Siberian Branch of Russian Academy of Science
BUDKER INSTITUTE OF NUCLEAR PHYSICS

P.G. Silvestrov

VORTICES IN A CYLINDER: LOCALIZATION AFTER DEPINNING

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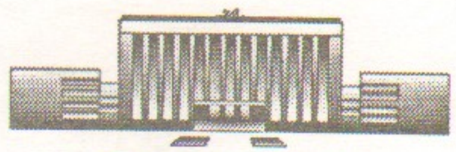

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Vortices in a cylinder: Localization after depinning

P.G. Silvestrov

Budker Institute of Nuclear Physics SB RAS
630090 Novosibirsk, Russia

Abstract

Edge effects in the depinned phase of flux lines in a thin superconducting cylindrical shell with columnar defects and electric current along the cylinder are investigated. Far from the ends of the cylinder vortices are distributed almost uniformly (delocalized). Nevertheless, near the edges these free vortices come closer together and form well resolved dense bunches. A semiclassical picture of this localization after depinning is described. For a large number of vortices their density $\rho(x)$ has square root singularity at the border of the bunch ($\rho(x)$ is semicircle in the simplest case). However, by tuning the strength of current, the various singular regimes for $\rho(x)$ may be reached. Remarkably, this singular behaviour reproduces the phase transitions discussed during the past decade within the random matrix regularization of $2d$-Gravity.

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Statistics of ensembles of flux lines in high-T_c superconductors has been a subject of numerous experimental and theoretical investigations (see for review Ref. [1]). An important modification of the superconductor is achieved by creation of artificial disorder in the form of columnar pins produced by heavy ion irradiation. The vortex distribution in the cylindrical superconductors with columnar defects and longitudinal electric current has attracted a renewed interest recently [2]. The current creates a transverse magnetic field, which attempts to wind vortices around the cylinder. For low current the vortices are trapped by the fluctuations in the density of defects and do not curl (transverse Meissner effect [3]). With the increase of current the transition to depinned phase takes place. The mapping of this transition onto the delocalization transition in $1d$ non-Hermitean quantum mechanics [2] has caused an immediate and wide interest [4]. However, as it was pointed out in Ref. [5], even after the transition to complex spectrum eigenfunctions of corresponding non-Hermitean Hamiltonian still exhibit the features of both localized and delocalized states. Physical consequences of this "localization after depinning" for vortices in the thin-shelled cylinder will be the subject of this paper. Our main prediction is illustrated by the Fig. 1. The depinned fraction of vortices is practically uniformly distributed over the surface far from the ends of cylinder. However, while approaching the ends, vortices come closer and form well resolved localized bunches. The density of vortices in this bunch at the edge coincides with the density of eigenvalues for the ensembles (Gaussian or non-Gaussian) of orthogonal random matrices. The phase transitions emerging while tuning the parameters in these random matrix models were a subject of intensive investigation in the past decade in the context of problems of $2d$-Gravity [6]. As we show in this paper, experimental investigation of vortices in cylindrical samples may open the way for the direct observation of such transitions in real low-dimensional systems.

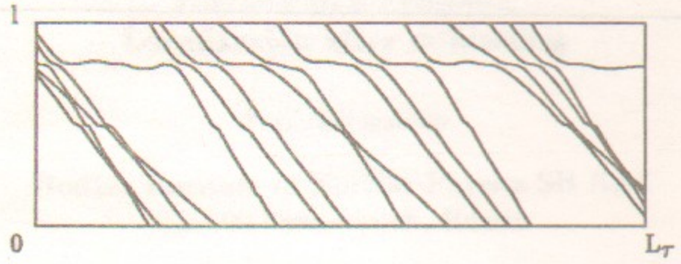

Figure 1: The vortex distribution on the cylinder ($l \equiv 0$). The Hamiltonian H has one eigenstate localized in the usual sense due to a strong local fluctuation of $V(x)$. In the typical transverse section far from the ends of the cylinder one vortex is trapped by this localized level, while others are almost uniformly distributed. However, near the ends these free vortices come closer together and form well resolved dense bunches.

The classical energy for a flux line in the cylinder may be written as an action of equivalent particle [7]

$$S = \int_0^{L_\tau} \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 - h \frac{dx}{d\tau} + V(x) \right] d\tau \ . \qquad (1)$$

The length and period of the cylinder are L_τ and l, $x + l \equiv x$. The potential $V(x)$ accounts for the interaction of vortex with the columnar defects and h is proportional to the longitudinal current. All our results are valid for random $V(x)$. However, in order to enhance the effect it is better to prepare the sample with smooth and inhomogeneous at the scale $\Delta x \sim l$ density of columnar pins. The partition function for classical vortex now takes the form of quantum evolution operator in imaginary time

$$Z_{L_\tau}(x_2, x_1) = \int \mathcal{D}x e^{-\beta S} = \langle x_2 | e^{-\beta H L_\tau} | x_1 \rangle \ , \qquad (2)$$

where $\beta = 1/T$ is the inverse temperature and we have introduced the non-Hermitean Hamiltonian

$$H = -\frac{1}{2} \left(\frac{1}{\beta} \frac{\partial}{\partial x} - h \right)^2 + V(x) \ . \qquad (3)$$

Thus the electric current in the cylinder acts like an imaginary vector potential ih. The path integral in Eq. (2) includes strings with given end points x_1

and x_2. In order to find the partition function for vortex with free ends one has to integrate over $x_{1,2}$ [2]. The evolution operator Eq. (2) may be written in the form

$$Z_{L_\tau}(x_2, x_1) = \sum_i \psi_i^R(x_2) \psi_i^L(x_1) e^{-\beta \varepsilon_i L_\tau} \ , \qquad (4)$$

where the left- and right- eigenvectors are defined via $H\psi^R = \varepsilon\psi^R$ and $H^T\psi^L = \varepsilon\psi^L$. The solutions are normalized via $\int_0^l \psi_i^L \psi_j^R dx = \delta_{ij}$. In this paper we will consider only the case $h^{-1} \ll l \ll L_\tau$. In particular this means that only the ground state contribution survives in the sum in Eq. (4).

Interacting vortices in this approach are equivalent to the interacting bosons. Moreover, we will consider only the case of strong δ-function repulsion of vortices(bosons).

Interesting for us physical information is contained in the slow harmonics of the potential $V(x)$. Therefore, in this paper we will restrict our attention on the case $dV/dx \ll V$. Also we will consider the case of strong depinning current $h^2 \gg V$.

$$\psi = \frac{\exp(-\sigma)}{\sqrt{l}} \ , \ \sigma = \beta \int_0^x (\sqrt{2(V - \varepsilon)} - h) dx' + \dots \ . \qquad (5)$$

The corrections to σ may be also easily found. The quantization condition, which allows one to find the set of complex eigenvalues ε_n, is $\sigma(l) - \sigma(0) = 2\pi n i$. It will be enough for us to consider only low lying excitations of the Hamiltonian Eq. (3). In this case

$$\varepsilon_n = -(ikT - h)^2/2 + \langle V \rangle \ , \quad k = 2\pi n/l \ , \qquad (6)$$
$$\sigma_n(x) = -ikx + \int_0^x \frac{V(x') - \langle V \rangle}{hT} dx' \ .$$

Here $\langle V \rangle = l^{-1} \int_0^l V dx$. The most exciting in this result is that $\sigma(x)$ acquires the nontrivial real part in the presence of columnar field $V(x)$ (in general $|Re\sigma| \gg 1$ if $l \gg h/V$). This means that even after the depinning transition eigenvectors of the non-Hermitean Hamiltonian Eq. (3) still remain strongly (exponentially!) localized. The localization described by the Eqs. (5,6) has almost nothing in common with usual localization at $h = 0$. To see this consider the simple example of very wide square-well potential: $V = U_0$ for $0 < x < l/2$ and $V = -U_0$ for $-l/2 < x < 0$. Instead of almost extended ground state in the absence of transverse field $\psi(h = 0) \sim \sin(2\pi x/l)$ at $-l/2 < x < 0$ one finds via the Eq. (6) the clearly localized even for $h^2 \gg U_0$ eigenvector $\psi_0^R \sim \exp(-U_0|x|/h)$ [9]. Moreover, we see from this

example that the imaginary vector potential first creates the localized states (maximum of localization is reached at $h^2 \sim V - \langle V \rangle$). However, with further increase of h the localization length grows up again. All eigenstates described by the Eq. (6) are localized near the minimum of $\sigma_0(x)$. This feature of "non-Hermitean" localization is also in sharp contrast with the usual Anderson case, where eigenstates with close energy strongly repel in the coordinate space.

The Eqs. (5,6) describe localization of eigenvectors by the long wavelength harmonics of potential $V(x)$ in strong imaginary vector potential. In the case of random disorder, however, a part of eigenfunctions (whose with the localization length $\xi < h^{-1}$) may be localized in the usual Anderson sense on the rare local fluctuations of $V(x)$ [2]. In this case, if there are only few vortices in the sample, they will be trapped by these true localized states. However, with increase of the number of fluxes, the fraction of depinned vortices emerges. Corresponding wave functions are described again by the analog of Eqs. (5,6). Just this situation is shown on the Fig. 1.

The localization described by the WKB formula Eq. (5) should have a simple classical explanation. Indeed, the action for the solution of classical equation of motion for the string connecting points x_1 and x_2 with n windings over the cylinder may be shown to have a form

$$S_{cl}^{(n)} = \varepsilon L_\tau + \int_{x_1}^{nl+x_2} (\sqrt{2(V-\varepsilon)} - h) dx \ , \qquad (7)$$

where the classical energy ε should be found from $\int_{x_1}^{nl+x_2} dx/\sqrt{2(V-\varepsilon)} = L_\tau$. Note that we use the reversed sign of the kinetic energy $\varepsilon = -\dot{x}^2/2 + V$ compared to usual classical definition. In order to find the most important classical configuration we have to determine the minimum of $S^{(n)}$ with respect to n. For $n \gg 1$ one has for $\Delta S_{cl} = S_{cl}^{(n+1)} - S_{cl}^{(n)}$

$$\Delta S_{cl} = \delta\varepsilon\,(L_\tau - L_\tau) + \int_0^l (\sqrt{2(V-\varepsilon)} - h) dx \ , \qquad (8)$$

where $\delta\varepsilon$ is the energy difference for two paths. Thus, the condition of extremum $dS^{(n)}/dn = 0$ coincides with the semi-classical quantization condition for the ground state of Hamiltonian Eq. (3). Consequently, the contribution of smallest $S_{cl}^{(n)}$ into the functional integral Eq. (2) is enough to reproduce the WKB result Eqs. (4,5) in the limit of large L_τ. The classical consideration allows also to find the typical tilt of the vortex trajectory $\langle \dot{x} \rangle = h + O(V^2/h^3)$.

The Hamiltonian Eq. (3) and its transpose are related via $H^T(h) = H(-h)$ [2]. This simple equality allows one to introduce the conserving current for left- and right- eigenvectors with the same energy $J = \psi^{L'}\psi^R - \psi^L\psi^{R'} + 2h\psi^L\psi^R$; $J' \equiv dJ/dx = 0$. For small h and random $V(x)$ in the thermodynamic limit $l \to \infty$ one has $J = 0$ for all low (real)energy states. Above some critical value $Re\varepsilon = \varepsilon_c$ the current become non-zero and the energy acquires nontrivial imaginary part. This appearance of conserving current $J \neq 0$ was usually considered as an indication of existence of delocalization transition in $1d$ non-Hermitean quantum mechanics. However, formally the existence of current shows the absence of exponential localization only for quantities bilinear in ψ^L and ψ^R. Individually ψ^L and ψ^R still may be localized. Indeed, for the WKB wave functions Eq. (5) one finds $\psi^L(x)\psi^R(x) = l^{-1}$, i.e. exponential growth of ψ^R is compensated by the same decrease of ψ^L. The probability to find vortex at the point x on the transverse slice τ of the cylinder due to the Eq. (2) has the form $\rho(x,\tau) = Z^{-1}\langle\Psi^f|e^{-\beta(L_\tau-\tau)H}|x\rangle\langle x|e^{-\beta\tau H}|\Psi^i\rangle$. For large $\tau, L_\tau - \tau \sim L_\tau$ one finds $\rho(x,\tau) \sim \psi_0^R(x)\psi_0^L(x)$. We see that the effect of localization is washed out inside the cylinder. However [2, 8], at the top of cylinder $\rho^R(x) = \rho(x, L_\tau) \sim \psi_0^R(x)$ and at the bottom $\rho^L(x) = \rho(x,0) \sim \psi_0^L(x)$ the vortex is strongly localized.

As we have told, the ensemble of vortices should be treated as the system of interacting bosons with single particle Hamiltonian Eq. (3). At least for low density of fluxes, the inter-boson interaction is equivalent to short-range repulsion $U = U_0\delta(x - x')$. For many vortices the single-particle wave functions $\psi_i^{R,L}$ and energy ε_i should be replaced by the many-particle ones $\Psi_i^{R,L}$ and the total energy E. The ground state for impenetrable bosons is found by the analogy with N-fermion solution [10]

$$\Psi^R = \prod_{i<j} \left(\prod_l sign(x_i - x_j + 2\pi l) \right) \frac{det|\psi_n^R(x_m)|}{\sqrt{N!}} \ . \qquad (9)$$

Here ψ_n^R are the eigenvectors of the single particle Hamiltonian Eq. (3) with lowest $Re\varepsilon_n$. For odd N the eigenvectors should be the usual periodic $\psi_n^R(x+l) = \psi_n^R(x)$. However, for even N one should use the antiperiodic boundary conditions $\psi_n^R(x+l) = -\psi_n^R(x)$. In both cases the total ground state energy $E = \sum \varepsilon_n$ entering the analog of the Eq. (4) is real. Also for the even and odd N and smooth $V(x)$ the $\psi_n^R(x)$ are well described by the Eqs. (5,6). Within

this approximation the wave function is further simplified

$$\Psi^R = \frac{1}{\sqrt{N!}} \prod_m \frac{e^{-\sigma_0(x_m)}}{\sqrt{l}} \prod_{i<j} 2 \left| \sin\left(\frac{\pi(x_i - x_j)}{l} \right) \right| . \qquad (10)$$

Furthermore, in the most interesting case, when the function $\exp(-\sigma_0(x_m))$ has a narrow ($\Delta x \ll l$) peak one may replace $|\sin(\pi(x_i - x_j)/l)|$ by $\pi|x_i - x_j|/l$. This means that the distribution of vortices described by the Ψ^R coincides formally with the distribution of eigenvalues of $N \times N$ Orthogonal Random Matrix Ensemble with the weight function $\exp(-Tr\sigma_0(M))$. The density of vortices $\rho(x)$ at the end of the cylinder is found after integration of the Ψ^R over the all x_i except for one. The saddle-point method (large N approximation) for calculation of such integrals was developed many years ago in Ref. [11]. In particular, in the most general case of quadratic minimum of σ_0 one has

$$\sigma_0 = \frac{\alpha(x - x_0)^2}{2hT} \; ; \; \rho = \frac{\alpha}{\pi hT} \sqrt{\frac{2hT}{\alpha} N - (x - x_0)^2} . \qquad (11)$$

Here we have shown explicitly the dependence of σ_0 on the transverse field h and temperature T. The coefficient α is determined only by the potential $V(x)$. With the increase of N the anharmonic contributions to σ_0 should be also taken into account. For example one may write (we put $x_0 = 0$)

$$\sigma_0 = \frac{1}{hT} \frac{\alpha x^2}{2} W(x/\lambda) \; , \; W(0) = 1 . \qquad (12)$$

Now α contains the information about the strength of interaction $V(x)$, while λ is the characteristic length for its variation. The vortices at the edge of the cylinder in this case also form a smooth dense bunch with square root $\rho(x)$ at the border (see Fig. 2a). The width of the bunch is $x_c \sim \sqrt{NhT/\alpha}$ and the anharmonic contributions became important starting from $N \sim N_c = \alpha\lambda^2/hT$. The new phenomena may take place if the function $\sigma_0(x)$ has more than one minimum. Some variants of a peculiar behavior of $\rho(x)$ in this case are illustrated by Figs. 2b and 2c. With increase of N in the case of two minimums the second small stable bunch of vortices is born at some value N_1. These two bunches join together at the second critical value N_2. The moment of consolidation of two bunches into one is shown on the Fig. 2b. The density of states close to the transition in the vicinity of meeting point x_0 is

$$\begin{aligned} \rho &= a(x - x_0)\sqrt{(x - x_0)^2 + \Delta[h - h_c]} \quad \text{for} \quad h < h_c , \\ \rho &= a((x - x_0)^2 + \Delta[h - h_c]/2) \quad \text{for} \quad h > h_c , \end{aligned} \qquad (13)$$

where a, Δ, x_0 and h_c vary smoothly with the change of N, T, or h. The number of vortices which may be kept in equilibrium in each well may be regulated by the external parameter h. This feature of the vortex distribution opens the way to create the metastable configurations like one shown on the Fig. 2c. For $h < h_c^*$ the vortices are confined (during exponentially long time) in the deepest well. Above h_c^* the fast decay into second well takes place. The critical configuration shown on the Fig. 2c is characterized by the novel singular behaviour of the density at the border $\rho(x) \sim (x - x_0)^{3/2}$. Just this type of critical behaviour corresponds to the continuous limit in the Random Matrix regularization of $2d$-Gravity.

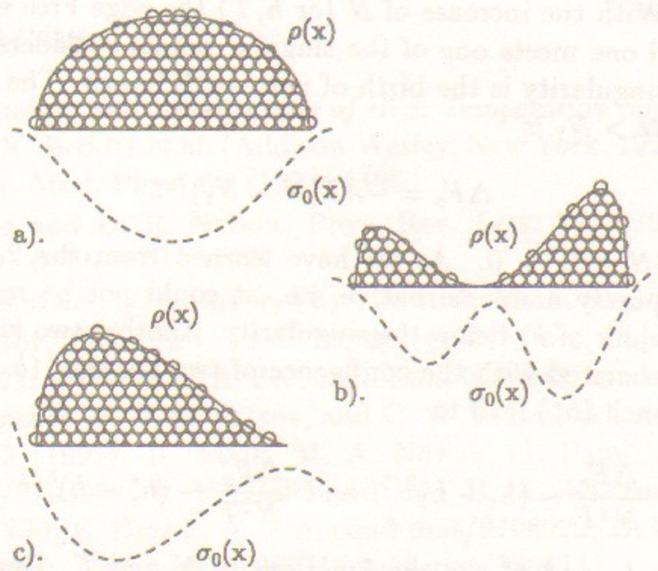

Figure 2: Examples of distribution of vortices at the end of cylinder: a). Semicircle in a single well b). Confluence of two bunches in a double well c). Opening of a decay of metastable bunch.

The catastrophic change in vortex distribution at the transitions should also change the thermodynamic characteristics of the ensemble of flux lines. After integration over the positions of the ends of vortices one finds the partition function and the Free energy

$$F = -T \ln(Z) = F_{L_r} + F_e^R + F_e^L . \qquad (14)$$

The contribution proportional to the total volume F_{L_τ} in our simple case has the form

$$F_{L_\tau} = NL_\tau \left\{ -h^2/2 + \langle V \rangle + (\pi NT/l)^2/6 + f_0 \right\} . \qquad (15)$$

Here $f_0(T)$ accounts for the short wave-length fluctuations of string and does not depend on h, $V(x)$ and N. The more interesting for us are the edge contributions F_e^R and F_e^L. For large N and the quadratic σ_0

$$F_e^R = N^2 T \left\{ \frac{3}{8} + \frac{1}{4} \ln \left(\frac{\alpha^R l^2}{4\pi^2 NhT} \right) \right\} + Ng_0 . \qquad (16)$$

Here again g_0 depends on the details of regularization of the functional integral Eq. (2). With the increase of N (or h, T) the edge Free energy changes smoothly until one meets one of the singular points considered above. The first possible singularity is the birth of new small bunch. The corresponding correction at $N > N_1$ is

$$\Delta F_e = -AT(N - N_1)^2 . \qquad (17)$$

Here $A = A(N, hT) > 0$. As we have learned from the $2d$-Gravity, this correction is purely nonperturbative, i.e. it could not be related with the analytic behaviour of F_e below the singularity. Another two kinds of singular corrections associated with the confluence of two bunches (h_c) and decay of metastable bunch (h_c^*) lead to

$$\frac{\Delta F_e}{N^2 T} \sim (h_c - h)^{3/2} \quad \text{and} \quad \frac{\Delta F_e^*}{N^2 T} \sim (h_c^* - h)^{5/2} . \qquad (18)$$

The constants h_c and h_c^* are the functions of N and T. Depending on the concrete way of realization of physical experiment one may write instead of the Eq. (18), for example, $\Delta F \sim (T_c - T)^{3/2}$ or $\Delta F \sim (N_2 - N)^{3/2}$.

In summary, the edge effects in vortex distribution in superconducting cylinders may provide us with the variety of new phenomena with clear experimental signature (see again the Figs. 1,2). Among them are the strong localization of vortices at the end of cylinder and various critical regimes for this localization available by tuning of the longitudinal current. Technically these effects arose due to the peculiar features of localization in non-Hermitean quantum mechanical Hamiltonian (3). From the pure theoretical point of view, the most exciting is the correspondence between distribution of flux lines at the end of cylinder and distribution of eigenvalues of the ensembles of random matrices. The ensemble of fluxes turns out to be the almost unique example of the system, where not only local (correlations of close

levels etc.), but also global features of random-matrix spectrum are of 100% importance. For example, the phase transitions in non-Gaussian Matrix ensembles have been a subject of enormous activity in last 10 years within the context of $2d$-Gravity [6]. However, to the best of my knowledge, in this paper the first proposal is presented of a real physical experiment where such a singular behaviour may be observed.

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П.Г. Сильвестров

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