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Budker INP 96-48

НОВОСИБИРСК

Nature of the Darwin Term and $(Z\alpha)^4 m^3/M^2$ Contribution to the Lamb Shift for an Arbitrary Spin of the Nucleus

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Abstract

The contact Darwin term is demonstrated to be of the same origin as the spin-orbit interaction. The $(Z\alpha)^4 m^3/M^2$ correction to the Lamb shift, generated by the Darwin term, is found for an arbitrary nonvanishing spin of the nucleus, both half-integer and integer. There is also a contribution of the same nature to the nuclear quadrupole moment.

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1. The literature, pedagogical included, abounds with assertions on the nature of the Darwin correction which are at least doubtful in our opinion. In particular, we cannot agree with the conclusion that the Darwin term is absent for a particle with spin 1, made in Ref.[1] (see also [2]). The subject becomes of real interest now for interpreting the high precision experiments in atomic spectroscopy [3, 4, 5].

To study the problem we consider in this note the Born amplitude for scattering of a particle with an arbitrary spin in an external electromagnetic field. In the case of a practical interest, that of an atom, this is the nucleus interaction with the electromagnetic field of electron. In this way we derive the general form of the Darwin term for an arbitrary nuclear spin and obtain the corresponding order $(Z\alpha)^4 m^3/M^2$ correction to the Lamb shift (here and below m is the electron mass, Z and M are, respectively, the charge and mass of the nucleus).

2. The wave function of a particle with an arbitrary spin can be written as (see, for instance, [6], §31)

$$\Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{1}$$

Both spinors,

$$\xi = \{\xi^{\alpha_1 \alpha_2 .. \alpha_p}_{\dot\beta_1 \dot\beta_2 .. \dot\beta_q}\}$$

and

$$\eta = \{\eta^{\beta_1 \beta_2 .. \beta_q}_{\dot\alpha_1 \dot\alpha_2 .. \dot\alpha_p}\},$$

are symmetric in dotted and undotted indices separately, and

$$p+q=2I,$$

where I is the particle spin. For a particle of half-integer spin one can choose

$$p=I+\frac{1}{2}\ ,\ \ q=I-\frac{1}{2}\ .$$

In the case of integer spin it is convenient to use

$$p=q=I.$$

Spinors ξ and η are choosen in such a way that under reflection they go over into each other (up to a phase). At $p\neq q$ they are different objects which belong to different representations of the Lorentz group. If $p=q$, these two spinors coincide. Nevertheless, we will use the same expression (1) for the wave function of any spin, i.e., we will introduce formally the object η for an integer spin as well, keeping in mind of course that it is expressed via ξ. It will allow us to perform calculations in the same way for both integer and half-integer spins.

In the rest frame both ξ and η coincide with a nonrelativistic spinor ξ_0, which is symmetric in all indices; in the rest frame there is no difference between dotted and undotted indices. The Lorentz transformation of ξ_0 is, up to the terms $\sim (v/c)^2$ included,

$$\xi=\left(1+\frac{\vec{\Sigma}\vec{v}}{2}+\frac{(\vec{\Sigma}\vec{v})^2}{8}\right)\xi_0\,,$$
$$\eta=\left(1-\frac{\vec{\Sigma}\vec{v}}{2}+\frac{(\vec{\Sigma}\vec{v})^2}{8}\right)\xi_0\,. \qquad (2)$$

Here

$$\vec{\Sigma}=\sum_{i=1}^{p}\vec{\sigma}_i-\sum_{i=p+1}^{p+q}\vec{\sigma}_i\,,$$

and $\vec{\sigma}_i$ acts on the ith index of the spinor ξ_0 as follows:

$$\vec{\sigma}_i\,\xi_0=(\vec{\sigma}_i)_{\alpha_i\beta_i}\,(\xi_0)_{\ldots\beta_i\ldots}\ . \qquad (3)$$

At the Lorentz transformation (2) for ξ, after acting of the operator $\vec{\Sigma}$ on ξ_0 the first p indices are identified with upper undotted indices, and the next q with lower dotted ones. The inverse situation takes place for η.

We will use however formula (2) as it is without distinguishing anymore between upper and lower, or dotted and undotted spinor indices. It allows us to introduce in a natural way the "standard" representation for the spinors, in close analogy with that for spin 1/2:

$$\phi=(\xi+\eta)/2\,;\ \ \chi=(\xi-\eta)/2\,.$$

In it the wave function is written as

$$\Psi=\begin{pmatrix}\left[1+(\vec{\Sigma}\vec{v})^2/8\right]\xi_0\\ \left(\vec{\Sigma}\vec{v}/2\right)\xi_0\end{pmatrix}. \qquad (4)$$

It is convenient to introduce as usual the object

$$\bar{\Psi}=(\phi^*,-\chi^*).$$

Then

$$\bar{\Psi}\,\Psi=\phi^*\phi-\chi^*\chi=\xi_0^*\,\xi_0$$

is an invariant. We will use the common noncovariant normalization of the particle number density

$$\rho=\frac{E}{M}\bar{\psi}\psi=1, \qquad (5)$$

where the wave function ψ is

$$\psi=\sqrt{\frac{M}{E}}\begin{pmatrix}\left[1+(\vec{\Sigma}\vec{v})^2/8\right]\xi_0\\ \left(\vec{\Sigma}\vec{v}/2\right)\xi_0\end{pmatrix}. \qquad (6)$$

Here E is the particle energy.

3. Let us go over now to the scattering amplitude itself. The order $1/M^2$ terms in it arise only in the time component of the electromagnetic current. Restricting to the formfactors of the lowest multipolarity, electric F_e and magnetic G_m, this component can be written for an arbitrary spin as

$$j_0=F_e\,\frac{E+E'}{2M}\bar{\psi}'\psi+\frac{G_m}{2M}\psi'^{\star}\,\vec{\Gamma}\vec{q}\,\psi, \qquad (7)$$

where

$$\vec{q}=\vec{p}\,'-\vec{p}.$$

The matrix

$$\vec{\Gamma} = \begin{pmatrix} 0 & \vec{\Sigma} \\ -\vec{\Sigma} & 0 \end{pmatrix} \tag{8}$$

is a natural generalization of the corresponding expression for spin 1/2 (valid both in the spinor and standard representations):

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} . \tag{9}$$

This generalization is fairly obvious in the spinor representation. Indeed, here, according to (9), $\vec{\sigma}$ connects a dotted index in the initial spinor ψ with undotted one in $\bar{\psi}$, and $-\vec{\sigma}$ connects an undotted index from ψ with dotted one of $\bar{\psi}$. And this is exactly what is being done by $\vec{\Gamma}$. It is straightforward now to prove the expression (8) for the standard representation. Let us mention also that formula (8) is confirmed by the final result which reproduces correctly the spin-orbit interaction, the form of the latter being well-known for an arbitrary spin (see, e.g., [6], §41).

The term with G_m in the current density is

$$j_{0\,m} = \frac{G_m}{2M}\xi_0'^{\star}\begin{pmatrix} 1, & \vec{\Sigma}\vec{v}'/2 \end{pmatrix}\begin{pmatrix} 0 & \vec{\Sigma}\vec{q} \\ -\vec{\Sigma}\vec{q} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \vec{\Sigma}\vec{v}/2 \end{pmatrix}\xi_0$$

$$= \frac{G_m}{4M^2}\xi_0'^{\star}\left(-(\vec{\Sigma}\vec{q})^2 + 4\,i\,\vec{I}[\vec{q}\times\vec{p}]\right)\xi_0 . \tag{10}$$

The spin operator here equals

$$\vec{I} = \frac{1}{2}\sum_{i=1}^{2I}\vec{\sigma}_i .$$

The first term, with F_e, in formula (7) reduces to an analogous structure:

$$j_{0\,ch} = F_e\,\frac{E'+E}{2\sqrt{EE'}}\,\xi_0'^{\star}\left(1+\frac{(\vec{\Sigma}\vec{v})^2}{8}+\frac{(\vec{\Sigma}\vec{v}')^2}{8}-\frac{(\vec{\Sigma}\vec{v}')(\vec{\Sigma}\vec{v})}{4}\right)\xi_0$$

$$= F_e\,\xi_0'^{\star}\left(1+\frac{(\vec{\Sigma}\vec{q})^2}{8M^2}-i\,\frac{\vec{I}[\vec{q}\times\vec{p}]}{2M^2}\right)\xi_0 . \tag{11}$$

Thus the total charge density is

$$j_0 = \xi_0'^{\star}\left(F_e-(2G_m-F_e)\frac{(\vec{\Sigma}\vec{q})^2}{8M^2}+(2G_m-F_e)\,i\,\frac{\vec{I}[\vec{q}\times\vec{p}]}{2M^2}\right)\xi_0 .$$

We neglect for the time being the charge radius of the nucleus, so that

$$F_e = F_e(0) = 1 .$$

The spin-orbit interaction dependence on the gyromagnetic ratio g is universal for any spin, this ratio enters through the factor $g-1$. Therefore, our magnetic formfactor is normalized as follows

$$G_m(0) = \frac{g}{2} .$$

Let us split now $(\vec{\Sigma}\vec{q})^2$ into the contact and quadrupole parts:

$$\Sigma_i\Sigma_j\,q_iq_j \;=\; \frac{\vec{q}^{\,2}}{3}\Sigma_i\Sigma_i + (q_iq_j-\frac{1}{3}\vec{q}^{\,2}\delta_{ij})\Sigma_i\Sigma_j \tag{12}$$

The first, contact term in (12) is

$$\vec{\Sigma}\vec{\Sigma} = \left(\sum_{i=1}^{p}\vec{\sigma}_i\right)^2 - 2\left(\sum_{i=1}^{p}\vec{\sigma}_i\right)\left(\sum_{i=p+1}^{p+q}\vec{\sigma}_i\right)+\left(\sum_{i=p+1}^{p+q}\vec{\sigma}_i\right)^2$$

$$= 3(p+q) + 2\left(\frac{p(p-1)}{2}+\frac{q(q-1)}{2}-pq\right) = 4I(1+\zeta); \tag{13}$$

$$\zeta = \begin{cases} 0, & \text{integer spin,} \\ 1/(4I), & \text{half-integer spin.} \end{cases}$$

When deriving formula (13), we use the symmetry in any pair of spinor indices, $\alpha_1\,\alpha_2$ (see (3)). This symmetry means that the corresponding spins, 1 and 2, add up into the total spin $S=1$. Therefore,

$$(\vec{\sigma}_1\vec{\sigma}_2)\,\xi_0 = \xi_0 .$$

The interaction operator is proportional to the Fourier transform of the Born amplitude (see, e.g., [6], §83). In this way we obtain from (13) the following contact interaction between a nucleus of charge Z and electron:

$$U(\vec{r}) = \frac{2\,\pi}{3}\,\frac{Z\alpha}{M^2}\,(g-1)\,I\,(1+\zeta)\,\delta(\vec{r}) . \tag{14}$$

The corresponding energy correction is

$$\Delta E_n = \frac{2}{3}\,\frac{m^3}{M^2}\,(g-1)\,I\,(1+\zeta)\,\frac{(Z\alpha)^4}{n^3}\,\delta_{0\ell} . \tag{15}$$

For the hydrogen atom ($I = 1/2$) this correction was obtained long ago in Ref. [7].

Let us consider now the quadrupole part of (12). Using again the complete symmetry of ξ_0, one can easily calculate the corresponding quadrupole interaction:

$$U_2(\vec{r}) = -\frac{1}{6}\nabla_i\nabla_j\frac{e}{r}\delta Q_{ij}\,. \tag{16}$$

Here

$$\delta Q_{ij} = -\frac{3}{4}\frac{Z\,e\,(g-1)}{M^2}\Lambda\left\{I_i\,I_j + I_j\,I_i - \frac{2}{3}\delta_{i\,j}\,I(I+1)\right\}; \tag{17}$$

$$\Lambda = \begin{cases} 1/(2I-1), & \text{integer spin,} \\ 1/(2I), & \text{half-integer spin.} \end{cases}$$

Expression (17) is a correction to the nuclear quadrupole moment. Its existence for $I = 1$ was pointed out in Ref. [1].

This correction to the quadrupole moment can be estimated as

$$\delta Q \approx -0.22\,(g-1)\,\frac{Z\,I}{A^2}\;e\,\text{mbarn}\,.$$

For the deuteron ($Z = 1$, $A = 2$, $g = 2\mu_d = 1.714$, $Q = 2.86\;e\,$mbarn) it equals $-0.04\;e\,$mbarn.

4. Let us come back now to the discussion of the contact term. There is some ambiguity in its definition related to the nuclear charge radius. The contribution of the latter produces a contact interaction also and enters physical observables in a sum with the expression $(g-1)\vec{q}^{\,2}\;I(1+\zeta)/(6M^2)$. In particular, the elastic cross-section of the electon-nucleus scattering at small $\vec{q}^{\,2}$ is, up to the terms $\vec{q}^{\,2}/M^2$ included,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\epsilon^2}\,\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\,\frac{1}{1+\,2\,\sin^2(\theta/2)\,\epsilon/M}$$

$$\left\{\left[1-\frac{1}{6}\langle r^2\rangle_F\,\vec{q}^{\,2} - (g-1)\,\frac{\vec{q}^{\,2}}{6M^2}I(1+\zeta)\right]^2 \right. \tag{18}$$

$$\left. +\,\frac{4}{3}\,G_m^2\;I(I+1)[2\,\tan^2(\theta/2)+1]\right\},$$

where $\langle r^2\rangle_F$ is defined through the expansion of the formfactor F_e:

$$F_e(q^2) \approx 1 - \frac{1}{6}\langle r^2\rangle_F\,\vec{q}^{\,2}. \tag{19}$$

Let us note here that the expression in square brackets in formula (18) reduces for the proton ($I = 1/2$) to

$$1 - \frac{1}{6}\langle r^2\rangle_F\,\vec{q}^{\,2} - (g-1)\,\frac{\vec{q}^{\,2}}{8M^2}, \tag{20}$$

and for the deuteron ($I = 1$) to

$$1 - \frac{1}{6}\langle r^2\rangle_F\,\vec{q}^{\,2} - (g-1)\,\frac{\vec{q}^{\,2}}{6M^2}. \tag{21}$$

However, the proton charge radius is commonly defined otherwise than in formula (20), namely, through the expansion of the so-called Sachs formfactor

$$G_e = F_e - \frac{\vec{q}^{\,2}}{4M^2}\,G_m\,.$$

Obviously, the charge radius defined through the formfactor G_e is

$$-\frac{1}{6}\langle r^2\rangle_G = \frac{\partial G_e}{\partial\vec{q}^{\,2}} = -\frac{1}{6}\langle r^2\rangle_F - \frac{g}{8M^2}.$$

Correspondingly, expression (20) is rewritten usually as

$$1 - \frac{1}{6}\langle r^2\rangle_G\,\vec{q}^{\,2} + \frac{\vec{q}^{\,2}}{8M^2}, \tag{22}$$

and the Darwin correction for the proton is defined as

$$\frac{\vec{q}^{\,2}}{8M^2}\,,$$

but not

$$-\frac{(g-1)\vec{q}^{\,2}}{8M^2}\,.$$

We could redefine the electric formfactor for the deuteron from F_e to G_e in such a way that here

$$-\frac{1}{6}\langle r^2\rangle_G = \frac{\partial G_e}{\partial\vec{q}^{\,2}} = -\frac{1}{6}\langle r^2\rangle_F - \frac{g}{6M^2},$$

so that the Darwin correction for the deuteron becomes

$$\frac{\vec{q}^{\,2}}{6M^2}\,,$$

instead of

$$-\frac{(g-1)\vec{q}^{\,2}}{6M^2}.$$

However, for a deuteron the common definition of the charge radius is neither F_e, nor G_e, but

$$-\frac{1}{6}\langle r^2\rangle_D = -\frac{1}{6}\langle r^2\rangle_F - \frac{g-1}{6M^2}.$$

Of course, under this definition the whole Darwin term is swallowed up by $\langle r^2\rangle_D$. No wonder therefore that the authors of Ref. [1], using $\langle r^2\rangle_D$ instead of $\langle r^2\rangle_E$ or $\langle r^2\rangle_G$, make the conclusion that for the deuteron, as distinct from proton, the Darwin correction is absent.

Clearly, this contradistinction of the deuteron to proton is based only on a rather arbitrary definition of the charge radius of the former; this contradistinction has no physical meaning, it has nothing to do with the nature of the Darwin term.

5. Thus, the Darwin interaction exists for any nonvanishing spin and is of the same nature as the spin-orbit interaction. In particular, as well as the spin-orbit interaction, the Darwin term is not directly related to the so-called Zitterbewegung. Of course, there is a certain difference between the spin-orbit and contact energy corrections. The former one has a classical limit together with $\langle 1/r^3\rangle$, while the latter, being proportional to $|\psi(0)|^2$, does not. However, this fact has nothing to do with relativity and negative energies, and therefore is certainly unrelated to the Zitterbewegung.

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Природа дарвиновской поправки и вклад порядка $(Z\alpha)^4 m^3/M^2$ в лэмбовский сдвиг для произвольного спина ядра

ИЯФ 96-48

Ответственный за выпуск А.М. Кудрявцев
Работа поступила 17.07.1996 г.

Сдано в набор 17.07.1996 г.
Подписано в печать 10.07.1996 г.
Формат бумаги 60×90 1/16 Объем 0.6 печ.л., 0.4 уч.-изд.л.
Тираж 170 экз. Бесплатно. Заказ N° 48

Обработано на IBM PC и отпечатано на
ротапринте ГНЦ РФ "ИЯФ им. Г.И. Будкера СО РАН",
Новосибирск, 630090, пр. академика Лаврентьева, 11.