously two times higher has to be the field at rod surface and four times – the current. Besides that, the skin-depth has also to be enlarged by two times, which means four times more current pulse duration.

The current is higher by ~ 3 times only and pulse duration conserves its initial value when the enlarged rod has an elliptic cross section with the big half-axle (directed along R) equal to $\sim 2r_0$ and the small one, defining the necessary skin-depth value, – to $\sim r_0$. A demerit here (not the crucial one!) consists in the different magnitudes of radial and axial field gradients, that is, as well, of k, ν and phase advance $\nu\phi_0$. This narrows a choice of the last value, and aggravates an asymmetry between the radial and axial cooling.

Second option is presented in fig. 3. The rod of enlarged radius here is composed of two insulated from each other conductors, supplied with currents of opposite direction. The first conductor is a lithium rod of radius r_0, placed inside the enlarged rod and shifted from its central line in direction to the bend center. The residual part of enlarged rod represents the second conductor, producing inside the first one a homogeneous dipole field of defined above value.

To get equal values of dipole and focusing fields by the same current densities in both conductors, the current through the inner rod has to be by three times less than that through the outer one. Meanwhile, it is important to supply the conductors in series, and to achieve this the inner rod is connected

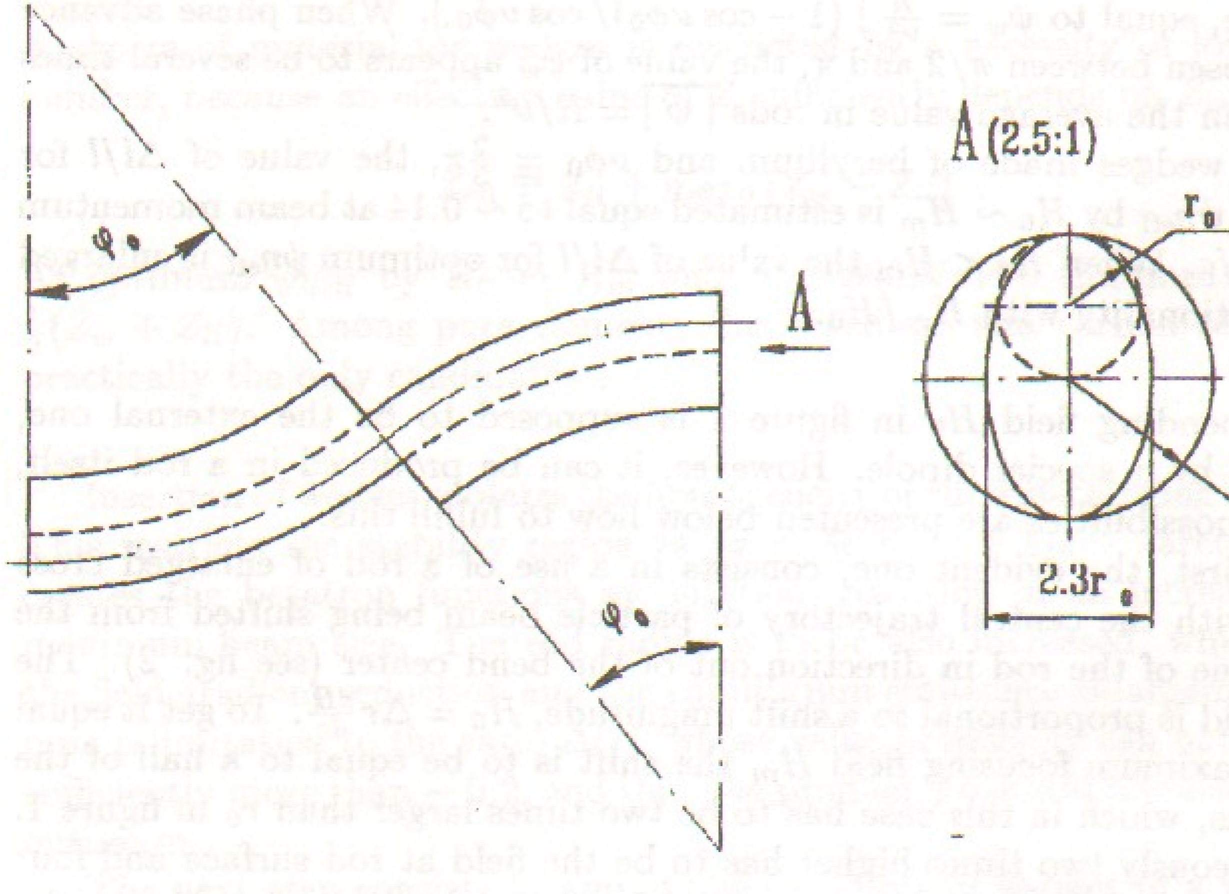

Figure 2:

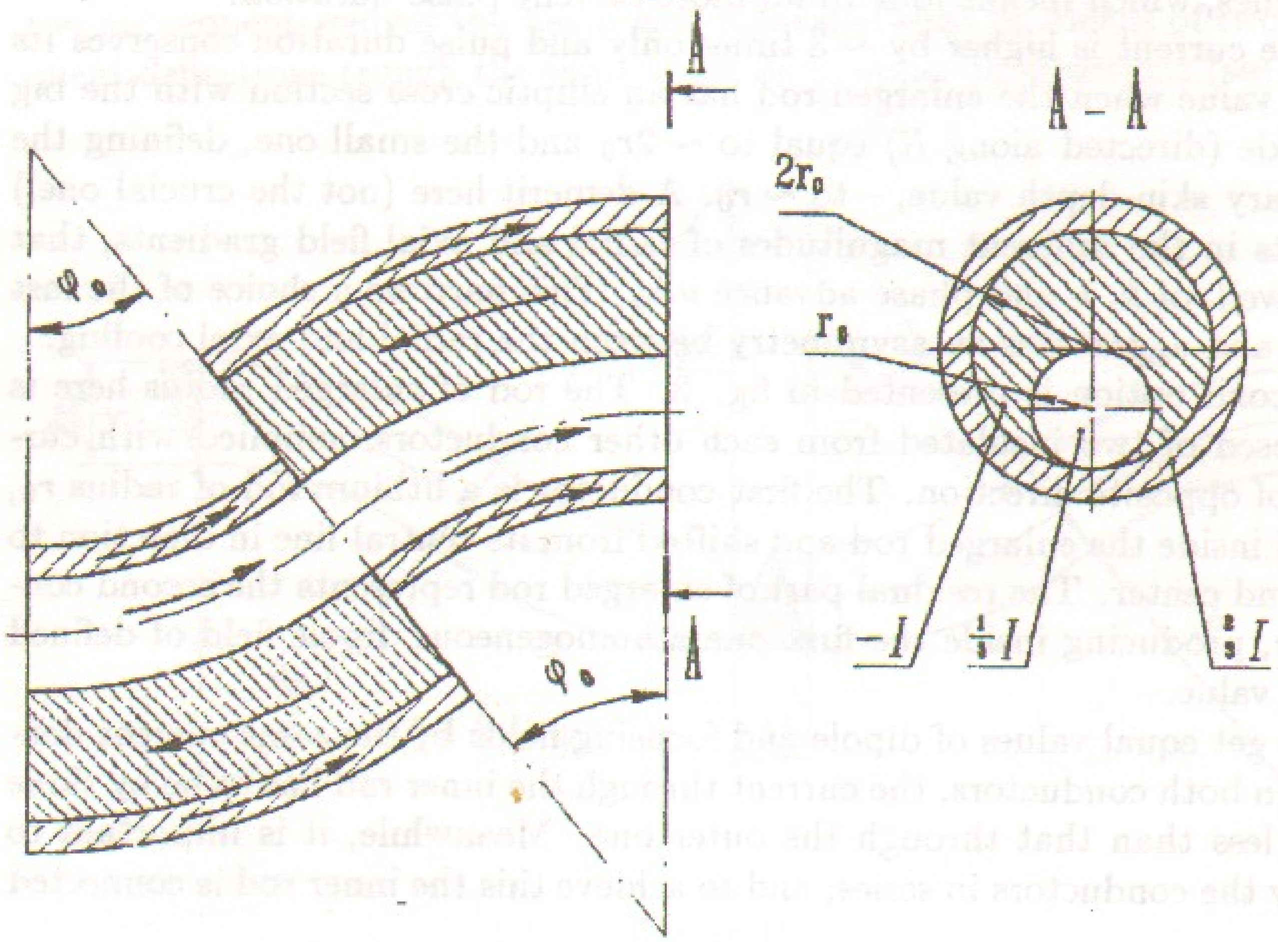

Figure 3:

to a cylindrical envelope, which shunts two thirds of the current. Skin-depth in this case is also defined by value of r_0.

A choice between two above options is defined as by technology problems as by geometry aspect. The first one, though comparatively simple in technology, does not look adequate the problem for sufficiently large beam emittance so that the diameter of conductor is comparable with the length, while in the second the current commutation problem is a complicated one [5].

5 Some results of cooling calculation and simulation

Figure 4 presents the cooling of 200 MeV/c muons in a beam line, composed of degrader units (see fig. 1), separated with linear accelerator sections, compensating the mean lost energy. The beam focusing by acceleration is considered to be the ideal one: the betatron phase advance between two degrader sections is equal to π for all particles, and the paraxial approach is valid for particle motion in accelerator.

The curves 1 and 2 show accordingly the radial and axial normalized emittances in cm·MeV/c and 3 – the longitudinal one in cm·MeV versus the number of degrader-accelerator cells. The thin lines present the numerical solution of equations (11), while the thick ones – the result of simulation with Moliere distribution used for probability of the Coulomb scattering angle and Vavilov distribution – for that of the energy loss.

Dashed line shows the ratio R_t of rod radius to the maximum r.m.s. beam, got from the numerical solution, while the dashed squares at figure bottom show the particles (from 100 initial) found lost by the simulation.

Close to constant value of $R_t \sim 2.5$, is provided through a gradual reducing of rod radius with cooling – by 2% per cell. The initial value of r_0 is ~ 0.8 cm, the final – ~ 0.35 cm. Simultaneously with rod radius, its length is also decreased in proportionality with $\sqrt{r_0}$ to keep constant the phase advance value $\nu\phi_0 = \frac{2}{3}\pi$. The wedge length is kept equal to 0.12 of lithium rod length, $\Delta l/l = 0.12$. The energy loss per cell is about $30 MeV$ in the beginning and about 20 MeV in the end. Both the dipole and maximum focusing fields are equal to 10 T.

The acceleration rate $e\varepsilon_0$ is taken equal to 0.5 MeV/cm, and $\bar{\lambda}$ to 5 cm. The initial r.m.s. momentum spread is ±4.5% and longitudinal coordinate – ±1 cm.

Figure 4:

The magnitude of 6-dimensional equilibrium emittance got in the end of 40-cells cooling is equal to $\sim 4.5 \cdot 10^{-6}\,\mathrm{cm}^3$ in a good accordance with estimation (12).

The final radial emittance appears to be by more than two times larger than the axial one, which proves that more than half of radial decrement is transferred to the longitudinal direction. The initial longitudinal emittance is close to the equilibrium value and thus only slightly decreases with cooling.

The effective cooling rate for radial emittance may be characterized by the square of rod radius reduction, that is by a value of $\sim 4\%$ per cell. Comparing this with the radial decrement one needs to notice, that emittance reduction by cooling is defined as $\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial s} \cong -\delta_r\left(1 - \frac{\epsilon_{eq}}{\epsilon}\right)$, which becomes sufficiently less in absolute value than $-\delta_r$ when emittance is close to the equilibrium value.

To avoid a significant difference in equilibrium values of radial and axial emittances by a close to optimum transfer of decrement to the longitudinal direction, the degrader units can be turned around the longitudinal axis through 90° several times over length of cooling. Figure 5 shows the result of such a procedure, performed after each eight cells. Close each other magnitudes of ϵ_r and ϵ_z are got practically without a loss in the final phase density.

6 Reprise

Presented in fig. 6 for a comparison, there is a result of cooling calculation by the same fields, initial emittances, accelerator parameters and R_t value, as in fig. 4, but without the wedges. Both transverse emittances in this case are close in magnitude to the axial one in fig. 4, but the longitudinal emittance is fast growing with length of cooling system. By a finite number of cells considered, there is not a big loss in 6-dimensional phase density. An increase in the normalized 6-d emittance, as compared to that in fig. 4, is described by a factor of $\sim 1.5 - 2$. However, the longitudinal emittance growth by ~ 3 times can hardly be considered acceptable with taken into account the finite size of accelerator separatrix and chromatic aberration in a lens, matching the amplitude functions in slowing medium and in accelerator.

An increase in transverse beam emittance, caused by chromatic aberration, is described as:

$$\Delta\epsilon_\perp / \epsilon_\perp \cong \frac{1}{2}\frac{\beta_{acc}}{\beta_{sl}}\frac{\langle \Delta E^2 \rangle}{(pv)^2}.$$

As far as caused by cooling reduction of transverse emittance is estimated as

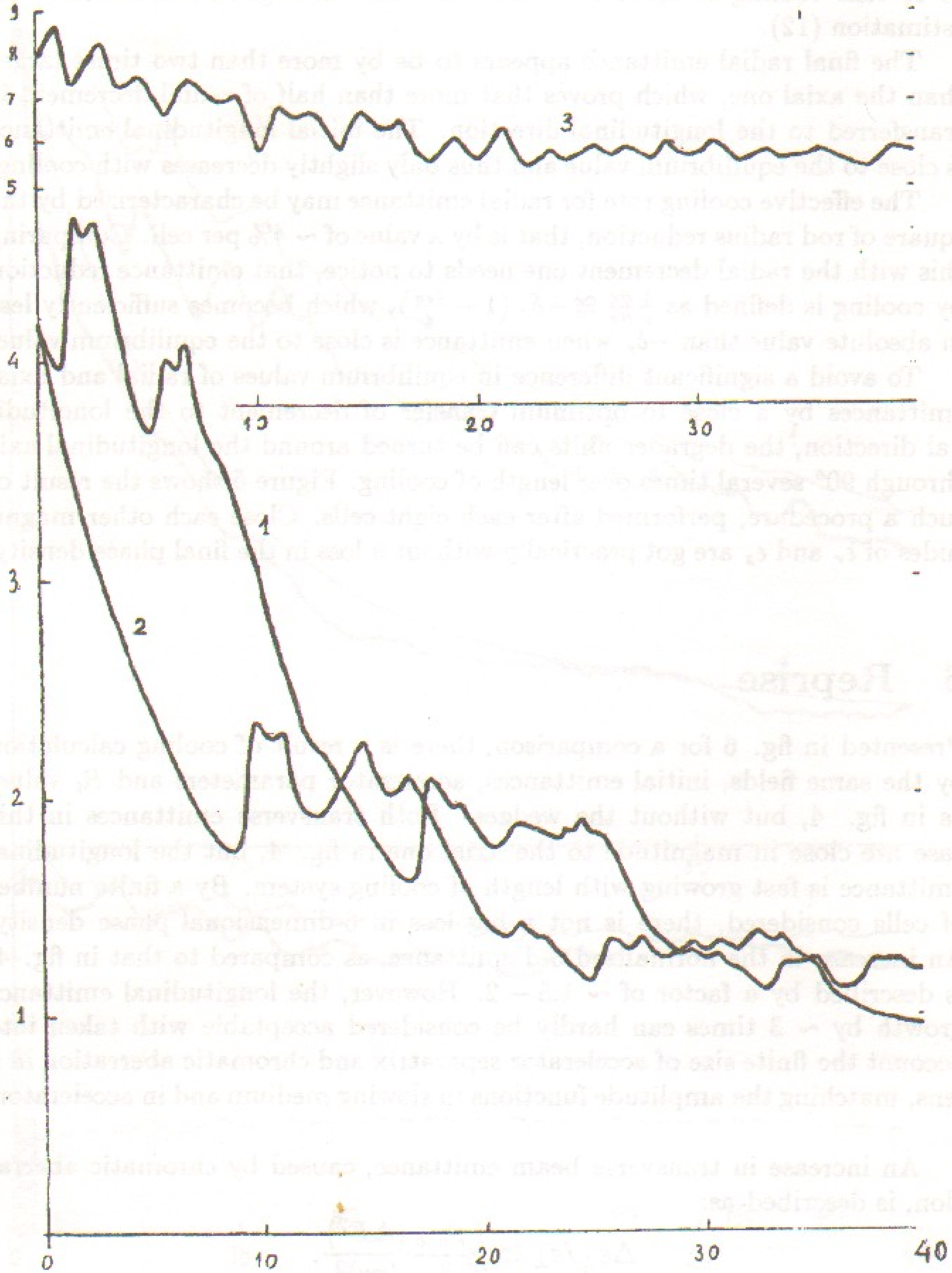

Figure 5:

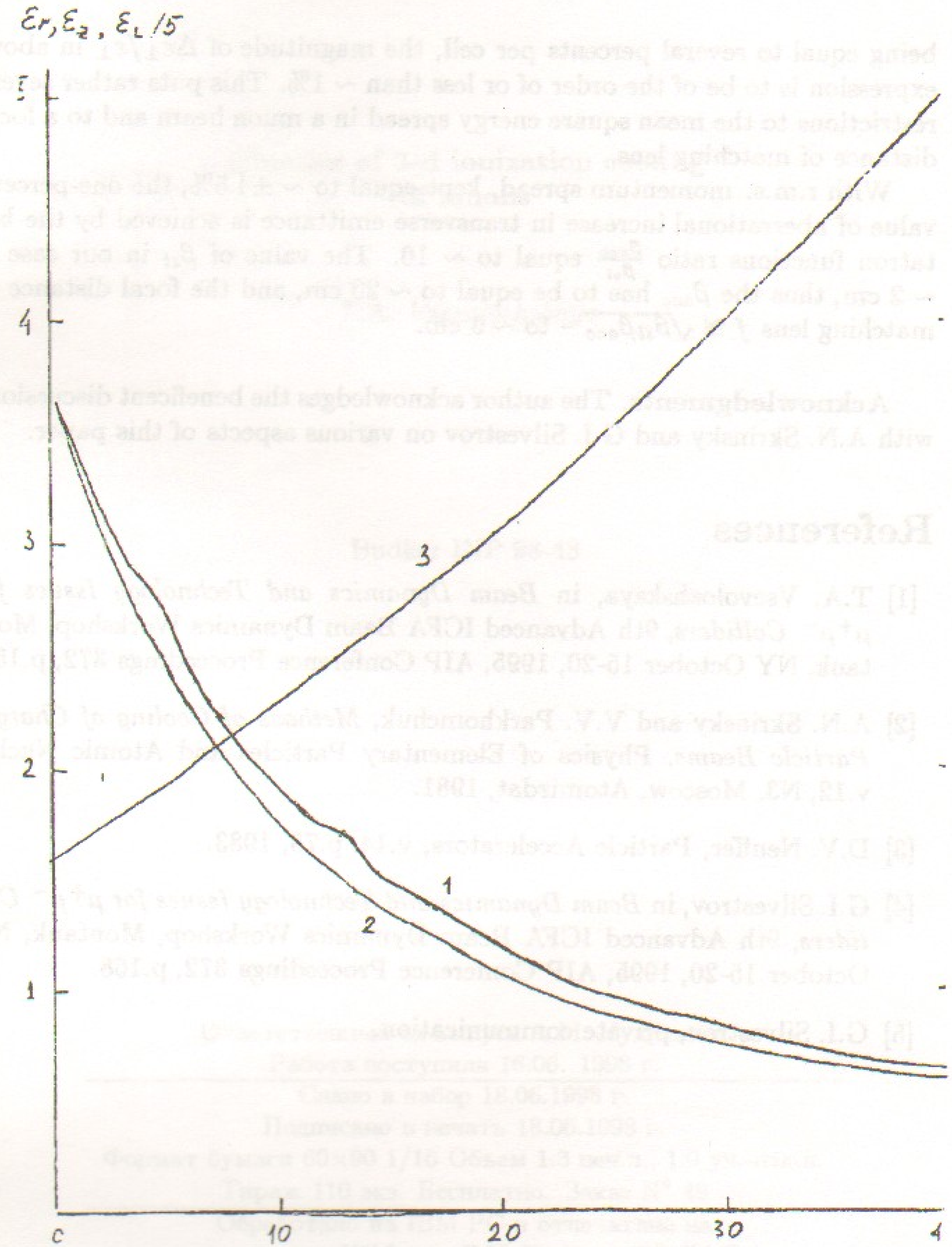

Figure 6:

being equal to several percents per cell, the magnitude of $\Delta\varepsilon_\perp/\varepsilon_\perp$ in above expression is to be of the order of or less than $\sim 1\%$. This puts rather severe restrictions to the mean square energy spread in a muon beam and to a focal distance of matching lens.

With r.m.s. momentum spread, kept equal to $\sim \pm 4.5\%$, the one-percent value of aberrational increase in transverse emittance is achieved by the betatron functions ratio $\frac{\beta_{acc}}{\beta_{sl}}$ equal to ~ 10. The value of β_{sl} in our case is ~ 2 cm, thus the β_{acc} has to be equal to ~ 20 cm, and the focal distance of matching lens $f \cong \sqrt{\beta_{sl}\beta_{acc}}$ – to ~ 6 cm.

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Kinetics of 3-d ionization cooling of muons

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