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TOWARDS POSSIBILITY OF SELF-MAINTAINED VACUUM TRAVERSIBLE WORMHOLE

Budker INP 96-87

НОВОСИБИРСК

Towards possibility of self-maintained vacuum traversible wormhole

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Abstract

We calculate renormalised vacuum expectation values of electromagnetic stress-energy tensor in the static spherically-symmetrical wormhole topology. We find that for metric tensor sufficiently slow varied with distance violation of the averaged weak energy condition takes place irrespectively of the detailed form of metric. This is a necessary condition for the electromagnetic vacuum to be able to support the wormhole geometry.

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1 Introduction

The possibility of existence of static spherically-symmetrical traversible wormhole as topology-nontrivial solution to the Einstein equations have been first studied by Morris and Thorne in 1988 [1]. They have found that the material which threads the wormhole should violate weak energy condition at the throat of the wormhole, that is, radial pressure should exceed the density. Moreover, Morris, Thorne and Yurtsever [2] have pointed out that averaged weak energy condition (i.e. that integrated over the radial direction) should also be violated. They have also suggested Casimir vacuum between conducting spherical plates as an example of such the material which is able to support the wormhole.

As the next step, it would be natural to check the possibility of existence of self-maintained vacuum wormhole threaded by vacuum fields in the wormhole geometry in the absence of any conductors. For that one must find vacuum expectation value of the stress-energy tensor as functional of geometry and solve Einstein equations with this tensor as a source. Both parts of this strategy (explicit finding stress-energy tensor and solving the equations) are very difficult problems. Calculation of vacuum expectations of the stress-energy tensor for different fields in the curved background has been considered in a number of works. Recently functional dependence of stress-energy on the metric is known only in the form of some anzats approximately valid for some class of metrics. For example, Page, Brown and Ottewill [3], [4], [5] have suggested approximate expression for stress-energy of the conformal massless scalar, spinor and vector fields in the Einstein spacetimes (where $R_{\mu\nu} = \Lambda g_{\mu\nu}$, $\Lambda = \mathrm{const}$). In some cases this approximation can rather accurately reproduce numerical results [6], [7] while in another cases it may disagree with precise values [8]. Frolov and Zel'nikov [9] have suggested another anzats for the stress-energy of conformally invariant massless fields in static spacetimes.

In the absence of exact expression for the stress-energy as functional of metric it seems to us that the most useful for our purposes would be an expansion over some parameter (which can be small, at least in principle). Consider quasiclassical expansion which is now expansion over derivatives of metric (over radial coordinate). In [10] the renormalised stress-energy tensor of electromagnetic vacuum has been calculated with the help of covariant geodesic point separation method of regularization [11]. It has been found to violate weak energy condition at the wormhole throat in the first nonvanishing order in the expansion over the derivatives of metric. This is a necessary condition of existence of self-maintained wormhole solution. Important is that this violation takes place irrespectively of the detailed form of metric. As for other fields, there may be those which oppose the maintenance of the certain wormhole metrics. Anderson, Hiscock and Samuel [12], [13] have developed a method for numerical calculation of the stress-energy tensor of a quantised scalar field with arbitrary curvature coupling and mass in a general static spherically symmetrical spacetime. Using this method, Anderson, Hiscock and Taylor [14] have found for five test wormhole geometries that both minimally or conformally coupled massive scalar field vacuum cannot maintain these geometries since it does not violate weak energy condition at the throat. Thus massive scalar field will hardly be of interest in the self-maintained wormhole aspect.

In the given note we continue studying the electromagnetic vacuum case and find that in the first nonvanishing order in the expansion over derivatives also averaged weak energy condition is violated in the wormhole topology and this violation takes place irrespectively of the detailed form of metric. Moreover, the Einstein equations formally admit some wormhole type solution, namely, that corresponding to the infinitely long wormhole (however, there is no authomatical smallness in the higher orders of expansion over derivatives in this case, and corresponding approximation well may be unacceptable).

2 Calculation

The notations for the metric functions $r(\rho)$, $\Phi(\rho)$ are read from the following expression for the line element:

$$ds^2 = r^2(\rho)(d\theta^2 + \sin^2\theta\, d\phi^2) + d\rho^2 - \exp(2\Phi)dt^2. \qquad (1)$$

We split the point in the radial distance ρ and consider the values of interest as bilocals at ρ, $\tilde{\rho}$ such that

$$\tilde{\rho} - \rho = \epsilon \to 0. \qquad (2)$$

We introduce also the variable z via

$$dz = \exp(-\Phi)d\rho. \qquad (3)$$

Local values taken at $\tilde{\rho}$, ρ are denoted by tilded and untilded letters respectively, as, e.g., $\tilde{\Phi}$ and Φ. The "physical" components are defined as

$$A_{\hat{\mu}\hat{\nu}\ldots\hat{\lambda}} \stackrel{\mathrm{def}}{=} |g_{\mu\mu}g_{\nu\nu}\ldots g_{\lambda\lambda}|^{-1/2} A_{\mu\nu\ldots\lambda} \qquad (4)$$

for any tensor A (these are notations of ref.[1]).

Standard summation over electromagnetic modes leads to the following expressions for the vacuum expectation values of the split-regularized stress-energy tensor [10]:

$$4\pi \begin{pmatrix} T_{\hat{t}\hat{t}} \\ T_{\hat{\rho}\hat{\rho}} \\ T_{\hat{\theta}\hat{\theta}} \end{pmatrix}^{\mathrm{reg}} = \sum_{l=1}^{\infty} \int_0^{\infty} \frac{2l+1}{\pi} dt \left\{ \left[\frac{\exp(\Phi - \tilde{\Phi})}{r^3 \tilde{r}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right. \right. \qquad (5)$$

$$+\frac{1}{r^2\tilde{r}^2}\begin{pmatrix}1\\-1\\1\end{pmatrix}\Bigg]\cdot l(l+1)$$

$$+\begin{pmatrix}1\\1\\0\end{pmatrix}\frac{\exp(-\Phi-\tilde{\Phi})}{r\tilde{r}}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \tilde{z}}-\frac{\partial}{\partial z}\right)\Bigg\} G(t,l;z,\tilde{z})$$

(the vacuum expectations are denoted just as components themselves: this will not lead to any confusion). The $G(t,l;z,\tilde{z})$ is half of the sum of "electric" and "magnetic" Green functions,

$$G=(G^E+G^M)/2, \tag{6}$$

obeying the equation

$$\left[-\frac{\partial^2}{\partial z^2}+t^2+l(l+1)\frac{\exp 2\Phi}{r^2}\right]G^{E,M}(t,l;z,\tilde{z})=\delta(z-\tilde{z}) \tag{7}$$

with the following boundary conditions on the conducting spherical plates:

$$G^E|_{z\in\Gamma}=0, \tag{8}$$

$$\left.\frac{\partial G^M}{\partial z}\right|_{z\in\Gamma}=0. \tag{9}$$

The surface Γ can be introduced as some auxiliary infra-red regulator and then shifted to infinity (this is a way to give a sence to summation over continuum set of energy levels in the infinite space). Then it proves that

$$G^E=G^M=G. \tag{10}$$

Solution to the nonuniform equation can be expressed in terms of the solutions to the corresponding uniform equation Ψ_+ and Ψ_- which fall off at $z\to+\infty$ and $z\to-\infty$, respectively. In turn, one of these, say Ψ_-, can be expressed in terms of another one, Ψ_+. As a result,

$$G(z,\tilde{z})=\Psi_+(z)\Psi_+(\tilde{z})\int_{-\infty}^{z}\frac{dz}{\Psi_+^2(z)} \tag{11}$$

at $\tilde{z}>z$. Let us denote $\Psi_+=\exp(-\Omega)$, write $dz=d\Omega/\Omega'$ and integrate by parts. Repeatedly performing this procedure gives

$$G(z,\tilde{z})=\exp\left[-\int_z^{\tilde{z}}\Omega'(y)dy\right]\sum_{k=0}^{\infty}\left[-\frac{1}{2\Omega'(z)}\frac{d}{dz}\right]^k\frac{1}{2\Omega'(z)} \tag{12}$$

where Ω obeys the equation

$$\Omega'^2-\Omega''=\omega^2(z)\equiv t^2+l(l+1)\exp\frac{(2\Phi)}{r^2}. \tag{13}$$

The solution to this equation is well-known from quantum mechanics as quasiclassical series which up to the second order reads:

$$\Omega'=\omega+\frac{\omega'}{2\omega}+\frac{\omega''}{4\omega^2}-\frac{3}{8}\frac{\omega'^2}{\omega^3}. \tag{14}$$

Substitute this into (12). Integration over dt and summation over l are finite due to the exponential factor $\exp(-\omega\Delta z)$. Therefore the divergence at

$$\Delta z\equiv\tilde{z}-z\to 0 \tag{15}$$

arises in a few first terms in quasiclassical expansion. Maximal divergence is Δz^{-4}, therefore the factors up to $O(\Delta z^4)$ at these divergences also contribute into finite part of the stress-energy. The resulting part of the Green function of interest tuns out to be

$$\begin{aligned}2G(z,\tilde{z})=\exp(-\omega\Delta z)\Bigg\{&\frac{1}{\omega}-\frac{1}{8}\frac{(\omega^2)''}{\omega^5}+\frac{5}{32}\frac{(\omega^2)'^2}{\omega^7}\\&+\Delta z\left[-\frac{1}{4}\frac{(\omega^2)'}{\omega^3}-\frac{1}{8}\frac{(\omega^2)''}{\omega^4}+\frac{5}{32}\frac{(\omega^2)'^2}{\omega^6}\right]\\&+\Delta z^2\left[-\frac{1}{4}\frac{(\omega^2)'}{\omega^2}-\frac{1}{8}\frac{(\omega^2)''}{\omega^3}+\frac{5}{32}\frac{(\omega^2)'^2}{\omega^5}\right]\\&+\Delta z^3\left[-\frac{1}{12}\frac{(\omega^2)''}{\omega^2}+\frac{5}{48}\frac{(\omega^2)'^2}{\omega^4}\right]+\Delta z^4\frac{1}{32}\frac{(\omega^2)'^2}{\omega^3}\Bigg\}.\end{aligned} \tag{16}$$

Upon substituting this into the expressions for the stress-energy (5) the terms like

$$\sum_{l=1}^{\infty}\left(l+\frac{1}{2}\right)[l(l+1)]^m\int_0^\infty dt\frac{\exp\left[-\Delta z\sqrt{t^2+l(l+1)V(z)}\right]}{[t^2+l(l+1)V(z)]^{n/2}} \tag{17}$$

arise. Here

$$V\equiv\frac{\exp(2\Phi)}{r^2}. \tag{18}$$

Introduce instead of t a new integration variable

$$q=\Delta z\sqrt{\frac{t^2}{l(l+1)}+V(z)} \tag{19}$$

and rewrite these terms in the form

$$\Delta z^{n-1}\int_{\Delta z\sqrt{V}}^{\infty}\frac{dq}{q^{(n-1)}\sqrt{q^2-\Delta z^2V}}\left(-\frac{d}{dq}\right)^{2m+1-n}\frac{f(q)}{q^2} \tag{20}$$

where

$$f(q)\equiv q^2\sum_{l=1}^{\infty}\left(l+\frac{1}{2}\right)\exp\left[-q\sqrt{l(l+1)}\right]. \tag{21}$$

The $f(q)$ is regular at $q\geq 0$. It turns out that coefficients in $T_{\mu\nu}^{\text{reg}}$ at Δz^{-n}, $n=0,1,2,3,4$ and at $\ln(L/\Delta z)$, $L=$ const, in the considered quasiclassical orders can be expressed in terms of $f^{(m)}(0)$, $m=0,1,2,3,4$. In turn, the latter are elementary calculable by taking into account a few first terms in the expansion of $\sqrt{l(l+1)}$ in the last formula over decreasing powers of $l+1/2$ [10].

The result of calculations made according to the above scheme gives for the two basic integrosums entering (5):

$$\sum_{l=1}^{\infty}\int_0^\infty\left(l+\frac{1}{2}\right)dt\,l(l+1)G=\frac{2}{\Delta z^4}\frac{1}{V^2}-\frac{2}{\Delta z^3}\frac{V'}{V^3} \tag{22}$$
$$+\frac{1}{\Delta z^2}\left(\frac{7}{4}\frac{V'^2}{V^4}-\frac{5}{6}\frac{V''}{V^3}\right)-\frac{1}{60}\left(\ln\frac{L}{\Delta z\sqrt{V}}-\frac{1}{2}\right)$$
$$+\frac{1}{72}\frac{V''}{V^2}-\frac{1}{72}\frac{V'^2}{V^3},$$

$$\sum_{l=1}^{\infty}\int_0^\infty\left(l+\frac{1}{2}\right)dt\frac{\partial}{\partial z}\left(\frac{\partial}{\partial\tilde z}-\frac{\partial}{\partial z}\right)G=-\frac{6}{\Delta z^4}\frac{1}{V}+\frac{4}{\Delta z^3}\frac{V'}{V^2} \tag{23}$$
$$+\frac{1}{\Delta z^2}\left(\frac{1}{3}-\frac{31}{12}\frac{V'^2}{V^3}+\frac{4}{3}\frac{V''}{V^2}\right)$$
$$+\frac{1}{60}\left(\ln\frac{L}{\Delta z\sqrt{V}}-1\right)V-\frac{1}{144}\frac{V'^2}{V^2}.$$

Here the derivatives of V are those over ρ and numerical constant under the logarithm sign is defined according to

$$\ln L=\frac{13}{12}+\ln M-\frac{5}{2}\int_0^\infty\frac{f^{\text{iv}}(q)-f^{\text{iv}}(0)\theta(M-q)}{q}dq \tag{24}$$

where $f^{\text{iv}}(0)=-2/5$ and $M\geq 0$ is arbitrary.

When substituting this into (5) one should also expand $\tilde r$, $\tilde\Phi$ at z over Δz and express Δz in terms of $\Delta\rho=\epsilon$. The result of calculations is

$$8\pi^2r^4T_{\hat t\hat t}^{\text{reg}}=-8\frac{r^4}{\epsilon^4}+\frac{4}{3}\frac{r^2}{\epsilon^2}\left(1-r'^2+2r''r\right) \tag{25}$$
$$-\frac{4}{3}\frac{r}{\epsilon}r'-\frac{1}{15}\ln\frac{Lr}{\epsilon}+\frac{13}{9}r'^2-\frac{8}{9}r''r,$$

$$8\pi^2r^4T_{\hat\rho\hat\rho}^{\text{reg}}=-24\frac{r^4}{\epsilon^4}+\frac{4}{3}\frac{r^2}{\epsilon^2}\left[1+r^2\left(2\Phi''+2\Phi'^2\right.\right. \tag{26}$$
$$\left.\left.-2\Phi'\frac{r'}{r}-\frac{r'^2}{r^2}+4\frac{r''}{r}\right)\right]-\frac{4}{3}\frac{r}{\epsilon}r'+\frac{1}{15}\left(\ln\frac{Lr}{\epsilon}-\frac{1}{2}\right)$$
$$+\frac{1}{9}r^2\left(-2\Phi''-2\Phi'^2+2\Phi'\frac{r'}{r}+11\frac{r'^2}{r^2}-6\frac{r''}{r}\right),$$

$$8\pi^2r^4T_{\hat\theta\hat\theta}^{\text{reg}}=8\frac{r^4}{\epsilon^4}+\frac{4}{3}\frac{r^2}{\epsilon^2}\left(-\Phi''-\Phi'^2+\Phi'\frac{r'}{r}-\frac{r''}{r}\right) \tag{27}$$
$$-\frac{1}{15}\left(\ln\frac{Lr}{\epsilon}-\frac{1}{2}\right)+\frac{1}{9}r^2\left(\Phi''+\Phi'^2-\Phi'\frac{r'}{r}+\frac{r'^2}{r^2}-\frac{r''}{r}\right).$$

The divergences at $\epsilon\to 0$ are reduced to renormalisation of the cosmological constant, Einstein gravity constant and coefficient at the

Weyl tensor squared in the effective gravity action [15]. Thus we should equate these coefficients to their experimental values and subtract from $T^{\rm reg}_{\mu\nu}$ the $T^{\rm div}_{\mu\nu}$ corresponding to the divergent part of the effective action and derived by Christensen [11]. Calculation according to the Christensen's formula for the electromagnetic field in our geometry gives:

$$
\begin{aligned}
8\pi^2 r^4 T^{\rm div}_{\hat t\hat t} &= -8\frac{r^4}{\epsilon^4} + \frac{4}{3}\frac{r^2}{\epsilon^2}\left(1 - r'^2 + 2r''r\right) \\
&\quad - \frac{4}{3}\frac{r}{\epsilon}r' - \frac{1}{15}\ln\frac{\Lambda}{\epsilon} + r'^2 - \frac{1}{3}r''r,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
8\pi^2 r^4 T^{\rm div}_{\hat\rho\hat\rho} &= -\; 24\frac{r^4}{\epsilon^4} + \frac{4}{3}\frac{r^2}{\epsilon^2}\Big[1 + r^2\left(2\Phi'' + 2\Phi'^2\right. \\
&\quad \left.\left. - \; 2\Phi'\frac{r'}{r} - \frac{r'^2}{r^2} + 4\frac{r''}{r}\right)\right] + \frac{1}{15}\left(\ln\frac{\Lambda}{\epsilon} - \frac{1}{2}\right) \\
&\quad + \; \frac{1}{9}r^2\left(-2\Phi'' - 2\Phi'^2 - 3\Phi'\frac{r'}{r} + 12\frac{r'^2}{r^2} - 6\frac{r''}{r}\right),
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
8\pi^2 r^4 T^{\rm div}_{\hat\theta\hat\theta} &= 8\frac{r^4}{\epsilon^4} + \frac{4}{3}\frac{r^4}{\epsilon^2}\left(-\Phi'' - \Phi'^2 + \Phi'\frac{r'}{r} - \frac{r''}{r}\right) \\
&\quad - \; \frac{1}{15}\ln\frac{\Lambda}{\epsilon} - \frac{1}{6}r^2\left(\Phi'' + \Phi'^2\right).
\end{aligned} \tag{30}
$$

The coefficient at the Weyl term is the only one which is both UV and IR divergent, and Λ is IR cut-off. We include both these divergences into the renormalisation of the coefficient at the Weyl term. The Christensen's procedure includes also forming half of the sum of the components $T_{\mu\nu}$ corresponding to point separations $\epsilon = \pm|\epsilon|$ (above formulas are given for $\epsilon > 0$), thereby only even powers of $|\epsilon|$ are left (and also $|\epsilon|$ rather then ϵ enters the logarithm). Subtracting $T^{\rm div}_{\mu\nu}$ from $T^{\rm reg}_{\mu\nu}$ we finally obtain

$$
8\pi^2 r^4 T^{\rm ren}_{\hat t\hat t} = -\frac{1}{15}\ln\frac{Lr}{\Lambda} + \frac{4}{9}r'^2 - \frac{5}{9}r''r, \tag{31}
$$

$$
8\pi^2 r^4 T^{\rm ren}_{\hat\rho\hat\rho} = +\frac{1}{15}\ln\frac{Lr}{\Lambda} + \frac{5}{9}\Phi'r'r - \frac{1}{9}r'^2, \tag{32}
$$

$$
\begin{aligned}
8\pi^2 r^4 T^{\rm ren}_{\hat\theta\hat\theta} &= -\frac{1}{15}\ln\frac{Lr}{\Lambda} + \frac{1}{30} + \frac{5}{18}r^2\left(\Phi'' + \Phi'^2\right) \\
&\quad - \frac{1}{9}\Phi'r'r + \frac{1}{9}r'^2 - \frac{1}{9}r''r.
\end{aligned} \tag{33}
$$

Here the ratio L/Λ is a constant which can be fixed only by experiment.

3 Discussion

Following [2] let us form the difference between the radial pressure $\tau = -T^{\rm ren}_{\hat\rho\hat\rho}$ and energy density $\varrho = T^{\rm ren}_{\hat t\hat t}$ and integrate it from $\rho = \rho_0 < 0$ to $\rho = +\infty$. The resulting value turns out to be definitely positive as the sum of the two positive terms:

$$
\begin{aligned}
\int_{\rho_0}^{\infty}(\tau - \varrho)\exp(-\Phi)d\rho &= -\frac{5}{72\pi^2}\frac{r'}{r^3}\exp(-\Phi)\Big|_{\rho=\rho_0} \\
&\quad + \frac{4}{3}\int_{\rho_0}^{\infty}\frac{r'^2}{r^4}\exp(-\Phi)d\rho > 0\,.
\end{aligned} \tag{34}
$$

That is, averaged weak energy condition for the electromagnetic vacuum in the wormhole topology is violated as well as the local weak energy condition at the throat is [10].

Thus, a necessary condition for the electromagnetic vacuum to be able to support the wormhole geometry is fullfilled. This requirement is stronger one than that of violation of the local weak energy condition *anywhere* (however, the fact that the local weak energy condition is violated *namely at the throat* does not follow, generally speaking, from violation of the averaged weak energy condition). Remarkably that it is fullfilled irrespectively of the detailed form of the metric $r(\rho)$, $\Phi(\rho)$ at least in the considered orders of quasiclassical expansion.

Moreover, the Einstein equations with $T^{\rm ren}_{\mu\nu}$ (31) as the source formally have the wormhole-type solution with $r = {\rm const}$ (infinitely long wormhole). However, the derivatives $\Phi^{(n)}$ are not small for this solution, and approximation based on the few first orders in the expansion over derivatives is questionable. Evidently, any progress in this direction is connected with finding more accurate and universal expression for the stress-energy tensor.

I am grateful to Professors V.L.Chernyak and I.B.Khriplovich for the interest to the work and the discussion.

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К осуществимости самоподдерживающиеся вакуумной проходимой "кротовой норы"

Budker INP 96-87

Ответственный за выпуск А.М. Кудрявцев
Работа поступила 16.10.1996 г.

Сдано в набор 10.12.1996 г.
Подписано в печать 11.12.1996 г.
Формат бумаги 60×90 1/16 Объем 0.7 печ.л., 0.6 уч.-изд.л.
Тираж 180 экз. Бесплатно. Заказ N° 87

Обработано на IBM PC и отпечатано на
ротапринте ГНЦ РФ "ИЯФ им. Г.И. Будкера СО РАН",
Новосибирск, 630090, пр. академика Лаврентьева, 11.