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НОВОСИБИРСК

Order $(Z\alpha)^4 \frac{m}{M} R_\infty$ Correction to Hydrogen Levels

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Abstract

The first in m/M and fourth in $Z\alpha$ (pure recoil) correction to a hydrogen energy level is found. This correction comprises two contributions, one coming from the atomic scale, the other from distances of the Compton wavelength order. Two different perturbation schemes are used to calculate the former. One of them exploits as unperturbed the solution to the Dirac-Coulomb problem, the nucleus' slow motion being the source of the perturbation. The alternative scheme treats both the electron and the nucleus as slow, while relativistic effects are considered perturbatively. The short-distance contribution is found in the Feynman and Coulomb gauges. Recent results for P levels are confirmed, in contrast with those for S levels. Numerically, the shift equals 2.77 kHz for the ground state and 0.51 kHz for $2S$ state.

1 Introduction

To determine the proton charge radius with a percent accuracy from the value of the hydrogen Lamb shift, the latter should be known both experimentally and theoretically with a precision of one kHz. Recently completed calculation of the order $m\alpha^2(Z\alpha)^5$ corrections [1] leaves among the effects of possible phenomenological interest the pure recoil correction of the order $m^2(Z\alpha)^6/M$ arising due to interference between the nucleus' recoil and relativistic effects in the motion of the electron. The present paper is devoted to the calculation of this correction for an arbitrary state of the hydrogen atom.

Recently this correction for P states was found [2]. In those states, as well as in all states with nonzero angular momenta, the correction proves to be saturated by a contribution coming from the atomic scale. Hence one can use there the standard quantum mechanical perturbation theory for the effective operators describing relativistic effects. Matrix elements of the effective operators arising in the perturbation theory converge at small distances thus testifying *a posteriori* that the used 'nonrelativistic' approach is correct at the given order of α for states with nonzero l.

An attempt to apply the same approach to S states, whose wavefunctions do not vanish at the origin, leads to matrix elements diverging at small distances. In fact, among the effective operators one finds those depending on r as r^{-3} and even r^{-4} [2]. As for the latter, for S states the operator r^{-4} is equivalent (modulo a nonsingular operator) to the sum of operators with the radial dependence r^{-3} and $\delta(\vec{r})/r$. It was shown in Ref.[3] that logarithmically divergent contributions are mutually cancelled. This cancellation means that for the states with vanishing angular momentum, the correction

we discuss splits naturally into two contributions – those of large and small distances – each gaining its value in its own scale. To calculate the former, one can use again the nonrelativistic approach, whereas the short distance contribution, residing in the Compton wavelength order scale, calls for a true relativistic approach.

The closed expression for the first recoil correction to an energy of the relativistic electron moving in the Coulomb field is outlined in Sec.2. This expression is used in Sec.3 for the evaluation of the long-distance contribution. It proves that the relativistic approach is more efficient even at the atomic scale. The contribution of short distances is found in Sec.4 employing the Feynman gauge. Sec.5 is devoted to checks of the obtained results. The long-distance contribution is recalculated there using the nonrelativistic approach, while the short-distance one is found in the Coulomb gauge. Finally, in Sec.6, we give the numerical values for the energy shifts and compare results of the present work with those obtained earlier in [2, 4].

Throughout the paper the relativistic units $\hbar = c = 1$ are used. Since we do not discuss radiative corrections, Z is also set equal to unity.

2 Methods of Calculation

One of the perturbation schemes we use at the present work starts from the Schrödinger equation in the Coulomb field, when both particles are considered as nonrelativistic in the zeroth approximation. To account for relativistic effects, a kind of the operator product expansion is built by calculation of scattering amplitudes for free relativistic particles. Thus arising effective operators are then the subject for the ordinary perturbation theory of the nonrelativistic quantum mechanics. This approach is rather well suited for long-distance contributions, which are due to effective operators saturated by nonrelativistic region and having non-local kernels. Formerly it was used in calculations of i) logarithmic in α corrections to the spectrum of the two-body system [5], ii) the order $m\alpha^6$ corrections to the positronium P levels [6], and iii) the order $m^2\alpha^6/M$ corrections to the hydrogen P levels [2]. Unfortunately, this approach becomes very tedious being applied to short-distance contributions, when effective operators with local kernels are represented by a number of diagrams.

An alternative approach deals with relativistic light particle (electron) moving in the field generated by the slow heavy one (nucleus). In the zeroth approximation, the heavy particle holds still being a source of the Coulomb field. Wavefunction of the system reduces to that of the light particle satis-

fying the Dirac equation. To first order in the heavy particle's inverse mass, the perturbation operator coincides with its nonrelativistic Hamiltonian:

$$V = \frac{\left(\vec{P} - |e|\vec{A}(\vec{R})\right)^2}{2M}. \qquad (1)$$

Here $\vec{P}$ is the operator of a nucleus' momentum. The vector potential $\vec{A}$ acts at the nucleus' site.

Unfortunately, one cannot calculate an energy shift induced by the perturbation (1) straightforwardly, i.e. taking merely its average. In fact, the operator (1) depends on the nucleus' dynamical variables while the argument of the unperturbed wavefunction is a position or momentum of the electron. To overcome this difficulty we use the gauge invariance of observables in the quantum electrodynamics [7]. Being reexpressed in terms of electron's variables, the average value of (1) should be retained gauge invariant. The new form of the average is now nearly evident:

$$\Delta E_{rec} = -\frac{1}{M}\int \frac{d\omega}{2\pi i}\left\langle \left(\vec{p} - \vec{D}\right) G_{E+\omega}\left(\vec{p} - \vec{D}\right)\right\rangle. \qquad (2)$$

Here $\vec{p}$ is the electron momentum operator, $\vec{D}$ is the integral operator describing the transverse quantum exchange. It has the kernel[1]

$$\frac{4\pi\alpha\vec{\alpha}_k}{k^2 - \omega^2}, \qquad \vec{\alpha}_k \equiv \vec{\alpha} - \frac{\vec{k}(\vec{\alpha}\vec{k})}{k^2}.$$

In (2), G is the Green's function for the Dirac equation in the Coulomb field, the average is taken over a Dirac-Coulomb eigenstate with an energy E. Actually, the "seagull" part of (2), that of the second order in $\vec{D}$, emerges naturally as a counterpart of the $\vec{A}^2$ term from (1) when we take the expectation value of this term over fluctuations of the electromagnetic field. All the other terms provide the invariance of (2) with respect to the gauge transformation,

$$\psi \to \exp(i\phi(\vec{r}))\psi, \qquad \vec{D} \to \vec{D} + i[\vec{p}, \phi]. \qquad (3)$$

One can easily convince oneself that just the same result can be obtained from the formula (11) of [7] with the help of the Dirac equation.

[1] In what follows we will write often a kernel rather than an appropriate operator for the sake of brevity.

The first attempt to obtain the relativistic expression for the recoil correction to hydrogen energy levels was undertaken in Ref.[8]. Complete expressions for various contributions in the Coulomb gauge were originally derived in the framework of the quasipotential approach in Ref.[9]. The sum of those contributions can be convinced to reduce to the right-hand side of (2).

3 Long-Distance Contribution

Present section is devoted to the calculation of the contribution to the energy shift which is saturated by the atomic scale and can thus be called the long-distance one. To check the results, two approaches described above have been applied in parallel. Here we describe in detail how the second, relativistic, approach works. The procedure of comparison with the results of the more cumbersome nonrelativistic approach will be postponed until the Section 5.

3.1 Pure Coulomb Contribution

In the relativistic approach the pure Coulomb contribution,

$$\Delta E_C = -\frac{1}{M}\int\frac{d\omega}{2\pi i}\,\langle\vec{p}G_{E+\omega}\vec{p}\rangle = \frac{1}{2M}\,\langle\vec{p}(\Lambda_+ - \Lambda_-)\,\vec{p}\rangle\,, \tag{4}$$

can naturally be represented as the sum of two terms [8],

$$\Delta E_C = \left\langle\frac{p^2}{2M}\right\rangle - \frac{1}{M}\,\langle\vec{p}\Lambda_-\vec{p}\rangle\,. \tag{5}$$

Here Λ_+ and Λ_- are the projection operators to sets of positive- and negative-energy Dirac-Coulomb eigenstates respectively. With the aid of the Dirac equation the mean value of $p^2/2M$ can readily be reexpressed in the following form [9]:

$$\left\langle\frac{p^2}{2M}\right\rangle = \frac{m^2-E^2}{2M} + \frac{m^2}{2M}\left\langle 2\left(\frac{E}{m}-\beta\right)\frac{\alpha}{r} + \frac{\alpha^2}{r^2}\right\rangle. \tag{6}$$

As for the second term in (5), responsible for virtual transitions into negative-energy states, the simple analysis shows that it doesn't contribute to the order of interest at the atomic scale. Actually, the trivial power counting on the right-hand side of the obvious inequality,

$$|\,\langle\vec{p}\Lambda_-\vec{p}\rangle\,| < \left|\frac{1}{4m^2}\,\langle[\vec{p},C]\Lambda_-[\vec{p},C]\rangle\right|, \tag{7}$$

where C is the Coulomb potential, shows that at the atomic scale, the product of commutators is already of the sixth order in α, so that the projector and the wavefunctions can sufficiently be replaced by their nonrelativistic counterparts. Since there is no negative-energy states in the nonrelativistic approximation, the atomic scale contribution to the initial average also vanishes in the order we consider.

3.2 Magnetic Contribution

After performing the integration over ω, the expression for the single transverse, or magnetic, contribution,

$$\Delta E_M = \frac{1}{M}\int\frac{d\omega}{2\pi i}\left\langle\vec{p}G\vec{D} + \vec{D}G\vec{p}\right\rangle, \tag{8}$$

turns into

$$\Delta E_M = -\frac{\alpha}{M}\mathrm{Re}\left\langle\vec{p}\left(\sum_+\frac{|m\rangle\langle m|}{k+E_m-E} - \sum_-\frac{|m\rangle\langle m|}{E-E_m+k}\right)\frac{4\pi\vec{\alpha}_k}{k}\right\rangle, \tag{9}$$

where $\sum_+$ denotes the sum over discrete levels plus the integral over positive-energy part of the continuous spectrum, while $\sum_-$ stands for the integral over negative-energy continuum.

For the transverse photon momenta in the atomic region, $k\sim m\alpha$, one can expand the first term in (9) to the power series in the ratio $(E-E_m)/k$. To zeroth order (in the approximation of the instant exchange), we have:

$$-\frac{\alpha}{M}\mathrm{Re}\left\langle\frac{4\pi\vec{\alpha}_k}{k^2}\Lambda_+\vec{p}\right\rangle = -\frac{m^2}{2M}\left\langle 2\left(\frac{E}{m}-\beta\right)\frac{\alpha}{r}+\frac{\alpha^2}{r^2}\right\rangle + \frac{\alpha}{M}\mathrm{Re}\left\langle\frac{4\pi\vec{\alpha}_k}{k^2}\Lambda_-\vec{p}\right\rangle. \tag{10}$$

The sum of the first term and (6) has very simple form [9],

$$\frac{m^2-E^2}{2M} = \frac{m^2\alpha^2}{2MN^2}, \tag{11}$$

where the standard notations for the Dirac-Coulomb problem are used,

$$N=\sqrt{(\gamma+n_r)^2+\alpha^2},\quad \gamma=\sqrt{\kappa^2-\alpha^2},$$

n_r is the radial quantum number, $\kappa=-1-\vec{\sigma}\vec{l}$. Notice that (11) reduces to the lowest order result for the states with $n_r=0$ only.

As far as we are seeking only for corrections of the even order in α, the next term of the expansion to be considered is

$$\Delta E_{ret} = -\frac{\alpha}{M}\mathrm{Re}\left\langle \vec{p}\sum_{+}(E_m-E)^2|m\rangle\langle m|\frac{4\pi\vec{\alpha}_k}{k^4}\right\rangle$$
$$= -\frac{\alpha}{M}\mathrm{Re}\left\langle [H,[H,\vec{p}]\,\Lambda_+\frac{4\pi\vec{\alpha}_k}{k^4}\right\rangle, \qquad (12)$$

where $H=\vec{\alpha}\vec{p}+\beta m+C$ is the Dirac Hamiltonian in the Coulomb field. This term describes the effect of retardation. Implying the corresponding operator by its kernel, we have

$$[H,\vec{p}]=\alpha\frac{4\pi\vec{k}}{k^2}, \qquad [H,[H,\vec{p}]]=\alpha\frac{4\pi\vec{k}(\vec{\alpha}\vec{k})}{k^2}.$$

To the lowest nontrivial order, matrix elements of $\vec{\alpha}$'s over positive-energy states can be replaced by the appropriate Pauli currents:

$$\Delta E_{ret}\approx -\frac{\alpha^2}{M}\left\langle\frac{4\pi\vec{k'}}{k'^2}\frac{2\vec{p}'\vec{k'}-k'^2}{2m}\frac{4\pi}{k^4}\frac{2\vec{p}_k+i\vec{\sigma}\times\vec{k}}{2m}\right\rangle. \qquad (13)$$

Here $\vec{k'}=\vec{q}-\vec{k}$, $\vec{q}=\vec{p}'-\vec{p}$, while $\vec{p}$ and $\vec{p}'$ are the arguments of the wavefunction and its conjugated respectively. Being converted to the spatial representation, the average above equals

$$\Delta E_{ret}=\frac{\alpha^2}{4m^2M}\left\langle -2\vec{p}\frac{1}{r^2}\vec{p}+\frac{7\vec{l}^2+2\vec{\sigma}\vec{l}}{2r^4}\right\rangle. \qquad (14)$$

Strictly speaking, in (13), the integral over $\vec{k}$ has an infrared divergent part. It is omitted from (14) since the photon momenta $k\sim m\alpha^2$ contribute to the previous order correction. To make sure that this is correct, one can regularize the divergency supplying the photon with a mass λ such that $m\alpha^2\ll\lambda\ll m\alpha$. The term proportional to $1/\lambda$ and omitted from (14) can be easily checked to cancel the respective term in the difference between (9) and the expression obtained from (9) by the replacement $k\to\sqrt{k^2+\lambda^2}$. On the other hand, just this difference determines the low-energy contribution to the order $m\alpha^5/M$ correction.

The last of contributions due to the single transverse exchange is generated by virtual transitions into negative-energy states and is covered by the

last terms in (9) and (10). The inequality similar to (7) shows that the nonrelativistic expansion of the last term in (9) starts with the seventh power of α. As for the negative-energy contribution to (10), to the lowest nontrivial order it reduces to

$$-\frac{\alpha^2}{2mM}\left\langle\frac{4\pi\vec{\alpha}_{k'}}{k'^2}\lambda_-(\vec{p}+\vec{k})\frac{4\pi\vec{k}}{k^2}\right\rangle\approx\frac{\alpha^2}{4m^2M}\left\langle\frac{4\pi\vec{\alpha}_{k'}\vec{k}}{k'^2}\frac{4\pi\vec{\alpha}\vec{k}}{k^2}\right\rangle, \qquad (15)$$

yielding in the spatial representation

$$\Delta E_{M-}=-\frac{\alpha^2}{4m^2M}\left\langle\frac{1}{r^4}-\frac{4\pi\delta(\vec{r})}{r}\right\rangle. \qquad (16)$$

Taken over S states, this average is logarithmically divergent at small distances (linear divergencies cancel each other). An ultraviolet divergency will be discussed later for the *total* long-distance contribution to the S level shift. In fact, due to the gauge dependence of an individual contribution (e. g., (16)), its divergent part alone has no physical meaning.

3.3 Seagull Contribution

Again, taking the integral over ω in the expression for the double transverse, or seagull, contribution,

$$\Delta E_S=-\frac{1}{M}\int\frac{d\omega}{2\pi i}\left\langle\vec{D}G_{E+\omega}\vec{D}\right\rangle, \qquad (17)$$

we obtain

$$\Delta E_S=\frac{\alpha^2}{2M}\left\langle\frac{4\pi\vec{\alpha}_{k'}}{k'}\sum_{+}\frac{|m\rangle\langle m|\left(1+\frac{E_m-E}{k'+k}\right)}{(E_m-E+k')(E_m-E+k)}\frac{4\pi\vec{\alpha}_k}{k}+\cdots\right\rangle, \qquad (18)$$

where the ellipsis stands for the negative-energy part which differs from the positive-energy one by the overall sign and signs before k and k'. One can easily check that the linear in $k/2m$ terms in the expansion of the negative-energy part cancel each other. But just these terms at the atomic scale could produce the energy correction of the necessary order. Hence it remains to consider the positive-energy part explicitly written in (18). In the leading nonrelativistic approximation,

$$\Delta E_{S+}=\frac{\alpha^2}{2M}\left\langle\frac{4\pi\vec{\alpha}_{k'}}{k'^2}\Lambda_+\frac{4\pi\vec{\alpha}_k}{k^2}\right\rangle, \qquad (19)$$

we again replace matrix elements of $\vec{\alpha}$'s over positive-energy states by the Pauli currents,

$$\Delta E_{S+} = \frac{\alpha^2}{2M}\left\langle \frac{4\pi}{k'^2}\frac{2\vec{p}'_{k'} + i\vec{\sigma}\times\vec{k}'}{2m}\frac{4\pi}{k^2}\frac{2\vec{p}_k + i\vec{\sigma}\times\vec{k}}{2m}\right\rangle, \tag{20}$$

and perform the Fourier transformation to obtain

$$\Delta E_{S+} = \frac{\alpha^2}{4m^2M}\left\langle 2\vec{p}\frac{1}{r^2}\vec{p} + \frac{1}{r^4} - \frac{3\vec{l}^2 + 2\vec{\sigma}\vec{l}}{2r^4}\right\rangle. \tag{21}$$

3.4 Total Long-Distance Contribution

Summing up (14), (16) and (21) we arrive at

$$\frac{\alpha^2}{4m^2M}\left\langle 2\frac{\vec{l}^2}{r^4} + \frac{4\pi\delta(\vec{r})}{r}\right\rangle. \tag{22}$$

Expanding (11) to the power series in α^2 and evaluating the average in (22) we obtain the long-distance contribution for a state with nonzero l:

$$\Delta E_{l>0} = \frac{m^2\alpha^6}{Mn^3}\left\{\frac{1}{8|\kappa|^3} + \frac{6\kappa}{|\kappa|(4\kappa^2-1)(2\kappa+3)} + \frac{3}{8n\kappa^2} - \frac{1}{n^2|\kappa|}\left(1 + 2\frac{\kappa^2(\kappa+1)}{(4\kappa^2-1)(2\kappa+3)}\right) + \frac{1}{2n^3}\right\}. \tag{23}$$

For $l = 1$ it reproduces the result of the paper [2]. As previously mentioned, the effective operators of the order under consideration contain singularities insufficient to compensate the vanishing of a wavefunction with nonzero l at $r \to 0$. That is why (23) is the total sought-for correction to $l > 0$ levels.

For S states the first term in (22) evidently vanishes due to the angular momentum operator $\vec{l}$ annihilating their wavefunctions. It is interesting to note that a naïve generalization of the result for states with nonzero l to S ones leads to the error – the vanishing of the angular average is compensated by the linear divergency of the radial one.

As for the second term, which formally contains the linear divergency, it is just a remnant of the short-distance contribution to the previous (fifth in α) order correction. To make sure that nothing is lost in the sixth order, let us regularize the ultraviolet divergency by subtraction of the potential

generated by the massive transverse exchange (the photon mass $\lambda \gg m\alpha$), from the potential of the ordinary transverse exchange entering (15):

$$\frac{4\pi\vec{\alpha}_{k'}}{k'^2} \to \frac{4\pi\vec{\alpha}_{k'}}{k'^2} - \frac{4\pi\vec{\alpha}_{k'}}{k'^2+\lambda^2}. \tag{24}$$

By going to the spatial representation we obtain the regularized version of the singular operator:

$$\frac{4\pi\delta(\vec{r})}{r} \to \frac{2\lambda}{3}4\pi\delta(\vec{r}).$$

Being averaged, it gives the energy correction of the order $m\lambda\alpha^5/M$. The latter should be cancelled by the linear in λ term in the expansion of the short-distance contribution to the order $m^2\alpha^5/M$ correction, calculated with the massive propagator of the transverse quantum (actually the expansion parameter is $\lambda/m \ll 1$). Along with linear in λ/m correction, one could expect the correction linear in $\alpha = m\alpha/m$. However the expansion parameter at large distances is $(p/m)^2 \sim \alpha^2$ so that the operator we discuss does not contribute to the order of interest. On the other hand, a linear in α correction to a local ($\propto \delta(\vec{r})$) operator can arise as an ordinary radiative one. In this case the correction is completely saturated by small distances. The next section is devoted to the calculation of such corrections.

So, in S states the long-distance contribution is exhausted by the $m^2\alpha^6/M$ term from the expansion of (11),

$$\Delta E^{ld}_{l=0} = \frac{m^2\alpha^6}{2Mn^3}\left(\frac{1}{4} + \frac{3}{4n} - \frac{2}{n^2} + \frac{1}{n^3}\right). \tag{25}$$

It vanishes in the ground state only.

4 Short-Distance Contribution

As we have mentioned in the Introduction, the long-distance contribution for S states is supplied by a short-distance one residing at scales of the Compton wavelength order. Since two contributions are well separated, each of them is gauge invariant, so that evaluating the short-distance one we can use a different, more appropriate gauge. Rather naturally the mostly convenient gauge is the Feynman one. The main formula (2) can be rewritten in this gauge by application of the Dirac equation or directly from the eq.(11) of [7]:

$$\Delta E_{tot} = -\frac{\alpha^2}{M}\int\frac{d\omega}{2\pi i}\left\langle\frac{4\pi}{k'^2+\lambda^2-\omega^2}\left(\vec{\alpha}+\frac{\vec{k'}}{\omega}\right)G_{E+\omega}\left(\vec{\alpha}-\frac{\vec{k}}{\omega}\right)\frac{4\pi}{k^2+\lambda^2-\omega^2}\right\rangle. \quad (26)$$

Momenta of the photons are assumed to flow both from the nucleus to the electron. The photon mass λ is introduced to establish control over infrared divergences reminiscent of lower-order and long-distance contributions. Those divergencies arise in the process of the approximate evaluation of the integrals in (26).

Taking the wavefunctions at the origin and replacing the Green's function by the first term of its expansion in the Coulomb field, we have

$$\Delta E_G = \frac{\alpha^3\psi^2}{M}\int\frac{d\omega}{2\pi i}\left\langle\frac{4\pi}{p'^2-\sqrt{}^2}\left(\vec{\alpha}-\frac{\vec{p'}}{\omega}\right)\frac{m+\omega+\beta m+\vec{\alpha}\vec{p'}}{p'^2-\Omega^2}\frac{4\pi}{q^2}\frac{m+\omega+\beta m+\vec{\alpha}\vec{p}}{p^2-\Omega^2}\left(\vec{\alpha}-\frac{\vec{p}}{\omega}\right)\frac{4\pi}{p^2-\sqrt{}^2}\right\rangle. \quad (27)$$

Here $\psi^2 \equiv |\psi(0)|^2$, the angle brackets denote integrations over $\vec{p}$ and $\vec{p'}$; $\vec{q} = \vec{p'} - \vec{p}$; and

$$\sqrt{} \equiv \sqrt{\omega^2-\lambda^2}, \qquad \Omega \equiv \sqrt{2m\omega+\omega^2}.$$

Contrary to the case of large distances, in the deep relativistic region the opposite order of integration is suitable – first over $\vec{p}$ and $\vec{p'}$, and then over ω. As for the former, it becomes rather trivial after conversion to the spatial representation. Preparatory to such the conversion, it is convenient to express all the scalar products containing different momenta in terms of their squares. Then some of the denominators can be cancelled. At this point we can drop those terms which do not contain Ω in their denominators. In fact, the only scale leaving in such terms is λ so that they cannot produce a short-distance contribution. In the spatial representation the initial two-loop integral with zero external momenta turns into a simple one-dimensional integral over r. The contour of the resulting ω-integration encloses the cut between the points $-2m$ and $-\lambda$ in the complex plane. After this last integration we obtain

$$\Delta E_G = \frac{\pi\alpha^3\psi^2}{Mm}\left(\frac{1}{\varepsilon}-\frac{8}{3\pi\sqrt{\varepsilon}}\int_1^\infty\frac{dx}{\sqrt{x(x^2-1)}}+4\ln 2-\frac{5}{2}\right), \quad (28)$$

where $\varepsilon = \lambda/2m$.

If the Green's function is taken to zeroth order in the interaction, we have to use the Dirac equation in order to account for the momentum dependence in the wavefunction:

$$\Delta E_\psi = 2\frac{\alpha^3\psi^2}{M}\int\frac{d\omega}{2\pi i}\left\langle\frac{4\pi}{p'^4}(2m+\vec{\alpha}\vec{p'})\frac{4\pi}{q^2-\sqrt{}^2}\left(\vec{\alpha}+\frac{\vec{q}}{\omega}\right)\frac{m+\omega+\beta m+\vec{\alpha}\vec{p}}{p^2-\Omega^2}\left(\vec{\alpha}-\frac{\vec{p}}{\omega}\right)\frac{4\pi}{p^2-\sqrt{}^2}\right\rangle. \quad (29)$$

Using the same procedure, we take $-r/2$ as the Fourier transform of $4\pi/p^4$. The linearly divergent constant we thus leave aside is actually proportional to $1/\alpha$ and contributes to the previous order correction. The result of the integration is

$$\Delta E_\psi = \frac{\pi\alpha^3\psi^2}{Mm}\left(-\frac{1}{2\varepsilon^2}+\frac{1}{\varepsilon}-\frac{8}{3\pi\sqrt{\varepsilon}}\int_1^\infty\frac{dx}{\sqrt{x(x^2-1)}}\right). \quad (30)$$

Regulator-dependent terms in (28) and (30) arise from the integrals saturated by the region of momenta $p \sim \lambda$ and frequency $\omega \sim \lambda$ (or $\sqrt{m\lambda}$) and are thus the remnants of the previous orders corrections or of the long-distance contribution. Truly relativistic contribution comes from the region $p \sim \omega \sim m$ and does not depend on the infrared cutoff:

$$\Delta E^{sd} = \frac{m^2\alpha^6}{Mn^3}\left(4\ln 2-\frac{5}{2}\right)\delta_{l0}. \quad (31)$$

It is pertinent to note here that this result is truly short-distance, i. e. it does not contain hidden long-distance contributions, which naïvely could arise due to cancellation of the same nonzero powers of λ from numerator and denominator – all positive powers of the photon mass were dropped out in the process of calculation. On the other hand, an emergence of such contributions would be self-contradictory. Actually, if an integral is saturated by distances of $1/\lambda$ order, then at $p \sim \lambda$, the integrand denominator has at least one power of momentum more than the product of the numerator and the measure of integration. In other words, any "long-distance" contribution (determined by the scale of λ) *has* to contain a positive power of the photon mass in its denominator.

5 Checking of the Results

5.1 Long-Distance Contribution

To be certain that the long-distance contributions are found correctly, all of them were rederived in the framework of the nonrelativistic approach which exploits the Schrödinger equation as a starting point. For the states with nonzero angular momenta we used the following procedure. All the contributions prove to have the same analytic structure in κ, namely

$$\Delta E = \frac{m^2\alpha^6}{Mn^3}\Sigma,$$

where

$$\begin{aligned}\Sigma &= \frac{1}{|\kappa|^3}\left(a_\infty + \frac{a_{1/2}}{\kappa-1/2} + \frac{a_{-1/2}}{\kappa+1/2} + \frac{a'_{-1/2}}{(\kappa+1/2)^2} + \frac{a_{-1}}{\kappa+1} + \frac{a_{-3/2}}{\kappa+3/2}\right)\\ &+\frac{1}{n\kappa^2}\left(b_\infty + \frac{b_{-1/2}}{\kappa+1/2}\right)\\ &+\frac{1}{n^2|\kappa|}\left(c_\infty + \frac{c_{1/2}}{\kappa-1/2} + \frac{c_{-1/2}}{\kappa+1/2} + \frac{c_{-3/2}}{\kappa+3/2}\right) + \frac{d}{n^3},\end{aligned} \tag{32}$$

Constants a, b, c, d evaluated in the nonrelativistic approach for individual contributions as their asymptotic values at $\kappa\to\infty$ or residues at corresponding poles were then compared with the respective results obtained in the relativistic approach. In the process of comparison, a number of 'nonrelativistic' contributions breaks down into groups according to the meaning of respective 'relativistic' ones. For example, the retardation part of the magnetic contribution (14) comprises three terms in the nonrelativistic approach: $\Delta E^{(1)}_{MC}$, $\Delta E^{(1)}_{MCC}$ and $\Delta E^{(1)}_{ret}$ (notations are from ref.[2]).

As we mentioned earlier, S states should be treated separately in order to avoid fictious contributions arising due to the compensation between vanishing angular averages and linearly divergent radial ones. All the ultraviolet divergencies in S states are checked to cancel each other. To this end we regularize the effective potentials which are too singular at $r\to 0$ and ensure that the total long-distance contribution for S states is independent of the regularization parameter.

5.2 Short-Distance Contribution

In order to compare the results of the present work with those of [4], the short-distance contribution was calculated using the Coulomb gauge also.

A mass of the magnetic quantum was used as the infrared regulator. The scheme of calculation is completely analogous to those used previously in the case of the Feynman gauge. The short-distance contributions are:

$$C_G = C_\psi = \frac{\pi\alpha^3\psi^2}{Mm}\frac{1}{2}; \tag{33}$$

$$M_G = -\frac{\pi\alpha^3\psi^2}{Mm}\left(\ln\frac{\lambda}{2m} + \frac{3}{2}\right), \tag{34}$$

$$M_\psi = -\frac{\pi\alpha^3\psi^2}{Mm}\ln\frac{\lambda}{2m}; \tag{35}$$

$$S_G = \frac{\pi\alpha^3\psi^2}{Mm}(4\ln 2 - 2), \tag{36}$$

$$S_\psi = \frac{\pi\alpha^3\psi^2}{Mm}2\ln\frac{\lambda}{2m}. \tag{37}$$

Here C, M, S denote Coulomb, magnetic and seagull contributions respectively. It is easy to check that the sum of these contributions coincides with (31).

6 Conclusion

Numerically, the correction to the energy equals 2.77 kHz for the ground state and 0.51 kHz for the $2S$ state. Being somewhat less than the naïve estimate ($m^2\alpha^6/M \approx 10.2$ kHz) it nevertheless is quite comparable with the accuracy of the near future measurements. It is also interesting to note that the corrections to $2S$ and $2P$ levels (with the radiative-recoil correction to $2P$ level [2] taken into account) are rather close to each other, so that the correction to their difference, 0.04 kHz, can be considered as negligibly small at the present level of the experimental accuracy [10].

Let us now set up a correspondence between the results of the present work and those of the other papers. The result for $l > 0$ levels appears to be firmly established [2, 11]. For S levels our result is contradictory to the recent results of the analytic [4] and numerical [11] calculations.

To elucidate the origin of the disagreement, we consider the correction to the ground state energy. It is easy to verify that our short-distance results

(33)–(37) are in one-to-one correspondence with respective "high-energy" contributions from [4]. A similar statement is true for long-distance contributions (low- and intermediate-energy ones in notations of [4]), with one exception. The coefficient –2 from Eq.(68) of Ref.[4] for the intermediate-energy contribution to the retarded exchange by the magnetic quantum, differs from our result, –1 (in the same units $m^2\alpha^6/M$), which arises after trivial averaging in (14) over the ground state. Unfortunately, we have not managed to reproduce the coefficient –2 starting from Eq.(67) of Ref.(14). Furthermore, several arguments can be brought forward, that the result (68),[4] for the retardation contribution looks at least suspicious. In particular, the logarithmic divergency in the order $m^2\alpha^6/M$ is known to appear due to the relativistic corrections to the *instant* transverse exchange. The result of the present work concerning the origin of this logarithmic divergency and the value of the corresponding coefficient is contained in (16) and is in complete agreement with those of Refs.[5] and [12]. As for the effect of retardation, it gives rise to the finite contribution only (in accord with (14)). But it follows from the result of Ref.[4] that just the retardation is the source of not only logarithmic, but even the linear divergency at small distances, while the long-distance relativistic correction to the instant transverse exchange does not contribute at all.

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Order $(Z\alpha)^4 \frac{m}{M} R_\infty$ Correction to Hydrogen Levels

А.С. Елховский

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