Instanton – Antiinstanton interaction and asymptotics of perturbation theory expansion in double well oscillator

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ABSTRACT

Instanton – antiinstanton pair is considered as a source of singularity at the Borel plane for the ground state energy of anharmonic oscillator. The problem of defining the short range instanton – antiinstanton interaction reduces to calculation of a smooth part of the Borel function, which cannot be found without explicit calculation of several terms of ordinary perturbation theory. On the other hand, the large order terms of perturbative expansion are dominated by large fluctuations in the functional integral like well separated instanton and antiinstanton.

The preasymptotics ($\sim 1/n$) of large order perturbation theory contribution to the ground state energy of anharmonic oscillator was found analytically. To this end the subleading long range asymptotics of the classical instanton – antiinstanton interaction, the one – loop quantum contribution to instanton – antiinstanton interaction and the second quantum correction to a single instanton density were considered.



## 1 Introduction

In this note we would like to consider how the large fluctuations in the Euclidean functional integral contribute to the ground state energy of double well anharmonic oscillator

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2(1-gx)^2 \quad . \tag{1}$$

Our practical aim will be the analytical derivation of the $\sim 1/n$ correction to the instanton – antiinstanton pair induced asymptotics of the perturbative expansion:

$$E = \sum E_n g^{2n} \; , E_n = -\frac{3^{n+1} n!}{\pi}\left(1 - \frac{53}{18n} + \ldots\right) . \tag{2}$$

After the works of L.N.Lipatov [1] it is generally recognized, that classical paths in the Feynman functional integral contribute significantly to the asymptotics of the ordinary perturbation theory. However the attraction between instanton and antiinstanton makes the high order estimates for the double well oscillator (1) somewhat more complicated. The usual trick to avoid this problem is the analytical continuation of the instanton – antiinstanton contribution over the coupling constant [2].

In present paper we prefer not to make the ambiguous analytical continuation at the intermediate steps of approximate calculation. Following t'Hooft [3] (see also [4]) we transform the functional integral to the Borel type integral by considering the action as a collective variable. In this approach the instanton – antiinstanton pair manifests itself as the singularity of the Borel transform. The problem of short range instanton – antiinstanton interaction reduces to that of calculation the shape of Borel transform far from singular point. On the other hand the smooth part of the Borel function is sensitive mostly to a first few terms of perturbative expansion. Thus not only instanton helps one to find the asymptotics of perturbation theory, but also the precise calculation of several terms of perturbative expansion allows to avoid the problem of short range instanton – antiinstanton attraction.

The type of singularity at the Borel plane is determined by interaction between the instanton and antiinstanton. Therefore one have to go beyond the dilute gas approximation in order to find the asymptotics of perturbative expansion. Up to now all authors have taken into account only the leading $\sim e^{-T}/g^2$ term of the classical instanton – antiinstanton interaction ($T$ is the distance between the pseudoparticles). In order to find the $\sim 1/n$ correction to (2) we have to consider the subleading $\sim Te^{-2T}/g^2$ correction to classical instanton – antiinstanton interaction, the one loop quantum correction $\sim Te^{-T}$ to the interaction and the second quantum correction $\sim g^2$ to the single instanton density.

## 2 Asymptotics of the perturbative expansion and nonperturbative effects

The ground state energy for the Hamiltonian (1) is given by the functional integral

$$E(g) = \lim_{L\to\infty} -\frac{1}{L}\ln Tre^{-HL} \quad , \quad Tre^{-HL} = N\int Dx(t)\, e^{-S} \ ,$$
$$S = \int_{-\frac{L}{2}}^{\frac{L}{2}} dt\left(\frac{1}{2}\dot{x}^2 \ + \ \frac{1}{2}x^2(1-gx)^2\right) . \tag{3}$$

Instanton – the well known classical path reads:

$$x_I(t) = \frac{1}{g(e^t+1)} \quad , \quad S[x_I] = \frac{1}{6g^2} \ . \tag{4}$$

This path connects two distinct vacua of the action (3) $x = 0$ and $x = 1/g$. We want to find the configurations, which make a large contribution to the high order terms of perturbation theory and therefore may be considered as a fluctuation (although large) around the trivial vacuum $x = 0$. The example



of such topologically trivial fluctuation is the instanton – antiinstanton pair. The pair is not the exact solution of the equation of motion and therefore we have to understand, what the specific configuration may be called the instanton – antiinstanton pair? But to a first approximation the interaction between pseudoparticles is model independent for a wide class of reasonable configurations (see e.g. [5])

$$S[x_{I-A}] = \frac{1}{g^2}(1/3 - 2e^{-T} + \cdots) \ . \tag{5}$$

The instanton – antiinstanton contribution to $Tre^{-HL}$ is given by:

$$(Tre^{-HL})^{I-A} \simeq L\int \frac{dT}{\pi g^2} exp\{-\frac{1}{g^2}(1/3 - 2e^{-T} + \cdots)\} \ . \tag{6}$$

Factor $L$ appears after integration over the center of the pair and $(\pi g^2)^{-1}$ comes from the square of single instanton density (see [6]). In order to find the ground state energy one have to sum over the "dilute gas of instanton – antiinstanton pairs". Hence the energy reads:

$$E(g) = -\int \frac{dT}{\pi g^2} exp\{-\frac{1}{g^2}(1/3 - 2e^{-T})\}. \tag{7}$$

This integral diverges at large $T$. The divergency will disappear if for large $T$ one replaces the summation over dilute gas of pairs by summation over dilute gas of individual pseudoparticles. Nevertheless even the expression (7) carries quite sufficient information about high orders of perturbation theory. Consider new variable $x = g^2 S$, where $S$ – is a classical action of the instanton – antiinstanton pair. The value of $x$ varies from $x = 0$ (the classical vacuum) to $x = 1/3$ and for widely separated pseudoparticles $x = 1/3 - 2e^{-T} + \ldots$ . Now formula (7) takes the form:

$$E = \int_0^{1/3} F(x) e^{-x/g^2}\frac{dx}{\pi g^2} \ , \ F(x) \approx -\frac{1}{(1/3 - x)}(1 + O(g^2)) + \ldots \ . \tag{8}$$

About the function $F(x)$ we know only that it has a pole at $x = 1/3$. However just the singularities of $F(x)$ at complex $x$ plane determine the behaviour of series $F(x) = \sum F_n x^n$ at large $n$ and thus the asymptotics of perturbative expansion of $E(g)$. The only assumption that pole at $x = 1/3$ (8) is the closest to origin singularity of $F(x)$ allows one to find the well known [7] leading term of the asymptotic expansion (2). Below we will find the weaker (logarithmic) singularity of function $F(x)$ at the same point $x = 1/3$. There is no direct way to calculate the smooth function $F(x)$ at $x < 1/3$, but after one have calculated precisely several terms of $E(g)$ perturbative expansion, the smooth part of $F(x)$ carries now new information.

At very large $T$ the dilute gas of many pseudoparticles is to be considered. In order to eliminate the double counting one have to subtract from (7) the contribution of noninteracting instanton and antiinstanton:

$$E_{\pm} = -\left(\int_{S(T)=0}^{T=\infty} \frac{dT}{\pi g^2} e^{-S(T)} - \int_{T=0}^{T=\infty} \frac{dT}{\pi g^2} e^{-1/3g^2}\right) \mp \frac{e^{-1/6g^2}}{g\sqrt{\pi}} . \quad (9)$$

Here the last term describes the nonperturbative splitting of the ground state due to tunneling through the barrier. Two signs correspond to the states having different parity with respect to transformation $x \to 1/g - x$. After passing to variable $x = g^2 S$ the eq. (9) takes the form:

$$E_{\pm} = -\left(\int_0^{1/3} \frac{e^{-x/g^2} - e^{-1/3g^2}}{1/3 - x} \frac{dx}{\pi g^2} - \frac{e^{-1/3g^2}\ln(6)}{\pi g^2}\right) \mp \frac{e^{-1/6g^2}}{g\sqrt{\pi}} . \quad (10)$$

This expression correctly accounts for the high order terms of perturbation theory and the main nonperturbative effects caused by single instanton or instanton – antiinstanton pair. However the first terms of perturbative expansion are completely wrong in (10). In order to make (10) quantitatively reliable one have to calculate precisely the $N$ first terms of the perturbation theory and subtract from (10) the $N$ first terms of its expansion in series in $g^2$:

$$E8_{\pm} = \sum_{n=0}^{N} g^{2n} E_n - \left\{\int_0^{1/3} \frac{e^{-x/g^2} - e^{-1/3g^2}}{1/3 - x} \frac{dx}{\pi g^2} - \frac{e^{-1/3g^2} ln(6)}{\pi g^2}\right.$$
$$\left. - \sum_{n=0}^{N} \frac{3^{n+1}(n)!}{\pi} g^{2n}\right\} \mp \frac{e^{-1/6g^2}}{g\sqrt{\pi}}(1 + \sum_{n=1} g^{2n} E_n^1) . \quad (11)$$

Here the last term $\sim \exp(-1/6g^2)$ formally is the largest nonperturbative correction, but this term cancels if one considers the "center of band" $E_{cb} = \frac{1}{2}(E_+ + E_-)$. The terms of series (2) begin to grow up at $n > 1/3g^2$. Respectively the approximate expression (11) reaches the best accuracy at $N = 1/3g^2$. Calculating the new terms of perturbation theory at $N > 1/3g^2$ one should only decrease the accuracy of total results.

In addition to the "center of band" the formula (11) allows to calculate also the individual energy levels. One have to calculate only a lot of



corrections $\sim g^{2n}$ to the instanton induced splitting of the ground state energy $\sim e^{-1/6g^2}/g$. The asymptotics of perturbative expansion of energy shift takes the form (see [8]):

$$\Delta E = \frac{2e^{-1/6g^2}}{g\sqrt{\pi}}(1 + \sum_{n=1} g^{2n} E_n^1) , \quad E_n^1 = -\frac{3^{n+2}n!}{\pi}(ln(6n) + \gamma) , \quad (12)$$

where $\gamma = 0.5772...$ is the Eurlers constant. The perturbative series for the energy shift looks very similar with that for the ground state energy. The "best accuracy" one can reach after summing this series (which is not worse than the smallest term of the series) is negligibly small ($\sim \exp(-1/2g^2)$). In order to reach the accuracy $\sim \exp(-1/3g^2)/g^2$ in $E_+$ or $E_-$ one have to break the summation in (12) at the term for which $g^{2n}E_n^1 \ll \exp(-1/6g^2)$. The three – instanton contribution to $\Delta E$, which generates the asymptotics (12) is considered in the Appendix (see (38)).

## 3 Corrections to effective action

Up to now we have taken into account only the leading term of interaction between instanton and antiinstanton $\Delta S \sim e^{-T}/g^2$ (5). The further correction to the action of the pair will be considered in this section.

The corrections of two types to the effective interaction are to be taken into account. The first is $\Delta S_c \sim e^{-2T}/g^2$ – the correction to the classical instanton – antiinstanton interaction. The second correction $\Delta S_G \sim e^{-T}$ appears after the Gaussian integration around the classical configuration. At first sight it seems that both these corrections do not contribute to the asymptotics of the perturbation theory. For example let us substitute the one loop correction $\Delta S_G \sim e^{-T}$ in the equation (7):

$$E(g) = -\int \frac{dT}{\pi g^2} \exp\left\{-\frac{1}{g^2}(1/3 - 2e^{-T})\right\}(1 + Be^{-T}) . \quad (13)$$

Here $B \sim 1$. Passing to the variable $x = g^2 S$ we get:

$$E(g) = -\int \frac{e^{-x/g^2}}{1/3 - x}\left[1 + \frac{1}{2}B(1/3 - x)\right]\frac{dx}{\pi g^2} . \quad (14)$$

The term proportional to $B$ cancels in the singularity of the Borel function $F(x)$ (8) and therefore does not contribute to the asymptotics of the perturbation theory. It is easy to show that correction to the classical action $\Delta S_c \sim e^{-2T}/g^2$ also does not contribute to the asymptotics.

Nevertheless as we will see below the corrections to the Gaussian integral and to the classical interaction appear which contain an additional large factor $T$: $\Delta S_G \sim Te^{-T}$ and $\Delta S_c \sim Te^{-2T}/g^2$. Taking into account of these corrections leads to a new singularity of the Borel function $\delta F \sim \log(1/3 - x)$, although weaker than the pole (8).

The quantum correction to the two instanton density is also to be taken into account:

$$\rho = \rho_I \rho_A = \frac{1}{\pi g^2} \exp(-2S_I)(1 + Ag^2) \, . \tag{15}$$

Because of the correction $Ag^2$ does not depend on $T$, the value of $A$ is twice the value of corresponding quantum correction to a single instanton density. The calculation of two loop correction even for single instanton is a complicated problem (see [10]), but one can avoid these calculations. The splitting of the ground state energy $\Delta E$ (12) just equals to the twice instanton density. The semiclassical method of finding the exponentially small splitting was given in [11]. The accuracy of semiclassical calculation allows to search for any of $\sim g^{2n}$ corrections to small $\Delta E$. The first term was found in [5]:

$$E_1^1 = \frac{A}{2} = -\frac{71}{12} \, . \tag{16}$$

## 3.1 The choice of classical trajectory

In order to introduce the collective variable describing the relative positions of instanton and antiinstanton we multiply the functional integral by the Faddeev-Popov unit:

$$1 = \int \delta(\langle \psi | x \rangle - \tau) \frac{\partial(\langle \psi | x \rangle - \tau)}{\partial T} dT \, , \; \langle \psi | x \rangle \equiv \int \psi(t) x(t) dt \, . \tag{17}$$

Here $T$ is the distance between pseudoparticles, $\tau = \tau(T)$ – some function and $\psi = \psi(t, T)$ – a vector in the functional space. The functional integration is to be performed over the hyperplanes orthogonal to the vector $\psi$. We would call the instanton – antiinstanton pair the configuration in the functional space which minimizes the action for a given value of collective variable $\tau(T)$. The equation of motion now reads:

$$\frac{\delta S[x_{I-A}]}{\delta x} = \xi \psi \, , \tag{18}$$

where $\xi$ is a Lagrange multiplier, which allows to satisfy the condition $\langle \psi | x_{I-A} \rangle = \tau$.



In order to describe the instanton – antiinstanton pair one have to introduce the second collective variable $t_0$ – the position of the center of the pair. But at least if $\psi(t)$ is symmetric under transformation $t - t_0 \to t_0 - t$, the second collective variable may be chosen in such a way, that it will not affect the classical configuration.

There is a great freedom in the choice of $\psi(t, T)$ and $\tau(T)$ (17). The only restriction is that after we have separated the collective variables the functional integral over all the rest (quantum) variables must be well convergent. Technically the functional integration reduces to calculation of $det \frac{\delta^2 S[x_{I-A}]}{\delta x^2}$ – the determinant of the second action variation in the field of the pair. Among the directions in the functional space there are two the "worse". These are the almost zero modes — the eigenfunctions of the operator $\frac{\delta^2 S}{\delta x^2}$ having anomalously small eigenvalues. The vector $\psi$ from (17) should not be orthogonal to these modes. Under this conditions any function $\psi(t)$ would lead to a reliable instanton – antiinstanton type solution of the equation (18). In the following we suppose that $\psi(t)$ is a symmetric function, which differs from zero only near the centers of separate instanton and antiinstanton and goes to zero very fast both inside and outside the pair.

## 3.2 Corrections to the classical trajectory and action

Thus over a wide range between pseudoparticles $\psi \simeq 0$ and eq. (18) takes the form:

$$-\ddot{x}_{I-A} + x_{I-A} = 3g x_{I-A}^2 - 2g^2 x_{I-A}^3 \, , \tag{19}$$

Here 'dot' means the time derivative. Let us represent $x_{I-A}$ as a sum of instanton and antiinstanton field plus some correction:

$$x_{I-A} \;=\; x_I(t + T/2) + x_A(t - T/2) + \delta x \, . \tag{20}$$

Here $x_I$ is an unperturbed instanton (4) placed at $t = -T/2$ and $x_A$ is an antiinstanton placed at $t = T/2$. Solving eq. (19) iteratively in the intermediate region between instanton and antiinstanton, we get:

$$x_{I-A} = \frac{e^{-T/2}}{g}(e^t + e^{-t}) + \frac{e^{-T}}{g}(e^{2t} + e^{-2t} + 6) + \ldots \, , \; \delta x \simeq 6e^{-T}/g \, . \tag{21}$$

Outside the instanton – antiinstanton pair, at $|t| - T/2 \gg 1$, $\delta x$ is neglegibly small.

The action accurate to terms $\sim Te^{-2T}/g^2$ reads:

$$S[x_{I-A}] \simeq S[x_I] + S[x_A] + S[\delta x] + \tag{22}$$

$$+2\Big(\int_{-\infty}^{0} dt \tfrac{\delta S[x_I]}{\delta x}(x_A+\delta x)+\int_{-\infty}^{0} dt \tfrac{\delta S[x_A]}{\delta x}\delta x+(\dot{x}_I+\dot{x}_A)\delta x|_{-\infty}^{0}$$
$$+\dot{x}_I x_A|_{-\infty}^{0}+\int_{-\infty}^{0} dt(3g^2 x_I^2 x_A^2-3g x_A^2 x_I-6g x_I x_A \delta x)\Big)$$
$$=\tfrac{1}{g^2}(1/3-2e^{-T}-12Te^{-2T}+O(e^{-2T}))\ .$$

Here we use $(\dot{x}_I+\dot{x}_A)\delta x|_0 \equiv 0$ and the equation of motion $\frac{\delta S[x_I]}{\delta x} \equiv 0$. Corrections to (22) are of the order of $\delta S \sim e^{-2T}/g^2$ and depend on the choice of $\psi$ (17),(18). But the contribution $\sim Te^{-2T}/g^2$ in the formula (22) is model independent.

### 3.3 Corrections to determinant

Now we have to find the determinant of the operator

$$M=\frac{\partial^2 S[x_{I-A}]}{\partial x^2}=-\frac{d^2}{dt^2}+1+U_I+U_A+\Delta U\ ,\ U_{I,A}(t)=-\frac{3}{2ch(t\pm T/2)}\ . \quad (23)$$

The correction $\Delta U$ equals to

$$\Delta U=-24e^{-T} \quad (24)$$

over the wide range between instanton and antiinstanton ($T/2-|t|\gg 1$) and goes to zero outside the pair ($|t|-T/2\gg 1$). Let us use the equality

$$det\left(\frac{M}{M_0}\right)=\exp\left(tr\, ln\left(1+\frac{U_I+U_A+\Delta U}{-d_{tt}^2+1}\right)\right)\ , \quad (25)$$

where $M_0=-d_{tt}^2+1$. Consider the logarithm in (25) as the power series in the potential $U$. Because of the Green function of the operator $M_0 : G_0(t,t')=e^{-|t-t'|}/2$ falls down exponentially at large $|t-t'|$, the interference of $U_1$ and $U_2$ in (25) (as well as $\Delta U$) may be neglected in the zeroth approximation. In this approximation the total determinant factorizes to the product of the single instanton determinants. There are two sources of the corrections to (25) of the order of $Te^{-T}$. These are the contribution of the first order in $\Delta U$ and the interference in the lowest order of $U_1$ and $U_2$:

$$-\int dt G(t,t)\Delta U \ = \ -12Te^{-T}\ , \quad (26)$$
$$-\frac{1}{2}\int dt dt' G^2(t,t')(U_I(t)U_A(t') \ + \ U_I(t')U_A(t))=-12Te^{-T}\ .$$



The full determinant up to model dependent corrections $\sim O(e^{-T})$ reads

$$det\left(\frac{M}{M_0}\right)=det^2\left(\frac{-d_{tt}^2+1+U_I}{-d_{tt}^2+1}\right)(1-24Te^{-T}+O(e^{-T}))\ . \quad (27)$$

## 4 Results

All the corrections (15),(22),(27) should be combined in expression like (9) or (7). Now if one pass again to the variable $x=g^2S(T)=1/3-2e^{-T}-12Te^{-2T}+\cdots$ the Borel function $F(x)$ (8) may be found in a form:

$$F(x)\simeq\left(\frac{-1}{1/3-x}+3ln(1/3-x)+\ldots\right)\left(1-\frac{71}{6}g^2\right)\ . \quad (28)$$

The straightforward calculation leads now to the asymptotics (2). The first attempt to calculate numerically the $1/n$ correction to the asymptotics was made in [7]. The authors of [7] were the first who have calculated the leading ($\sim n!$) asymptotics (2) of the perturbative expansion of ground state energy of the double well oscillator (1). However their numerical value for the $1/n$ correction was incorrect, being 103/36 instead of 53/18. In Table 1. about 10 first terms of approximate expansion (2) are compared with the exact values [7]. Starting from $n=5$ the error in $E_n$ never exceeds 12%.

**Table 1**

| n | $E_n$ | $3^n n!\frac{3}{\pi}\frac{1}{E_n}$ | $3^n n!\frac{3}{\pi}(1-\frac{53}{18n})\frac{1}{E_n}$ |
|---|---|---|---|
| 3 | 44.5 | 3.47 | 0.06 |
| 4 | 626.6 | 2.96 | 0.78 |
| 5 | $1.1031\ 10^4$ | 2.52 | 1.04 |
| 6 | $2.2888\ 10^5$ | 2.19 | 1.11 |
| 7 | $5.4198\ 10^6$ | 1.94 | 1.12 |
| 8 | $1.4360\ 10^8$ | 1.76 | 1.11 |
| 9 | $4.2015\ 10^9$ | 1.62 | 1.09 |
| 10 | $1.3448\ 10^{11}$ | 1.52 | 1.07 |
| 11 | $4.6755\ 10^{12}$ | 1.44 | 1.06 |

The exact values of the coefficients of perturbative expansion $E_n$ (see [7]) and the ratio of approximate to exact value found in the leading ($\sim n!$) approximation and with $\sim 1/n$ correction taken into account. It is seen that starting from $n=5$ the accuracy of the approximate formula (2) is not worse than 12%.

It is convenient to integrate by part the $\sim \ln(1/3-x)$ term in $F(x)$ (28). Now the expression for the "center of band" transforms to (compare with (11)):

$$E_{cb} = \sum_{n=0}^{N} g^{2n} E_n - \left\{ \int_0^{1/3} \frac{e^{-x/g^2} - e^{-1/3g^2}}{1/3-x} \frac{dx}{\pi g^2} - \frac{e^{-1/3g^2} ln(6)}{\pi g^2} - \sum_{n=0}^{N} \frac{3^{n+1} n!}{\pi} g^{2n} \right\} \left(1 - \frac{53}{6} g^2\right) \qquad (29)$$

The accuracy of this formula depends on $N$ - the number of the terms of perturbative expansion, which were found exactly. In order to clarify this issue let us rewrite the integral in (29) through the principal value integral:

$$\int_0^{1/3} dx \frac{e^{-x/g^2} - e^{-1/3g^2}}{1/3-x} = P\int_0^{\infty} dx \frac{e^{-x/g^2}}{1/3-x} + e^{-1/3g^2}(ln(3g^2) - \gamma) \ .$$

Now it is easy to subtract explicitly the $N$ terms of the approximate expansion:

$$P\int_0^{\infty} \frac{e^{-x/g^2}}{1/3-x} \frac{dx}{\pi g^2} - \sum_{n=0}^{N} \frac{3^{n+1} n!}{\pi} g^{2n} \equiv P\int_0^{\infty} \frac{(3x)^N e^{-x/3g^2}}{1/3-x} \frac{dx}{\pi g^2} \ . \qquad (30)$$

Formula (29) reaches the best accuracy when one minimizes the value of (30). This minimum occurs at $|N - 1/3g^2| \sim 1$. For such $N$ the integral in r.h.s. of (30) (as well as the minimum term of the perturbative expansion) is of the order of $\sim 1/g e^{-1/3g^2}$. The error in $E_{cb}$ (29) is much smaller than the minimum term of the series $\delta E_{cb} \sim g^0 e^{-1/3g^2}$. The integral in r.h.s. of (30) is extremely easy to estimate if $N \equiv 1/3g^2$ with some (large) value $N$. For such $g$ the "best value" of $E_{cb}$ reads:

$$E_{cb} \approx \sum_{n=0}^{N} g^{2n} E_n + \frac{3Ne^{-N}}{\pi} \left\{ ln(6N) + \gamma + \frac{1}{3}\sqrt{\frac{2\pi}{N}} \right\} \left(1 - \frac{53}{18N}\right) \qquad (31)$$

Except for the new factor $1 - 53/18N$ (or $1 - 53g^2/6$) our results (31), (29) agree with that presented in the literature (see [9]). In Table 2 we compare the exact (numerical) values of $E_{cb}$ with that found from the approximate formula (31) for various values of $N \equiv 1/3g^2$. It is seen that the error in (31) is of the order of $\sim e^{-N} \equiv e^{-1/3g^2}$.



Table 2

| $N \equiv 1/(3g^2)$ | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|
| $E_{cb}^{exact}$ | 0.4439 | 0.43797 | 0.44832 | 0.459178 | 0.467156 |
| $E_{cb}^{theor}$ | 0.4367 | 0.44367 | 0.44933 | 0.459307 | 0.467173 |

The exact numerical energy $E_{cb}^{exact}$ and the approximate value $E_{cb}^{theor}$ found from the formula (31) as a function of $g^2$. As we expect the accuracy is $\sim e^{-N}$.

**Asknowledgements.** Authors are grateful to V.L.Chernyak for helpful discussions. The work of P.G.S. was supported by International Science Foundation.

## 5 Appendix

In this Appendix we would like to consider the three instanton contribution to the asymptotics of $\sim g^{2n}$ corrections to the nonperturbative splitting $\Delta E = E_- - E_+$ of the Hamiltonian (1) ground state energy. Previously this asymptotics was found within the usual quantum mechanical approach in [8], and via analytical continuation of the multiinstanton contribution over $g$ in [5]. Although the trick with analytical continuation [2] leads to the correct asymptotics, the author of [5] have not shown, what the specific three instanton effects contribute to $\Delta E$.

Following [5] consider the functional integral with inverted ($x \to 1/g - x$) boundaries:

$$e^{-E_+ L} - e^{-E_- L} = TrPe^{-HL} = N\int_{x(-\frac{L}{2})+x(\frac{L}{2})=1/g} Dx(t) e^{-S} \quad ; \quad L \gg 1 \ . \qquad (32)$$

Only the $n$-instanton configurations with odd $n$ contribute to $TrPe^{-HL}$. The single instanton contribution is trivial:

$$\left(TrPe^{-HL}\right)^{(1-ins)} = (2e^{-L/2}) L \frac{e^{-1/6g^2}}{\sqrt{\pi g}} . \qquad (33)$$

There are two distinct three-instanton contributions linear in $L$. First is the contribution from compact molecule – like three-instanton object having two interior degrees of freedom. The second contribution appears due to the effect of "excluded volume". Consider the pair of instanton and antiinstanton separated by interval $T$ from each other and the single instanton outside the pair. While the integral over positions of the pair gives the factor $L$, the instanton can "run" only over the volume $L-T$. Thus the interference $\sim LT$ should be considered in addition to the factorized $\sim L^2$ contribution of the pair and single instanton.

Like it was done for the instanton pair (9) we have to subtract the factorized three-instanton and instanton plus pair contributions from the contribution of three-instanton molecule . After such regularization the molecule gives

$$I_1 = (2e^{-L/2})\frac{1}{(\pi g)^{3/2}} L e^{-1/6g^2} \int\int dt dt' \Big( e^{-S(t,t')} - e^{-S_p(t)} - e^{-S_p(t')} + e^{-1/3g^2} \Big) , \qquad (34)$$

where $t, t' > 0$ – are the distances between the first and second and second and third pseudoparticles. $S(t,t') = 1/g^2(1/3 - 2e^{-t} - 2e^{-t'} + \cdots)$ is the three-instanton action (with $1/6g^2$ excluded) and $S_p(t) = 1/g^2(1/3 - 2e^{-t} + \cdots)$ is the action of instanton – antiinstanton pair (5). Let us restrict the integration in (34) by condition $t, t' < T_{reg}$, where intermediate scale $1 \ll T_{reg} \ll e^{1/6g^2}$ should cancel in the final result. Consider the integral

$$K = \int\int dt dt' \Big( e^{-S(t,t')} - e^{-1/3g^2} \Big) . \qquad (35)$$

Here the integration should be performed over the range $S(t,t') > 0$. After passing to variables $t$ and $x = g^2 S(t,t')$ the integration over $t$ may be done explicitly

$$K = \int_0^{1/3} dx \frac{e^{-x/g^2} - e^{-1/3g^2}}{1/3 - x} 2\left( ln\left(\frac{1/3 - x}{2}\right) + T_{reg} \right) . \qquad (36)$$

All the rest contributions in (34) are proportional to $T_{reg}$ and naturally lead to cancellation of the proportional to $T_{reg}$ part of (36).

The effect of excluded volume leads to the linear in $L$ three-instanton contribution of the form

$$I_2 = (2e^{-L/2})\frac{1}{(\pi g)^{3/2}} L e^{-1/6g^2} \int_{S_p(T)=0}^{\infty} dT(-T)\Big( e^{-S_p(T)} - e^{-1/3g^2} \Big) . \qquad (37)$$

After the change of variables $T \rightarrow x = g^2 S_p(T)$ the integral in (37) transforms to that of (36).



Finally the splitting of the ground state energy reads:

$$\Delta E = \frac{2}{(\pi g)^{3/2}} e^{-1/6g^2} 3 \int_0^{1/3} dx \frac{e^{-x/g^2} - e^{-1/3g^2}}{1/3 - x} ln(\frac{1/3 - x}{2}) \qquad (38)$$

Expanding $ln(1/3 - x)/(1/3 - x)$ in powers of $x$ one finds the series (12).

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