The State Scientific Center of Russia
The Budker Institute of Nuclear Physics
SB RAS

G. Casati, B.V. Chirikov,
I. Guarneri and F.M. Izrailev

QUANTUM ERGODICITY AND LOCALIZATION IN GENERIC CONSERVATIVE SYSTEMS: THE WIGNER BAND RANDOM MATRIX MODEL

Budker INP 95-98

НОВОСИБИРСК

Quantum ergodicity and localization in generic conservative systems: the Wigner Band Random Matrix model

G. Casati[1], *B.V. Chirikov*[1,2], *I. Guarneri*[1] *and F.M. Izrailev*[1,2]

The State Research Center of Russian Federation
Budker Institute of Nuclear Physics SB RAS
630090, Novosibirsk, Russia

Abstract

First theoretical and numerical results on the global structure of the energy shell, the Green function spectra and the eigenfunctions, both localized and ergodic, in a generic conservative quantum system are presented. In case of quantum localization the eigenfunctions are shown to be typically narrow and solid with their centers randomly scattered within the semicircle energy shell while the Green function spectra are extended over the whole shell but sparse.

[1]Universitá di Milano, Sede di Como, Via Lucini 3, 22100 Como, Italy.
[2]Budker Institute of Nuclear Physics

One of the main results in the study of the so-called quantum chaos has been the discovery of quantum dynamical localization as a mesoscopic quasi-classical phenomenon. This phenomenon has been widely studied and confirmed by many researchers for dynamical models described by maps (see, e.g., Refs.[5,8]). Contrary to a common belief, maps describe not only time-dependent systems but also conservative ones (in the form of Poincare's maps). On the other hand, to our knowledge, there were no direct studies of quantum dynamical localization in bounded conservative models; moreover, the very existence of quantum localization is challenged by some researchers. The localization in conservative systems would restrict quantum distributions to smaller regions of phase space than classically allowed, and would therefore introduce significant deviations from ergodicity.

We have addressed this problem on the Wigner Band Random Matrix (WBRM) model which was introduced by Wigner 40 years ago [1] for the description of complex conservative quantum systems like atomic nuclei. Due to severe mathematical difficulties the random matrix theory (RMT) immediately turned to the much simpler case of statistically homogeneous (full) matrices for which impressive theoretical results have been achieved (see, e.g., Refs.[2]). However, the full matrices describe a local chaotic structure only, and this limitation is often inacceptable, for instance, in the case of atoms [3,4].

Generally speaking, RMT is a statistical theory of systems with discrete energy (and frequency) spectrum. Since the latter is a typical property of quantum dynamical chaos [5], RMT provides a statistical description of quantum chaos and, what is very important, one which does not involve any coupling to a thermal bath which is a standard element in most statistical theories. Moreover, a *single* matrix from a given statistical ensemble represents

the typical (generic) *dynamical* system of a given class characterized by a few matrix parameters. This makes an important bridge between dynamical and statistical description of quantum chaos.

To the extent that Band Random Matrices can be taken as the models for generic few–freedom conservative systems which are classically strongly chaotic (in particular, ergodic) on a compact energy surface, the results presented in this Letter provide the first characterization of the structure of quantum chaos in momentum space for quantum systems of this class.

We consider real Hamiltonian matrices of a rather general type (more specific random matrix models have been recently proposed in Ref.[6])

$$H_{mn} = E_n\,\delta_{mn} + v_{mn} \qquad (m,n = 1,..,N)\,, \tag{1}$$

where off–diagonal matrix elements $v_{mn} = v_{nm}$ are statistically independent Gaussian random variables, with $< v_{mn} >= 0$ and $< v_{mn}^2 >= v^2$ if $|m-n| \leq b$, and are zero otherwise. In a classical picture, WBRMs like (1) would correspond to classical Hamiltonians of the form:

$$H = H_0 + V\,, \tag{2}$$

where the perturbation V is usually assumed to be sufficiently small while the unperturbed Hamiltonian H_0 is completely integrable. In the quantum model the matrix (1) is given in the basis of the unperturbed eigenstates ϕ_n of $\hat{H}_0$. Correspondingly, the fluctuations of unperturbed energy levels E_n are taken as Poissonian. Although in completely integrable quantum system there is a quantum number for each freedom, we suppose that the unperturbed states are ordered according to increasing energy ($E_n \approx n/\rho$), and we thereby label them by a single number n. The most important characteristic of WBRM is the average level density ρ:

$$\rho^{-1} = \langle E_n - E_{n-1} \rangle\,. \tag{3}$$

Here and below the averaging is understood either over disorder that is over many random matrices or within a single sufficiently large matrix. Both ways are equivalent owing to assumed independence of matrix elements.

In the classical case, the unperturbed energy E_0 is not constant along a chaotic trajectory of the full Hamiltonian with a given total energy $H = E$. Instead, it sweeps a range of values, or 'energy shell' $\Delta E_0 = \Delta V$, and is distributed inside this shell according to a measure $W_E(E_0)$. The form of $W_E(E_0)$ depends on the form of the perturbation V. We will call this measure *ergodic* because it is determined by the ergodic (microcanonical) measure on

the energy surface $H = E$. The quantum analog of this measure characterizes the distribution of the ergodic eigenfunction (EF) in the unperturbed basis.

Conversely, if we keep the unperturbed energy E_0 fixed the bundle of trajectories of the total Hamiltonian H, which reach the surface $H_0 = E_0$, has a distribution in the total energy E described by a measure $w_{E_0}(E)$. In the quantum case this measure corresponds to the energy spectrum of the Green function (GFS) at energy E_0, and has received different names, such as "strength function" [1], "local spectral density of states", "spectral measure" of the unperturbed eigenstate at energy E_0.

An expression for the latter measure has been given by Wigner [1]. For a typical perturbation represented by a WBRM the average measure $w(E) = \langle w_{E_0}(E) \rangle$ depends on the Wigner parameter

$$q = \frac{(\rho\, v)^2}{b} \tag{4}$$

and has the following limiting forms [1] (see also Refs.[8,9,10]):

$$w(E) = \begin{cases} \frac{2}{\pi E_{sc}^2}\sqrt{E_{sc}^2 - E^2}, & |E| \leq E_{sc}, \quad q \gg 1 \\ \frac{\Gamma/2\pi}{E^2 + \Gamma^2/4} \cdot \frac{\pi}{2 \cdot \arctan(1/\pi q)}, & |E| \leq E_{BW}, \quad q \ll 1 \end{cases} \tag{5}$$

Outside the specified energy intervals both distributions have exponentially small tails. Formulae (5) are valid provided $\rho v \gtrsim 1$ which is the condition for strong coupling of neighboring unperturbed states by the perturbation. In the opposite case $\rho v \lesssim 1$ the effect of the perturbation is small, and we have the so-called perturbative localization.

In the limit $q \gg 1$ we have the semicircle (SC) law, and the width of the energy shell $\Delta E = 2E_{sc} = 4v\sqrt{2\,b} = 4\sqrt{2q}E_b \gg E_b$ where $E_b = b/\rho$ is the half width (in energy) of the band. In the opposite limit $q \ll 1$ we have the Breit - Wigner (BW) distribution of width $\Delta E = 2E_{BW} = 2E_b$ with the main part inside a width $\Gamma = 2\pi\rho v^2 = 2\pi q E_b \ll E_b$. In all these expressions E is measured with respect to the center of the distribution. Since $q \ll 1$ requires $\rho v \ll \sqrt{b}$, in the BW regime the perturbation is not strong enough to couple all states within one bandwidth. This means that the BW regime corresponds in fact to a sort of partial perturbative localization.

The numerical results presented below are contained in the EF matrix C_{mn} which connects exact eigenfunctions ψ_m, obtained by diagonalization of the Hamiltonian matrix (1), to the unperturbed basis states ϕ_n:

$$\psi_m = \sum_n C_{mn} \cdot \phi_n\,. \tag{6}$$

In what follows the eigenvalues E_m are ordered, so that $E_m \approx m/\rho$.

From the matrix C_{mn} we have found both the statistical distribution $W_m(n) = C_{mn}^2$ of the eigenstates ψ_m on the unperturbed ones ϕ_n, and the distribution $w_n(m)$ of the unperturbed eigenstates on the exact ones. The meaning of these distributions is similar to that of the classical W and w discussed above. We have then analyzed both distributions, and have compared their structures to each other and to the SC distribution paying special attention to localization. By localization we mean a situation in which the eigenfunctions are concentrated on a scale significantly smaller than the maximal one consistent with energy conservation. Indeed, the size of the region which is populated by an eigenfunction (termed localization length in the following) is bounded from above by the *ergodic localization length* $d^{(e)} = c\rho\Delta E$ which measures the maximum number of basis states coupled by the perturbation. This length characterizes the full width of the energy shell ΔE. The factor c depends on the definition of localization width (see Eq.(7) below). In other words, in a conservative quantum system there is always localization in energy due to the existence of a finite ΔE [7]. This fact, which is sometimes a source of confusion, is just a trivial consequence of energy conservation. Here we are interested in localization *inside* the shell [7] which can be caused by quantum effects. In this connection, the matrix size N is an irrelevant parameter, provided $N \gg d^{(e)}$ is large enough to avoid boundary effects. The quantum model (1) is thus defined by the 3 physical parameters: ρ, v, and b.

The localization length can be defined in several ways. We have used the so–called inverse participation ratio (see, e.g., Ref.[5]):

$$d_m^{-1} = \frac{1}{3}\sum_n W_m^2(n); \qquad d^{-1} = \sum_n W^2(n) \tag{7}$$

and similarly for w. The numerical factor $1/3$ accounts for the fluctuations in individual distributions $W_m(n)$, $w_n(m)$ but not in the average ones like $W(n)$, $w(m)$, and the others (see below). The fluctuations are assumed to be Gaussian and independent [4].

In order to suppress large fluctuations in individual distributions of both types, $W_m(n)$ and $w_n(m)$, we have taken averages over 300 of them chosen around the center of the spectrum. Since different distributions cover different regions of the n (respectively, m) space, prior to averaging they have to be shifted into a common region. This we have done in two different ways, namely, by counting the site label n in $W_m(n)$ starting either from the center of the energy shell, i.e., from the reference site m (and vice-versa in the case

of $w_n(m)$) or from the center $n_c(m)$ of $W_m(n)$ defined by

$$n_c(m) = \sum_n W_m(n) \cdot n\,. \tag{8}$$

The two types of the average are denoted by $\langle W_m(n)\rangle$ and $\overline{W}_m(n)$, respectively. In particular, $\overline{W}_m(n)$ yields the average shape of an eigenstate (see Fig.2).

First, we shall discuss distributions $W_m(n)$. In Ref.[7] it was shown that the localization length obeys a scaling law of the form

$$\beta_d = \frac{d}{d^{(e)}} \approx 1 - \mathrm{e}^{-\lambda} < 1, \tag{9}$$

where

$$\lambda = \frac{ab^2}{d^{(e)}} = \frac{ab^{3/2}}{4\sqrt{2}c\rho v}\,. \tag{10}$$

Here $a \approx 1.2$, and factor c can be directly calculated from the limiting expression (5) for w which gives $c \approx 0.92$.

The empirical relation (9) has been found in Ref.[7] to hold in the whole interval $\lambda \leq 2.5$, and was confirmed in the present studies up to $\lambda \approx 7$.

The parameter λ was introduced in Ref.[11] to describe the energy level statistics, and was explained in Ref.[7] as the *ergodicity parameter*. When it is large the localization length approaches its maximal value $d^{(e)}$ which means that the eigenfunctions become ergodic, i.e., delocalized over the whole energy shell. Notice that in the BW region the ergodic localization length $d^{(e)} = \pi\rho\Gamma = 2\pi^2 bq$, and $\lambda \approx ab/2q\pi^2 \gg 1$ [8] since $q \ll 1$ (and $b \gg 1$ in quasiclassics). Hence, the localization is only possible in the SC domain which is the main object of the present studies.

In the case $\lambda \gg 1$ (Fig.1) we have found that the averaged distributions $\langle W_m(n)\rangle$, $\overline{W}_m(n)$ are fairly close to the SC law: a remarkable result, because that law was theoretically predicted for the *other* distribution, namely, for the GFS $\langle w_n(m)\rangle$. We presume that the deviations from the SC law which are observed in the distribution $\overline{W}_m(n)$ are due to a not very large value of the ergodicity parameter ($\lambda = 3.6$). The numerical values of the localization parameter (9) are $\beta = 0.94$ and $\beta = 1.08$ for the two types of averages, respectively. This is in a reasonable agreement with the average $\beta_d = 0.97$ computed from (9) for $\lambda = 3.6$. For finite q the average distributions of both types are bordered by the two symmetric steep tails which apparently fall down even faster than the simple exponential (see below).

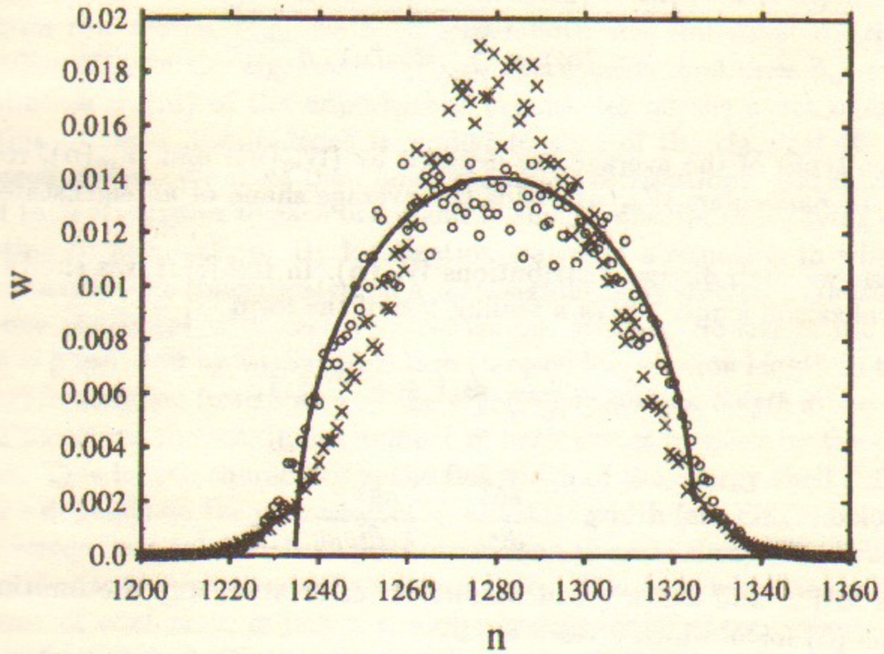

Fig. 1. Structure of ergodic eigenfunctions ($\lambda = 3.6$) from a single matrix with parameters: $N = 2560$, $v = 0.1$, $b = 16$, $\rho = 40$, $q = (\rho v)^2/b = 1$. The thick line is semicircle law (5). Crosses were obtained by averaging 300 eigenfunctions with respect to their centers; circles, by averaging the same eigenfunctions with respect to the centers of their energy shells. All the distributions are close to one another apart from fluctuations.

The structure of EFs is completely different in the case $\lambda \ll 1$ (Fig.2). Whereas individual eigenstates exhibit large fluctuations, the main part of the average distribution $\overline{W}_m(n)$ with respect to the centers $n_c(m)$) shows a clear evidence for exponential localization, with localization length in agreement with the empirical formula (9). The width of the main part is small ($\beta = 0.24$) which is again close to average $\beta_d = 0.21$ for $\lambda = 0.23$. We have found that the main part of the distribution can be represented reasonably well by a simple expression:

$$\overline{W}_m(n) \approx \frac{2/\pi l}{\cosh(2n/l)}, \qquad (11)$$

where the parameter l is related to the localization length by $l = 4\pi^{-2}d$. The direct fit in Fig.2 gives $l = 45.2$, $d = 112$, and $\beta = 0.23$.

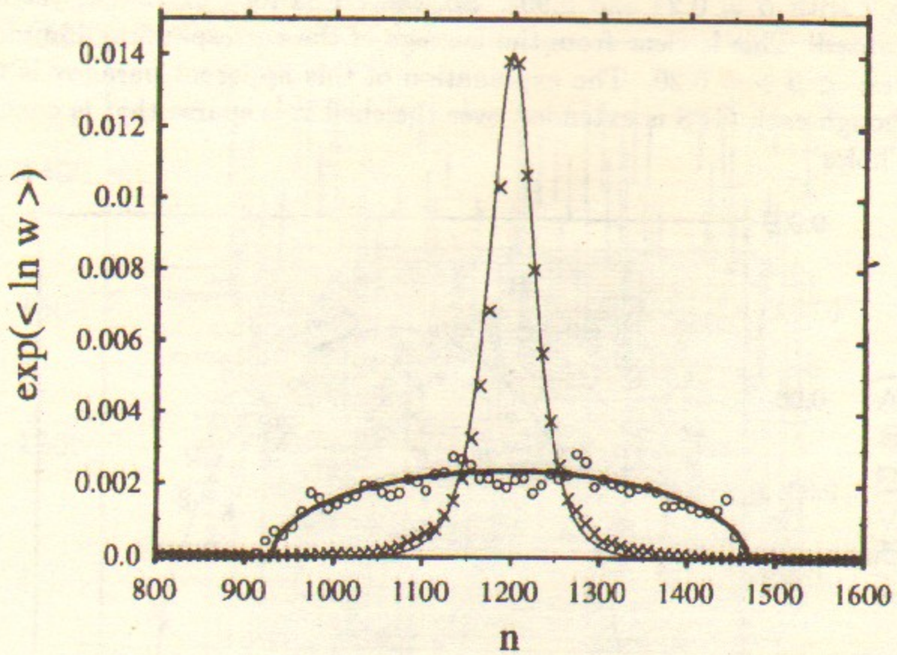

Fig. 2. Same as in Fig.1 for localized eigenfunctions ($\lambda = 0.23$) from a single matrix with parameters: $N = 2400$, $v = 0.1$, $b = 10$, $\rho = 300$, $q = 90$. The average with respect to centers $n_c(m)$ of eigenfunctions $W_m(n)$ (crosses) shows a clear localization with $\beta = 0.24$ while the other average (circles) with $\beta = 0.99$ remains close to semicircle. The thin line is empirical relation (11) with fitted $l = 45.2$.

If, instead of averaging the EFs with respect to their centers, we average them with respect to the center of the energy shell, a nice SC (with some tails) reappears (Fig.2, $\beta = 0.99$) in spite of localization. This shows that, in the average, the EFs homogeneously fill up the whole energy shell. In other words, their centers are randomly scattered whithin the shell (see also Fig.4). The latter type of averaging provides a new method for the direct empirical evaluation of ergodic $d^{(e)}$, and hence of the important localization parameters β_d and λ (9).

Now we turn to the analysis, in the case $\lambda \ll 1$, of the other type of distribution, the GFSa $w_n(m)$ which is obtained from the columns of the matrix C_{mn}. The structure of this distribution is quite different from that of EFs (represented by matrix rows). Averaging with respect to their centers or with respect to the shell center now yields similar results, which well fit the SC distribution in both cases (Fig.3, $\beta = 0.97$ and 0.99, respectively,

cf. Fig.2 with $\beta = 0.24$ and 0.99). So, the GFSa look extended, yet they are localized! This is clear from the average of the corresponding individual β–values: $< \beta >= 0.20$. The explanation of this apparent paradox is that even though each GFS is extended over the shell it is sparse that is contains many 'holes'.

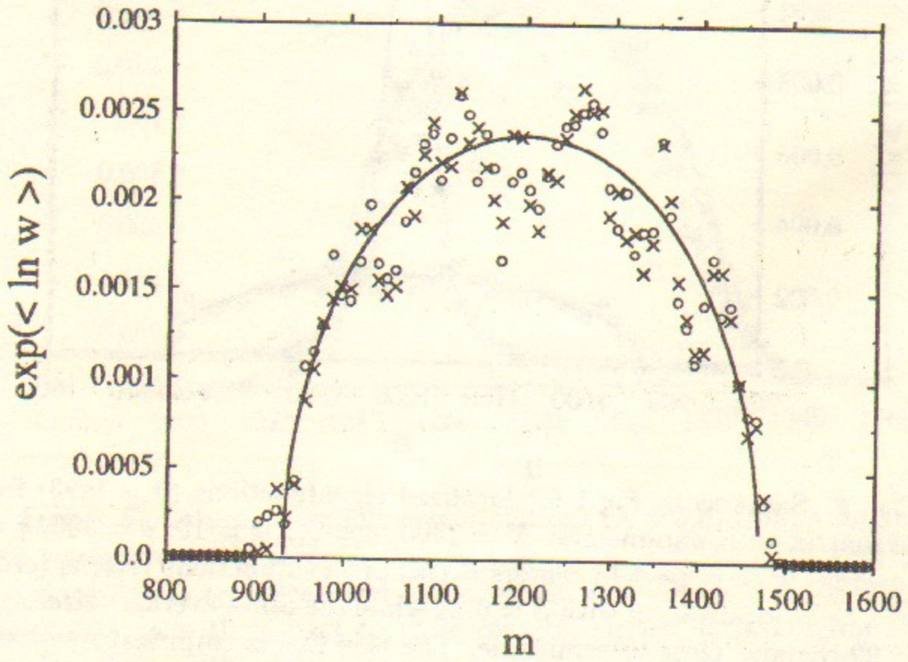

Fig. 3. Structure of the Green function spectra from a single matrix with the same parameters as in Fig.2. The same averages are shown, and unlike that case they are close to each other and to the semicircle law.

The difference in the structure between EFs and GFSa is clear from Fig.4 where solid vertical bars show the main parts of EFs, and where the GFSa are represented by horizontal dashed lines whose sparsity immediately follows from scattered and localized EFs. Our physical interpretation of this structure is the following. Spectral sparsity decreases the level density of the operative EFs (that is, the ones which are actually excited in a given initial state). This is the essential mechanism of quantum localization via decreasing the relaxation time scale [5,8]. Yet, the initial diffusion and relaxation are still classical, similar to the ergodic case, which requires extended GFSa. On the other hand, EFs are directly related to the steady–state distribution, both being solid because of the homogeneous diffusion during the statistical relaxation.

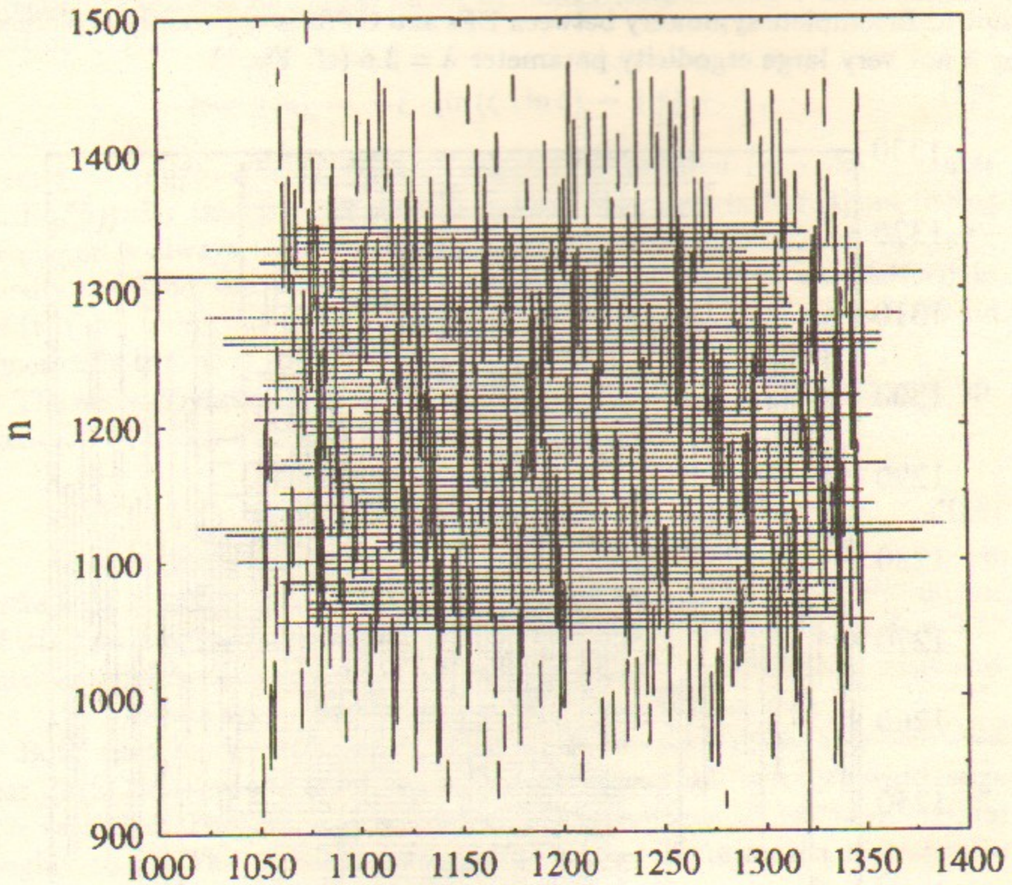

Fig. 4. A comparison of the structure of eigenfunctions and of Green function spectra in the localized case. Solid vertical bars represent the widths Δn of individual eigenfunctions over the unperturbed basis. Horizontal dotted lines show the size Δm of the Green function spectra for individual basis states. All eigenfunctions within a part of energy shell are shown while only each fifth Green function spectrum is left to distinguish the whole structure. Although all spectra have comparable sizes, close to the size of the energy shell, they are very sparse ($\beta = 0.20$) due to the strong localization and irregular scattering of eigenfunctions inside the energy shell.

For comparison, in Fig.5 the structure of ergodic EFs and GFSa is also shown. Incomplete symmetry between EFs and GFSa is apparently explained by a not very large ergodicity parameter $\lambda = 3.6$ (cf. Fig.1).

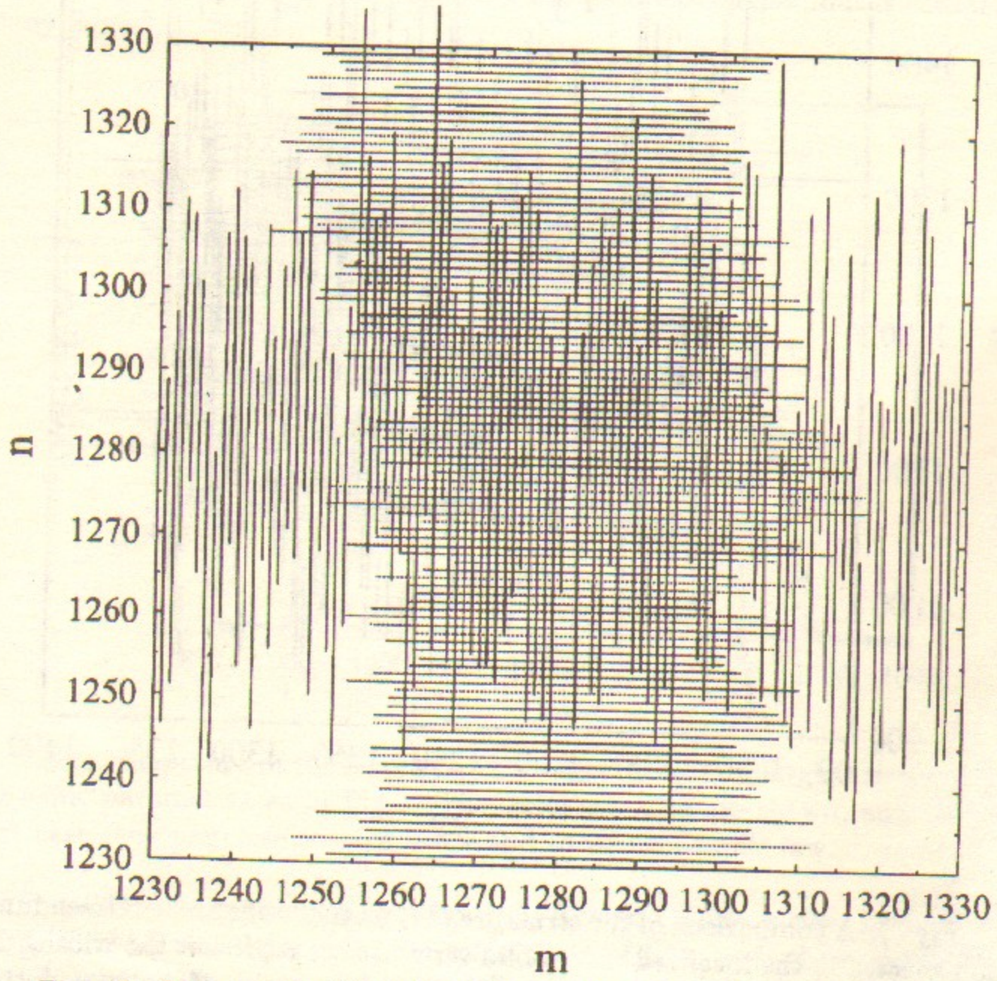

Fig. 5. Same as in Fig.4 for the ergodic case. Incomplete symmetry between eigenfunctions and Green functions spectra is apparently due to a not very large ergodicity parameter $\lambda = 3.6$.

We finally come back to the structure of the distribution tails which mark a difference from the limit SC law (5). Two somewhat different theoretical relations are currently available for the discription of the tails. Both are solutions of different versions of the original Wigner integral equation for $w(E)$ [1]. The first one was obtained in Ref.[4] and reads (with our minor

correction of the last numerical factor by numerically solving a trascendental equation in [4]):

$$\ln w_a(m) \to -\xi \cdot [\ln(\zeta \cdot \ln \zeta) - 1.6] + A\,, \tag{12}$$

where $\xi = |m|/b$, $\zeta = \xi^2/q$, and where m is counted from the w center (cf. Eq.(5)). By the virtue of the integral equation used in Ref.[4] one fitting parameter is always required (A, for a separate normalization of the tail). Usually we fixed the lower part of w_a. Actually, Eq.(12) was derived in Ref.[1] (and corrected in Ref.[4]) for the BW regime. Yet, we have found numerically that it is equally applicable in the SC region as well.

The second relation can be derived from the results in Ref.[12] and, approximately, it has the form:

$$\ln w_a(n) \to -\frac{f}{q^p} \cdot (|\Delta\xi|)^{5/4} + A\,, \tag{13}$$

where $b \cdot \Delta\xi = m \pm \rho E_{sc}$ is counted now from the SC borders. This relation has 3 fitting parameters: normalization parameter A as before, factor $f \approx 3.42$ which is slightly different from the theoretical value 2.541, and exponent $p \approx 2/15$ which is very close to the theoretical prediction 1/8.

Both relations are asymptotic ($\xi \to \infty$) but do work reasonably well even near the SC border, and are practically indistinguishable in a very wide range of w variation. The characteristic width of the exponential tail in m is equal roughly to b. The physical mechanism of the tail formation is a specific quantum tunneling via intermediate basis states [4].

In conclusion, we have found the global structure of eigenfunctions and of the Green function spectra in a generic conservative classically chaotic quantum system. We have provided numerical evidence for the existence of both the ergodic (delocalized) regime in which the eigenfunctions have maximal size and the average shape close to the semicircle law as well as the localized regime in which the size of eigenfunctions is much smaller than the semicircle width. In ergodic case statistical symmetry between eigenfunctions and Green function spectra is shown. The quantum localization was found to impose a crude asymmetry in that the eigenfunctions, remaining solid, become short and randomly scattered within the semicircle energy shell while the Green function spectra remain extended over the whole shell but become sparse. Our results suggest similar investigations for the realistic Hamiltonians of conservative classically chaotic quantum systems.

The authors benefitted from interesting discussions with P.G. Silvestrov.
Support by the NATO linkage grant LG930333 and by I.N.F.M. is acknowledged. Partial support by grant RB7000 is also gratefully acknowledged by F.M.I.

B.V.C. and F.M.I. are grateful to their colleagues in the University of Milano at Como for the hospitality during the completion of this work.

References

[1] E.Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957).

[2] T. Brody, J. Flores, J. French, P. Mello, A. Pandey, and S. Wong, Rev. Mod. Phys. **53**, 385 (1981); M. Mehta, *Random Matrices: An Enlarged and Revised Second Edition* (Academic Press, New York, 1991).

[3] B.V. Chirikov, Phys. Lett. A **108**, 68 (1985).

[4] V.V. Flambaum, A.A. Gribakina, G.F. Gribakin, and M.G. Kozlov, Phys. Rev. A **50**, 267 (1994).

[5] G. Casati and B.V. Chirikov, The legacy of chaos in quantum mechanics, in: *Quantum Chaos: Between Order and Disorder*, edited by G. Casati and B.V. Chirikov (Cambridge University Press, Cambridge, 1995).

[6] V.V.Flambaum, F.M.Izrailev and G.Casati, preprint DYSCO58 Como 1995.

[7] G. Casati, B.V. Chirikov, I. Guarneri and F.M. Izrailev, Phys. Rev. E **48**, R1613 (1993).

[8] G. Casati and B.V. Chirikov, Physica D **86**, 220 (1995).

[9] Y.V. Fyodorov, O.A.Chubykalo, F.M.Izrailev, G.Casati, Wigner random banded matrices with sparse structure: Local spectral density of states, preprint DYSCO 95.

[10] D.M.Leitner and M.Feingold, J.Phys. A **26**, 7367 (1993).

[11] M. Feingold, D. Leitner, M. Wilkinson, Phys. Rev. Lett. **66**, 986 (1991); J. Phys. A **24**, 1751 (1991).

[12] P.G. Silvestrov, Statistics of quasi 1D Hamiltonian with slowly varying parameters: Painleve again, preprint Budker INP 95–21, Novosibirsk, 1995.

G. Casati, B.V. Chirikov,
I. Guarneri and F.M. Izrailev

Quantum ergodicity and localization in generic conservative systems: the Wigner Band Random Matrix model

И. Гварнери, Ф.М. Израйлев,
Дж. Казати, Б.В. Чириков

Квантовая эргодичность и локализация в консервативных системах: модель полосковых случайных матриц Вигнера

Budker INP 95-98

Ответственный за выпуск С.Г. Попов
Работа поступила 22.11 1995 г.

Сдано в набор 4.12. 1995 г.
Подписано в печать 4.12 1995 г.
Формат бумаги 60×90 1/16 Объем 1,3 печ.л., 1,0 уч.-изд.л.
Тираж 250 экз. Бесплатно. Заказ N 98

Обработано на IBM PC и отпечатано на
ротапринте ГНЦ РФ "ИЯФ им. Г.И. Будкера СО РАН",
Новосибирск, 630090, пр. академика Лаврентьева, 11.