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ASHTEKAR CONSTRAINT SURFACE AS PROJECTION OF HILBERT-PALATINI ONE

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Ashtekar constraint surface as projection of Hilbert–Palatini one

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Abstract

The Hilbert-Palatini (HP) Lagrangian of general relativity being written in terms of selfdual and antiselfdual variables contains Ashtekar Lagrangian (which governs the dynamics of the selfdual sector of the theory on condition that the dynamics of antiselfdual sector is not fixed). We show that nonequivalence of the Ashtekar and HP quantum theories is due to the specific form (of the "loose relation" type) of constraints which relate self- and antiselfdual variables so that the procedure of (canonical) quantisation of such the theory is noncommutative with the procedure of excluding antiselfdual variables.
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1 Introduction

Ashtekar formulation of general relativity (GR) [1] was shown to follow from Hilbert-Palatini (HP) tetrad-connection Lagrangian upon partial use of classical equations of motion together with certain gauge fixing [2]. Alternatively, one can first introduce Ashtekar Lagrangian and show that it results in HP Lagrangian upon using equations of motion (see, e.g., reviews [3] and, for more detail, [4]). However it may be, equations of motion for selfdual sector from HP and Ashtekar Lagrangians coincide [5].

Another question is that of correspondence between the two theories, HP and Ashtekar ones, on quantum level. Since the two Lagrangians differ by more than pure divergence, these theories may be not equivalent in the framework of canonical quantisation. Of course, one can present examples in which the two Lagrangians which do not coincide up to the total divergence nevertheless provide equivalent theories (the most evident are examples of locally trivial theories such as 2+1 dimensional gravity). Therefore it is interesting to study in what cases such the equivalence takes place and when it does not.

In this note we show that canonically quantised HP and Ashtekar theories are inequivalent. To see this it is most convenient to represent

HP theory as a "sum" of two copies of Ashtekar theory: selfdual and antiselfdual ones, and study commutators (Dirac brackets) between field variables. The self- and antiselfdual sectors are not independent: these are related by some (class II) constraints (which for the pseudoEuclidean metric signature become well-known reality conditions, but survive also in the purely real Euclidean case). Because of the specific form of class II constraints (which are of the type of "loose relation" for antiselfdual variables to be excluded) the commutators of the same variables are different depending on what theory is quantised, the total HP or only it's Ashtekar subset. It should be mentioned that this fact is not connected with the infiniteness of the number of the degrees of freedom of the theory and may take place even for rather trivial systems, as it is shown in the simple example below.

In particular, the quite natural feature of standard GR that the quantum states cannot describe transition through unphysical points with degenerate metric shows up in quantum HP theory as singularity of quantum commutators of field operators at these points, whereas Ashtekar theory is completely nonsingular at these points and allows such the transitions.

2 A simple example

To describe the situation, let us consider the theory whose phase space is coordinatised by N canonical pairs (p, q) and N primed canonical pairs (p', q') subject to $2N_2$ class II constraints of the form

$$\Theta_A(p, q, p', q') \stackrel{\text{def}}{=} \theta_A(p, q) - \theta_A(p', q') = 0. \qquad (1)$$

Besides, there are $N_1 = N - N_2$ pairs of class I constraints

$$\Phi_i(p, q) = 0, \quad \Phi_i(p', q') = 0. \qquad (2)$$

The usual way of quantising the theory with class I constraints is to impose these constraints on the Hilbert space of states. As for the class II constraints, these cannot be imposed on states due to their noncommutativity; the only consistent way is to take them into

account in operator sense by projecting the fields appearing in Poisson brackets orthogonally to the surface of class II constraints. Thus we get Dirac brackets as consistent choice for commutators when performing canonical quantisation:

$$\{f, g\}_{\mathrm{D}} \stackrel{\text{def}}{=} \{f, g\} - \{f, \Theta_A\}(\Delta^{-1})^{AB}\{\Theta_B, g\}, \qquad (3)$$

where $\{f, g\}$ is Poisson bracket and Δ^{-1} is matrix inversed to that of the Poisson brackets of the class II constraints:

$$(\Delta^{-1})^{AB}\{\Theta_B, \Theta_C\} = \delta^A_C. \qquad (4)$$

Now, if N_2, half the number of class II constraints Θ_A coincides with the number N of the canonical pairs (p', q') to be excluded then the matrix of the Poisson brackets of Θ_A is easily invertible:

$$\{\Theta, \Theta\}^{-1} = \frac{1}{2}\{\theta, \theta\}^{-1}_{\eta} = \frac{1}{2}\{\eta, \eta\}_{\theta}. \qquad (5)$$

Here $\eta = p$ or q and index at brackets means the set of variables treated as canonical ones w.r.t. which the brackets are taken. Then simple algebra shows that commutators coincide with Poisson brackets for the kinetic term $2p\dot{q}$ which should arise in the Lagrangian upon excluding primed variables (in suggestion that class II constraints are nondegenerate and give $p' = p, q' = q$, at least locally):

$$\begin{aligned} L &= 2p\dot{q} - H \\ \{f, g\}_D &= \frac{1}{2}\{f, g\}_{p,q}. \end{aligned} \qquad (6)$$

Thus, if the number of (irreducible) class II constraints equals precisely the number of canonical variables to be excluded, the latter can be excluded on quantum level.

If the number of class II constraints is less than the number of canonical variables to be excluded, we have a freedom in finding the latter; on the one hand, due to the symmetry generated by the class I constraints the resulting Lagrangian is invariant w.r.t. this freedom.

On the other hand, the matrix of the Poisson brackets cannot be inverted in a simple way as in (5) and Dirac brackets may be much more complex as compared to the Poisson ones. For example, consider the following Lagrangian:

$$L = p_1\dot{q}_1 + p_2\dot{q}_2 + p_1'\dot{q}_1' + p_2'\dot{q}_2' + \lambda\Phi(p,q) + \lambda'\Phi(p',q') + \mu_1\Theta_1 + \mu_2\Theta_2 \quad (7)$$

where λ, λ' are Lagrange multipliers at the first class constraints

$$\Phi(p,q) = p_1 - p_2, \quad \Phi(p',q') = p_1' - p_2' \quad (8)$$

and μ_1, μ_2 are the multipliers at the second class constraints

$$\Theta_1 = p_1 + p_2 - p_1' - p_2', \quad \Theta_2 = q_1 + q_2 - q_1' - q_2'. \quad (9)$$

These class II constraints provide the following Dirac brackets between the coordinates and momenta:

$$\{\begin{pmatrix} p_1 \\ p_2 \\ p_1' \\ p_2' \end{pmatrix}, (q_1, q_2, q_1', q_2')\}_D = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}. \quad (10)$$

(The brackets of the coordinate-coordinate and momentum-momentum type are zero).

Also we can try first to exclude p', q' here. The number of constraints Θ_1, Θ_2 is unsufficient to uniquely express p', q' in terms of p, q. However, the class I constraint $\Phi(p',q')$ generates the symmetry w.r.t. the transformation

$$q_1' \to q_1' - \epsilon, \quad q_2' \to q_2' + \epsilon, \quad (11)$$

which makes it possible to exclude primed variables from the Lagrangian to give

$$L = 2p_1\dot{q}_1 + 2p_2\dot{q}_2 + \lambda(p_1 - p_2). \quad (12)$$

The commutators are defined here, up to an overall factor, by the Poisson brackets

$$\{\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, (q_1, q_2)\}_D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \quad (13)$$

This looks simpler than the corresponding upper lefthand 2×2 block in (10).

Qualitatively, we can say that looseness of the constraints Θ_1, Θ_2 in the latter example admits nontrivial additional degree of freedom (11) which modifies the commutators.

3 Ashtekar formalism

Finally, let us turn to the central point of interest, namely, Ashtekar formalism considered as reduced HP theory. Consider this for both cases, namely those of Euclidean and pseudoEuclidean metric signature (these cases will differ by minor modifications). The formalism itself is now well-elaborated (see, e.g., the reviews [3], [4]), and here we only define our notations. The original HP action

$$S = \frac{1}{8}\int d^4x\, \epsilon_{abcd}\epsilon^{\mu\nu\lambda\rho} e^a_\mu e^b_\nu [\mathcal{D}_\lambda, \mathcal{D}_\rho]^{cd}, \quad (14)$$

where $\mathcal{D}_\lambda = \partial_\lambda + \omega_\lambda$ (in fundamental representation) is covariant derivative, and $\omega^{ab}_\mu = -\omega^{ba}_\mu$ is element of $so(3,1)$, Lie algebra of $SO(3,1)$ group in the pseudoEuclidean case or an element of $so(4)$, Lie algebra of $SO(4)$ in the Euclidean case. Raising and lowering indices is performed with the help of metric $\eta_{ab} = \mathrm{diag}(\pm 1, 1, 1, 1)$, while $\epsilon^{0123} = +1$. $\alpha, \beta, \ldots = 1,2,3$ or $\mu, \nu, \ldots = 0,1,2,3$ are coordinate indices and $a, b, \ldots = 0,1,2,3$ or $i, j, \ldots = 1,2,3$ are local ones. The canonical variables are ω_α (coordinates) and conjugate momenta π^α:

$$\pi^\alpha_{ab} = \frac{1}{2}\epsilon_{abcd}\epsilon^{\alpha\beta\gamma} e^c_\beta e^d_\gamma. \quad (15)$$

The antisymmetric tensor fields A_{ab} are split into selfdual ${}^+A$ and antiselfdual ${}^-A$ parts, which in pseudoEuclidean case take the form

$$A = {}^+A + {}^-A, \quad {}^\pm A = \frac{1}{2}(A \pm i\frac{1}{2}\epsilon^{ab}{}_{cd}A^{cd}). \quad (16)$$

Each tensor part is then embedded into 3D vector space (complex in pseudoEuclidean case) by expanding over basis of (anti-)selfdual matrices

$$ {}^{\pm}\Sigma^k_{ab} = \pm i(\delta^k_a\delta^0_b - \delta^k_b\delta^0_a) + \epsilon_{kab}, \tag{17} $$

so that

$$ {}^{\pm}A^{ab} = {}^{\pm}A^k\,{}^{\pm}\Sigma^{ab}_k/2 \stackrel{\rm def}{=} {}^{\pm}\vec{A}\cdot{}^{\pm}\vec{\Sigma}^{ab}/2 \tag{18} $$

In particular, 3-vector images of (anti-)selfdual constituents of area bivector π^α in terms of tetrad take the form

$$ {}^{\pm}\vec{\pi}^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma}(\mp i\vec{e}_\beta\times\vec{e}_\gamma - e^0_\beta\vec{e}_\gamma + e^0_\gamma\vec{e}_\beta). \tag{19} $$

In the Euclidean case these definitions are modified by replacing $i = \sqrt{-1}$ by 1 so that everything becomes real; besides, we change overall sign of ${}^{\pm}\Sigma^a_{kb}$ in order that these would obey algebra of Pauli matrices times i in both cases. The (anti-)selfdual projections of area bivector onto the (now real) 3D vector space in terms of tetrad then read

$$ {}^{\pm}\vec{\pi}^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma}(\mp\vec{e}_\beta\times\vec{e}_\gamma - e^0_\beta\vec{e}_\gamma + e^0_\gamma\vec{e}_\beta). \tag{20} $$

The phase space of HP theory can be coordinatised by 9 canonical pairs $({}^+\vec{\pi}^\alpha, {}^+\vec{\omega}_\alpha)$ and 9 canonical pairs $({}^-\vec{\pi}^\alpha, {}^-\vec{\omega}_\alpha)$. Each sector of variables, selfdual or antiselfdual ones, is subject to the same set of class I constraints, $\Phi_i({}^+\pi, {}^+\omega)$ and $\Phi_i({}^-\pi, {}^-\omega)$ where 7 functions $\Phi_i(\pi,\omega)$ take the form

$$ \mathcal{D}_\alpha\vec{\pi}^\alpha, \quad \epsilon_{\alpha\beta\gamma}\vec{\pi}^\beta\cdot\vec{R}^\gamma, \quad \epsilon_{\alpha\beta\gamma}\vec{\pi}^\alpha\times\vec{\pi}^\beta\cdot\vec{R}^\gamma. \tag{21} $$

(Gaussian, diffeomorphism and Hamiltonian constraints, correspondingly). Here $\mathcal{D}_\alpha(\cdot) = \partial_\alpha(\cdot) - \vec{\omega}_\alpha\times(\cdot)$, $2\vec{R}^\alpha = -\epsilon^{\alpha\beta\gamma}[\mathcal{D}_\beta, \mathcal{D}_\gamma]$, and $(\vec{\pi}^\alpha, \vec{\omega}_\alpha)$ are $({}^+\vec{\pi}^\alpha, {}^+\vec{\omega}_\alpha)$ or $({}^-\vec{\pi}^\alpha, {}^-\vec{\omega}_\alpha)$. Besides, there are class II constraints ensuring the tetrad form (15) of area tensors π^α_{ab},

$$ {}^+\vec{\pi}^\alpha\cdot{}^+\vec{\pi}^\beta - {}^-\vec{\pi}^\alpha\cdot{}^-\vec{\pi}^\beta = 0, \tag{22} $$

and those following by differentiating these in time,

$$ {}^+\vec{\pi}^\gamma\cdot{}^+\vec{\pi}^{(\alpha}\times{}^+\mathcal{D}_\gamma\,{}^+\vec{\pi}^{\beta)} - {}^-\vec{\pi}^\gamma\cdot{}^-\vec{\pi}^{(\alpha}\times{}^-\mathcal{D}_\gamma\,{}^-\vec{\pi}^{\beta)} = 0. \tag{23} $$

The $(\alpha\ldots\beta)$ means the sum of objects with indices $\alpha\ldots\beta$ and $\beta\ldots\alpha$. The (22) and (23) relate selfdual and antiselfdual sectors. Of course, the part of class I constraints is then the consequence of others. For example, it is sufficient to set diffeomorphism and Hamiltonian constraints in only one of the two sectors, selfdual or antiselfdual one, in order that these would hold modulo other constraints in also another sector. This fact is important for establishing the correct number (two) of the degrees of freedom of gravity system (but it is unimportant in our analysis).

Thus we arrive at the situation described in section 2 of this paper where now $(p,q) = ({}^+\pi, {}^+\omega)$, $(p', q') = ({}^-\pi, {}^-\omega)$, the class II constraints (1) take the form (22) and (23) and class I constraints are of the type (21). In the simple model given in that section the class I constraints (8) play the role of Gaussian ones in the problem at hand, while the class II constraints there (9) resemble the recent (22) and (23) (especially this is seen if one linearizes the latter in the vicinity of the flat background). Therefore the structure of the Dirac brackets is similar to that of the simple model; in particular

$$ \{{}^+\pi^\alpha_i, {}^+\omega^k_\beta\}_{\rm D} = \frac{3}{4}\delta^\alpha_\beta\delta^k_i - \frac{1}{4}g_{\beta\gamma}\,{}^+\pi^\gamma_i\frac{1}{\det\|g_{\alpha\beta}\|}\,{}^+\pi^{\alpha k} \tag{24} $$

and

$$ \{{}^+\pi^\alpha_i, {}^-\omega^k_\beta\}_{\rm D} = \frac{1}{4}\delta^\alpha_\beta\,{}^+\pi^\gamma_i\frac{1}{\det\|g_{\alpha\beta}\|}g_{\gamma\epsilon}\,{}^-\pi^{\epsilon k} + \frac{1}{4}g_{\beta\gamma}\,{}^+\pi^\gamma_i\frac{1}{\det\|g_{\alpha\beta}\|}\,{}^-\pi^{\alpha k}. \tag{25} $$

If, however, we first exclude antiselfdual variables from the HP Lagrangian using the class II constraints, we find much more simple expression for nonzero brackets for the rest variables:

$$ \{{}^+\pi^\alpha_i, {}^+\omega^k_\beta\}_{\rm D} = \{{}^+\pi^\alpha_i, {}^+\omega^k_\beta\} = \frac{1}{2}\delta^\alpha_\beta\delta^k_i. \tag{26} $$

Thus, nonequivalence of Ashtekar and HP formulations of gravity on quantum level has quite transparent (but not quite trivial) reason and is connected with the "loose relation" type of class II constraints which allows to express antiselfdual variables in terms of selfdual ones only up to an SO(3) rotation. In other words, Ashtekar formulation cannot be obtained from usual tetrad (or metric) general relativity on quantum level by equivalent transformations, and this can be explained by quantum fluctuations of this SO(3) rotation not taken into account in Ashtekar theory. Important is that this circumstance holds for both Euclidean and pseudoEuclidean cases. The only difference between these cases is that in pseudoEuclidean case the (anti-) selfdual parts become complex, and the class II constraints can be interpreted as some reality conditions. But the reality of the Euclidean case does not mean, as we have seen, that problems connected with class II constraints disappear. In view of this, the strategy of quantising gravity which consists in considering the Euclidean Ashtekar theory and then making some generalised Wick transform to pseudoEuclidean case [6] does not allow to avoid these problems.

The different quantisations might give different explanations of why there are no metric signature changing transitions in the real world. The fact of absence of such transitions of metric passing through unphysical points of zero determinant may be more conceivable in the usual HP theory where commutators are singular at such points. One can say heuristically that the SO(3) rotation which connects selfdual and antiselfdual parts strongly fluctuate at these points leading to infinitely large quantum uncertainty of the tetrad and/or connection there. Situation in the Ashtekar theory in the considered aspect is less evident, because this theory is completely nonsingular at the degenerate metrics. However, one can hope that the set of states sufficient to describe real world is contained in the completely defined quantum Ashtekar theory.

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