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INDUCED SURFACES AND THEIR INTEGRABLE DYNAMICS

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Induced Surfaces and
their Integrable Dynamics

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A B S T R A C T

A method is considered to induce surfaces in the three dimensional (pseudo) Euclidean space via the solutions of the two dimensional linear problems 2D LPs and their integrable dynamics (deformations) via the 2+1-dimensional nonlinear integrable equations associated with these 2D LPs. Coordinates X^i of the induced surfaces are defined as the integrals over certain bilinear combinations of the wavefunctiones ψ of these 2D LPs. General formulation as well as the three concrete examples are considered. Some properties and features of such inducing are discussed. Three-dimensional Riemann spaces associated with the 2+1-dimensional nonlinear integrable equations are considered too.

1 Introduction

Dynamics of surfaces, interfaces, fronts is a key ingredient in a number of interesting phenomena in physics. They are, in particular, surfaces waves, propagation of flame fronts, growth of crystals, dynamics of vortex sheets, deformation of membranes, formation of Saffman–Taylor fingers, many problems of hydrodynamics connected with motion of boundaries between regions of differing densities and viscosities (see e.g. [1–3]). Such a dynamics can be modelled by the nonlinear partial differential equations (PDEs) which describe the evolution of surfaces in time (see [1–3] and recent papers [4,5]). Solvable cases when the corresponding PDEs can be integrated analytically are, of course, of the great interest.

In mathematics the differential geometry of surfaces has been completed, in essence, at the end of XIX and the beginning of this century (see e.g. [6–10]). Basic differential equations (Gauss equations) which describe surfaces in the three-dimentional space have been studied in details from various points of view. One of the classical problems of the differential geometry was the study of the connection between differential geometry of submanifolds and nonlinear partial differential equations (PDEs). The Liouville and sine–Gordon equations which describe minimal and pseudospherical surfaces, respectively, are the best known examples. In particular, Liouville has found [11] the general solution of the equation $\phi_{xy} = \exp\phi$. Bianchi [7] and Bäcklund [12] introduced symmetry transformation of new type (now known as the Bäcklund transformation) for the sine-Gordon equation $\phi_{xy} = \sin\phi$ which allows to construct new pseudospherical surfaces from a given one.

Less known example is given by the equation $\phi_{xy} = \exp\phi - \exp(-2\phi)$ which describes surfaces with the so-called Tzitzeica property [13] (see e.g. [14]).

A new tool to solve nonlinear PDEs was discovered in 1967 by Gardner, Green, Kruskal and Miura [15]. This method (inverse spectral transform (IST) method) allows effectively solve a number of nonlinear PDEs with two and three independent variables which appear in various fields of physics and applied mathematics (see e.g. [16–18] and for multidimentional integrable PDEs [20–22]) A key element of the IST method is the representation of the nonlinear PDE as the compatibility condition of certain system of linear equations for so-called eigen (wave) function. Nonlinear PDEs integrable by the IST method possess a number of remarkable properties: soliton solutions, infinite number of conservation laws, infinite symmetry groups, Bäcklund and Darboux transformations, special Hamiltonian structures and so on (see [16–21]). In particular, it was shown that the Liouville and sine–Gordon equations are integrable by the IST method.

The sine–Gordon and Liouville equations were the first nonlinear PDEs which revealed a deep connection between the differential geometry and soliton equations. After that such a connection has been discussed many times. It was observed in [23] that the Gauss–Weingarten equations of the surface created by a special motion of a relativistic string can be viewed as a pair of spectral problems whose compatibility condition gives the so-called Lund–Regge system. A geometric interpretation of the 2×2 matrix spectral problem in terms of pseudospherical surfaces has been given in [24]. The formulations of the 1+1-dimensional soliton equations in terms of vector bundles has been discussed in [25]. The characterisation of the modified Korteweg-de Vries equation as the relation between local invariants of certain foliation on a surface of constant nonzero Gaussian curvature has been proposed in [26]. The detailed study of the nonlinear integrable equations which describe pseudospherical surfaces was given in the series of papers [27–30]. An extension of the sine-Gordon equation and Bäcklund transformation for the negative constant sectional curvature submanifolds in the Euclidean space R^{2n-1} has been considered in [31,32]. The IST method has been applied to such generalisation in [33,34]. Multidimensional Gauss–Codazzi equations and corresponding nonlinear equations have been considered in [35]. The Darboux and Lame systems which describe, respectively, the triply conjugate and triply orthogonal systems of surfaces [9], their exact solutions and transformation properties have been discussed recently in [36,37,22].

A different approach which uses the powerful tool of the IST method to construct explicitly surfaces has been proposed in [38,39]. Within such "soliton surfaces approach" one starts with the system of 1+1-dimensional

linear problems $\psi_x = P\psi$, $\psi_t = Q\psi$ and then construct explicit formulas for the immersions of one-parameter families of surfaces labeled by the spectral parameter. Emphasize that in this approach a surface is associated with the given solution of the 1+1-dimensional integrable PDE which arises as the compatibility condition of the above linear system. Interesting results in this direction have been obtained recently in [40].

In the present paper we consider the method of inducing surfaces in the three-dimensional Euclidean (and pseudo-Euclidean) space via the solutions of the two-dimensional linear problems (2D LPs). Given 2D LP, one defines the variable coordinates X^i ($i = 1,2,3$) of a surface as some integrals over certain bilinear combinations of solutions ψ of the 2D LP and solutions ψ^* of the adjoint 2D LP. This approach is not completely new. Particular examples of such construction were known already in the XIX century. They are the Weierstrass–Enneper formulas for minimal surfaces (see e.g. [6–8]) and the formulas of Lelieuvre [41] for surfaces refered to their asymptotic lines. Here we consider both the case of general 2D LP and particular cases including the generalization of the Weierstrass–Enneper formulas. 2D LPs on the whole plane induce the unbounded surfaces while periodic 2D LPs induce compact surfaces. Using the exact explicit solutions of 2D LP, which can be found, in particular, by the dressing method, one constructs the corresponding induced surface by explicit formulas. Emphasize that our approach is essentially different from that proposed by A. Sym in [38,39]: here a surface is induced by a single 2D LP and there is no nonlinear PDE associated with this surface.

Another our aim is to present a method to construct integrable dynamics of induced surfaces. Given 2D LP we consider the time evolution which is described by the LP of the type $\psi_t = A\psi$ and which is compatible with the 2D LP. The compatibility of the time evolution with the 2D LP first guarantees the preservation in time of the formulas for inducing surfaces. Second, it gives rise, as typically to the IST method, to the 2+1-dimensional nonlinear integrable PDE both for the coefficients of the 2D LP and for the wave-function ψ. This nonlinear evolution equation induces the corresponding evolution of the induced surface. Such a time dynamics of surface is integrable one. Standard IST procedure for the solution of the initial value problems allows us to solve (linearize) the initial value problem for the induced dynamics of surfaces. Exact explicit solutions of the inducing linear problems give rise to the time evolutions of the induced surfaces given by the exact explicit formulas. As far as concerning the differential geometry of surfaces our approach provides a method to construct integrable deformations of induced surfaces.

In the paper we consider several concrete integrable evolutions of surfaces which are induced by the well-known 2+1-dimensional integrable nonlinear PDEs. Resonantly interacting waves equations, the Nizhnik–Veselov–Novikov (NVN) equation, the modified NVN equation, the Kadomtsev– Petviashvili (KP) and Davey–Stewartson (DS) equations are among them. We consider some general and particular features of the proposed method of inducing surfaces and their integrable dynamics. The one-dimensional limit of the above constructions is also discussed. It reproduces the known results on the integrable dynamics of curves.

Three-dimensional Riemann spaces associated with the 2+1- dimensional integrable nonlinear PDEs are considered too.

The paper is organized as follows. In section 2 we remind the basic notions and equations of the differential geometry of surfaces. The Weierstrass-Enneper and Lelieuvre formulas are presented in section 3. General method of inducing surfaces and their integrable dynamics is discussed in section 4. Dynamics of surfaces induced by the NVN hierarchy of equations via the Lelieuvre formula is considered in section 5. The generalised Weierstrass–Enneper inducing and the corresponding dynamics is presented in section 6. Inducing the surfaces by the general matrix 2D LP and their dynamics in considered in section 7. Different particular cases are discussed in the next section 8. The one-dimensional limit of the above constructions is analyzed in section 9. The interralation between the 2+1-dimensional integrable PDEs and three-dimensional Riemann spaces is considered in section 10.. Some properties and features of the induced surfaces and their integrable dynamics are discussed in the Conclusion.

2 Surfaces in R^3

Here we will remind briefly some basic elements of the theory of surfaces.

So we consider a surface in the three-dimensional Euclidean space R^3. We will denote the local coordinates of the surface by u^1, u^2. The surface can be defined by the formulae (see e.g. [6–8])

$$X^i = x^i(u^1, u^2), \quad i = 1,2,3\,, \tag{2.1}$$

where X^i $(i = 1,2,3)$ are the coordinates of the variable point of the surface and x^i (u^1, u^2) are scalar functions. The basic characteristics of the surface are given by the first Ω_1 and second Ω_2 fundamental forms

$$\Omega_1 = ds^2 = g_{\alpha\beta} du^\alpha du^\beta\,, \Omega_2 = d_{\alpha\beta} du^\alpha du^\beta\,, \tag{2.2}$$

where $g_{\alpha\beta}$ and $d_{\alpha\beta}$ are symmetric tensors, α, β take values 1,2 and here and below the summation over repearted indices is assumed. The quantities $g_{\alpha\beta}$

and $d_{\alpha\beta}$ are calculated by the formulas [6–8]

$$g_{\alpha\beta} = \frac{\partial X^i}{\partial u^\alpha} \cdot \frac{\partial X^i}{\partial u^\beta}\,, d_{\alpha\beta} = \frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta} \cdot N^i \quad (\alpha, \beta = 1,2)\,, \tag{2.3}$$

where N^i are the components of normal vector:

$$N^i = (\det g)^{-1/2} \varepsilon^{ike} \frac{\partial X^k}{\partial u^1} \frac{\partial X^l}{\partial u^2} \quad (i = 1,2,3)\,, \tag{2.4}$$

and ε^{ike} is totally antisymmetric tensor, $\varepsilon^{123} = 1$.

The metric $g_{\alpha\beta}$ completely defines the intristic properties of the surface. The Gaussian curvature K and mean curvature K_m of the surface are calculated by the formulae $K = R_{1212}(\det g)^{-1}$ where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor and $K_m = g^{\alpha\beta} d_{\alpha\beta}$. We will considered here mainly the real surfaces. Embedding of the surface into R^3 is described both by $g_{\alpha\beta}$ and $d_{\alpha\beta}$ and it is governed by the Gauss–Codazzi equations

$$\frac{\partial^2 X^i}{\partial u^\alpha \partial u^\beta} - \Gamma^\gamma_{\alpha\beta} \frac{\partial X^i}{\partial u^\gamma} - d_{\alpha\beta} N^i = 0\,, \tag{2.5}$$

$$\frac{\partial N^i}{\partial u^\alpha} + d_{\alpha\gamma} g^{\gamma\beta} \frac{\partial X^i}{\partial u^\beta} = 0 \quad (i = 1,2,3;\, \alpha, \beta = 1,2)\,, \tag{2.6}$$

where $\Gamma^\gamma_{\alpha\beta}$ are the Christofel symbols.

Among the global characteristic of surfaces we mention the integral curvature (see e.g. [6–8])

$$\chi = \frac{1}{2\pi} \int_s K (\det g)^{1/2} du^1 du^2\,, \tag{2.7}$$

where K is the Gaussian curvature and the integration in (2.7) is performed all over the surface. For compact oriented surfaces

$$\chi = 2(1 - n)\,, \tag{2.8}$$

where n is a genus of the surface.

Families of parametric curves on the surface form the systems of curvilinear local coordinates on the surface. It is often very convenient to use special types of parametric curves on surfaces as coordinates. The following types of special parametric curves are the basic one (see e.g. [6–10]):

1. Orthogonal curves. The first fundamental form looks like $\Omega_1 = g_{11} du^{1^2} + g_{22} du^{2^2}$ and the corresponding coordinates are the curvilinear orthogonal coordinates.

2. Minimal lines (curves of zero length). In this case $g_{11} = g_{22} = 0$, i.e.

$$\Omega_1 = 2g_{12}\, du^1 du^2 \,. \tag{2.9}$$

For real surfaces minimal lines are complex and $\Omega_1 = 2\lambda(z,\bar{z})dzd\bar{z}$ where bar means the complex conjugation and λ is a real function. The Gauss–Codazzi equation for a surface referred to its minimal lines are reduced to the Liouville equation

$$\frac{\partial^2 \phi}{\partial u^1 \partial u^2} = -Ke^{\phi} \,, \tag{2.10}$$

where $\phi = \log g_{12}$.

3. Conjugate lines. They are the curves orthogonal with respect to the second fundamental form. So in this case $d_{12} = 0$ and

$$\Omega_2 = d_{11} du^{1^2} + d_{22} du^{2^2} \,. \tag{2.11}$$

The coordinates X^i of a surface referred to its parametric conjugate lines obey, as it follows from (2.5), the Laplace equation

$$\frac{\partial^2 X^i}{\partial u^1 \partial u^2} - \Gamma^1_{12} \frac{\partial X^i}{\partial u^1} - \Gamma^2_{12} \frac{\partial X^i}{\partial u^2} = 0 \,. \tag{2.12}$$

The Laplace transformation for equations (2.12) gives rise to the 2D Toda lattice equations [6]. The Laplace equation (2.12) is also the basic equation of the so-called theory of conjugate nets (see e.g. [42])

4. Asymptotic lines. In this case $d_{11} = d_{22} = 0$ and

$$\Omega_2 = 2d_{12} du^1 du^2 \,. \tag{2.13}$$

In particular, for the surfaces of constant negative curvature $-\mu^2$ one has $g_{11} = g_{22} = 1$, $g_{12} = \cos\omega$, $d_{12} = \mu \sin\omega$ and the function ω obeys the sine–Gordon equation

$$\frac{\partial^2 \omega}{\partial u^1 \partial u^2} = \mu^2 \sin\omega \,. \tag{2.14}$$

5. Lines of curvature (orthogonal and conjugate) and geodesics. These types of curves play a fundamental role in the theory of surfaces. But we shall not discuss them here.

As we see from the formulas (2.10)–(2.14) these special types of local coordinates are very useful for the revealing the role of nonlinear PDEs in the theory of surfaces. They will be also convenient for our purpose of inducing the surfaces and their integrable dynamics.

3 The old Weierstrass–Enneper and Lelieuvie formulas

In connection with the study of minimal surfaces, for which mean curvative $K_m = 0$, Weierstrass and independently Enneper (see e.g. [6–10]) discovered the following formulas.

Let $\psi(z)$ and $\phi(z)$ are arbitrary holomorphic functions. Then let us introduce the functions W_1, W_2 and W_3 via

$$\frac{\partial W_1}{\partial z} = i\,(\psi^2 + \phi^2)\,, \quad \frac{\partial W_2}{\partial z} = \psi^2 - \phi^2\,, \quad \frac{\partial W_3}{\partial z} = -2\psi p \,. \tag{3.1}$$

We define

$$X^1 = \mathrm{Re}\, W_1 = \mathrm{Re} \int i(\psi^2 + \phi^2)dz \,,$$
$$X^2 = \mathrm{Re}\, W_2 = \mathrm{Re} \int (\psi^2 - \phi^2)dz \,, \tag{3.2}$$
$$X^3 = \mathrm{Re}\, W_3 = -\mathrm{Re} \int 2\psi\phi dz \,.$$

Then the functions $X^i(z,\bar{z})$ $(i = 1,2,3)$ define a minimal surface. The parametric lines z=const, $\bar{z}$=const$'$ are the minimal lines. Note that the functions ϕ and ψ are defined by the Cauchy–Riemann equations

$$\frac{\partial \psi}{\partial \bar{z}} = 0\,, \quad \frac{\partial \phi}{\partial \bar{z}} = 0 \,. \tag{3.3}$$

The Lelieuvre formula [41] is our second example. One starts with the pertubed string equation

$$\frac{\partial^2 \psi}{\partial u^1 \partial u^2} + p(u^1,u^2)\psi = 0 \,, \tag{3.4}$$

where p and ψ are scalar functions. Let $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$ are the three linearly independent solutions of equation (3.4) with given $p(u^1,u^2)$. We define X^i $(i = 1,2,3)$ by the formulas [41]

$$\frac{\partial X^i}{\partial u^1} = \varepsilon^{ike}\psi^{(k)}\frac{\partial \psi^{(e)}}{\partial u^1}\,, \quad \frac{\partial X^i}{\partial u^2} = -\varepsilon^{ike}\psi^{(k)}\frac{\partial \psi^{(e)}}{\partial u^2} \quad (i,k,e = 1.2.3) \,. \tag{3.5}$$

Then the functions $X^i(u^1,u^2)$ (i=1,2,3) defines a surface referred to its asymptotic lines which are the parametric lines u^1=const, u^2 = const$'$ [41]. The Gaussian curvature of this surface is $K = -(\psi^{(i)}\psi^{(i)})^{-2}$.

One more example of similar type is contained as an exercise in [8] (p.82). Let θ_1 and θ_2 are the two linearly independent solutions of the equation

$$\frac{\partial^2\theta}{\partial u^1\partial u^2} - \frac{\partial}{\partial u^1}(\log\lambda)\cdot\frac{\partial\theta}{\partial u^2} - \lambda^2(u^1,u^2)\theta = 0\,, \tag{3.6}$$

where $\lambda(u^1,u^2)$ is a scalar real function. Then the functions $X^i(u^1,u^2)$ are given by the quadratures

$$X^1 + iX^2 = \int\left(\theta_1^2 du^1 + \frac{1}{\lambda^2}\left(\frac{\partial\theta_1}{\partial u^2}\right)^2 du^2\right)\,,$$
$$X^1 - iX^2 = \int\left(\theta_2^2 du^1 + \frac{1}{\lambda^2}\left(\frac{\partial\theta_2}{\partial u^2}\right)^2 du^2\right)\,, \tag{3.7}$$
$$X^3 = i\int\left(\theta_1\theta_2 du^1 + \frac{1}{\lambda^2}\frac{\partial\theta_1}{\partial u^1}\frac{\partial\theta_2}{\partial u^2}du^2\right)$$

define a surface referred to its minimal lines [8].

The generalization of the basic idea of these three examples of the inducing surfaces by the solutions of linear PDEs is the first main goal of our paper.

4 General method of inducing surfaces and their integrable dynamics

So we start with the linear PDE with two independent variables u^1, u^2 and, in general, with matrix-valued coefficients

$$L(\partial_{u^1}, \partial_{u^2})\psi = 0\,, \tag{4.1}$$

where L is the linear operator and ψ is a matrix-valued function. We shall refer to (4.1) as the two-dimentional linear problem (2D LP). The 2D LP formally adjoint to (4.1) with respect to the standard bilinear form $<\phi,\psi>\stackrel{\text{def}}{=} \iint du^1du^2\text{tr}(\phi\psi)$ will be denoted as

$$L^*(\partial u^1, \partial u^2)\psi^* = 0\,. \tag{4.2}$$

It is well-known that

$$\psi^*L\psi - \psi L^*\psi^* = \frac{\partial P_1}{\partial u^1} - \frac{\partial P_2}{\partial u^2}\,, \tag{4.3}$$

where P_1 and P_2 are certain bilinear combinations of ψ and ψ^*. So if $\psi^{(i)}$ and $\psi^{*(k)}$ are the solutions of (4.1) and (4.2), then one has

$$\frac{\partial P_1^{(ik)}}{\partial u^1} = \frac{\partial P_2^{(ik)}}{\partial u^2}\,. \tag{4.4}$$

This equation implies the existence of the quantities $\omega^{(ik)}$ such that

$$P_1^{(ik)} = \frac{\partial\omega^{(ik)}}{\partial u^2}\,,\quad P_2^{(ik)} = \frac{\partial\omega^{(ik)}}{\partial u^1}\,. \tag{4.5}$$

In virtue of (4.4) the quantities $X^i = \gamma^{ike}\omega^{(ke)}$, given by the quadratures

$$X^i = \gamma^{ike}\int_\Gamma (P_2^{(ke)}du^1 + P_1^{(ke)}du^2) \tag{4.6}$$

do not depend on the choice of the curve of integration Γ. Here γ^{ike} are arbitrary constants.

A basic step is now to treat u^1, u^2 as the local coordinates on the surface and three quantities X^1 $(i = 1,2,3)$ of the type (4.6) as the coordinates of the variable point of the surface in the three-dimentional Euclidean space R^3. So given 2D LP (4.1) (and its adjoint (4.2)), one induces a surface by the formulae of the type (4.6).

Any three linearly independent solutions $\psi^{(i)}$ of (4.1) and , respectively, of (4.2) induce a surface. Linear PDEs (4.1), (4.2) have infinitely many linearly independent solutions. Hence 2D LP (4.1) with the fixed coefficients $\{p(u^1,u^2)\}$ generates an infinite family of surfaces in R^3. A variation of p gives rise to the various surfaces too. Finally, different 2D LPs induce the surface of different types.

In the case of coordinates u^1, u^2 varying all over the plane we have unbounded surfaces in R^3. The periodic 2D LPs (4.1) induce the compact surfaces. There are several powerful methods to solve linear PDEs with variables coefficients: method of the operators of transformations (see e.g. [43]), dressing method (see e.g. [16,20–22]) and others. Exact explicit solutions of the 2D LP (4.1) define by the explicit quadratures (4.6) the induced surfaces. So the method under consideration provides us the diverse possibilities to study various properties of surfaces in R^3.

A principal advantage in this approach consists in the possibility to formulate for such induced surfaces the time evolutions (deformations) which are integrable in certain sence. So let us assume that all quantities in the 2D LP (4.1) depend on the new time variable t. Time evolution of these quantities can be defined in the different ways. We shall consider such time evolutions which preserve in time the formulas of inducing (4.6). For the invariance of (4.6) in time it is sufficient to consider the deformations which preserve 2D LP (4.1) in time. Following the main idea of IST method (see e.g. [15–22]) we shall fix the time evolution by the linear PDE of the type

$$M(\partial_t, \partial_{u^1}, \partial_{u^2})\psi = 0\,, \tag{4.7}$$

where $\partial_t \equiv \frac{\partial}{\partial t}$, $\partial_{u^\alpha} \equiv \frac{\partial}{\partial u^\alpha}$, and M is some linear operator. In the cases which we shall consider $M = \partial_t + A$ where A is some linear operator.

The compatibility of 2D LP (4.1) and LP (4.7) guarantees the invariance of the inducing (4.6) in time t. On the other hand, this compatibility condition, as it is well known, is equivalent to the nonlinear PDE for the coefficients $\{p(u^1, u^2, t\}$ of the problem (4.1)

$$p_t = F\{p, p_t, p_{u^1}, p_{u^2}, \ldots\}\,, \tag{4.8}$$

where F is some nonlinear function. Equation (4.8) is just the nonlinear PDE integrable by the IST metod with the help of the auxiliary linear problems (4.1) and (4.7). The IST method allows us to study such an integrable nonlinear PDEs in great details. In particular, it allows to solve (linearize) the initial-value problem for the integrable equation (4.8) $p(u^1, u^2, t = 0) \rightarrow p(u^1, u^2, t)$ and to construct very wide classes of its exact explicit solutions [16-22]. The corresponding formulas, with the use of the inducing (4.6), provide us the solution of the initial-value problem for the evolution of the induced surfaces:

$$X^i(u^1, u^2, t = 0) \rightarrow X^i(u^1, u^2, t) \quad (i = 1, 2, 3)\,. \tag{4.9}$$

It gives us also the explicit exact formulas which describe the continuous deformations of surfaces in time t.

Thus, the 2+1-dimensional integrable equations (4.8) induce via (4.6) the integrable evolutions (dynamics or deformations) of the induced surfaces.

It is well-known that there exists infinite hierarchies of integrable equations associated with the given 2D LP (4.1). For them the operator M in (4.7) is of the form $M_n = \partial_t + A_n$ where A_n is an n-order differential operator ($n = 1, 2, 3, \ldots$). Each member of the hierarchy induces the corresponding dynamics of surfaces. So one has the infinite hierarchies of integrable dynamics of the induced surfaces.

To calculate the compatibility condition for the system (4.1), (4.7) explicitly one needs to fix the properties of the coordinate u^1, u^2. In the present

paper we shall consider the case when the local coordinates u^1, u^2 of a surface are not affected by the time evolution, i.e.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial u^\alpha} = \frac{\partial}{\partial u^\alpha}\frac{\partial}{\partial t} \quad (\alpha = 1, 2)\,. \tag{4.10}$$

The condition (4.10) implies certain constraints of the admissable deformations of surfaces. Namely, we assume that the points of the surface taken in the two different values of time are always in one-to-one correspondence. In such a case the coordinate curves on the surface can be choosen so that u^1 and u^2 will have the same values on the two surfaces at corresponding points (see e.g. [6–8])

Now let us compare our approach with that proposed in [39]. "Soliton surfaces approach" of [39] is the method of inducing surfaces too. But it starts with the set of the one-dimensional problems

$$\frac{\partial\psi}{\partial u^\mu} = g_\mu(u; \lambda)\psi\,, \quad \mu = 1, 2, \ldots \tag{4.11}$$

where $g_\mu(u; \lambda)$ are matrix-valued functions and λ is a spectral parameter. The variable coordinates X^i of a surface are induced by the formula

$$\frac{\partial X}{\partial u^\mu} = \psi^{-1}\frac{\partial g_\mu}{\partial\lambda}\psi\,, \quad \mu = 1, 2, \ldots \tag{4.12}$$

where X belong to some matrix algebra.

Each of the LP (4.11) is the one-dimensional one, has finite number of linearly independent solutions, and the compatibility condition of the system (4.11) is equivalent to the nonlinear integrable equation. In the 1+1-dimensional case it is one of the well-known 1+1-dimensional integrable equations. So in the "soliton surfaces approach" a surface is induced by the system of the one-dimensional LPs or, equivalently, by the nonlinear integrable equation.

In our approach, in contrast, a surface is induced by a single 2D LP (4.1). So our inducing procedure does not contain any nonlinear PDE within it. In our approach the 2+1-dimensional nonlinear PDEs induce the integrable dynamics (deformations) of the induced surfaces.

5 Lelieuvre's inducing and NVN integrable dynamics

In this and next sections we shall consider concrete examples of the general scheme of inducing proposed in the previous section.

We start with the LP

$$\frac{\partial^2\psi}{\partial u^1 \partial u^2} + p(u^1, u^2)\psi = 0\,, \qquad (5.1)$$

where p and ψ are scalar real functions. The Lelieuvre formulas (3.5) define the coordinates X^i of a surface by the quadratures [41]

$$X^i = \varepsilon^{ike} \iint_\Gamma \left(\psi^{(k)} \frac{\partial \psi^{(e)}}{\partial u^1}\, du^1 - \psi^{(k)} \frac{\partial \psi^{(e)}}{\partial u^2}\, du^2 \right)\,, \qquad (5.2)$$

where the integration in (5.2) is performed along an arbitrary curve Γ. In virtue of (3.5) and (5.1) the integral (5.2) does not depend on the choice of Γ. Note that the LP (5.1) is self-adjoint. As a result the formula (5.2) is quadratic in ψ. In the Lelieuvre inducing the local coordinates u^1, u^2 of the surface are the asymptotic lines. For the negative Gaussian curvature K asymptotic lines are real and distinct. For positive Gaussian curvature K asymptotics lines are complex and complex conjugate to each other.

So the 2D LP (5.1) with real u^1, u^2 induces via (5.2) surfaces with $K < 0$. The 2D LP (5.1) with $u^2 = \bar{u}^1$ induces the surfaces with $K > 0$.

Using (5.2), one, in particular, gets the metric $g_{\alpha\beta}$:

$$g_{11} = (\psi^{(k)}\psi^{(k)}) \frac{\partial \psi^{(e)}}{\partial u^1} \frac{\partial \psi^{(e)}}{\partial u^1} - \left(\psi^{(k)} \frac{\partial \psi^{(k)}}{\partial u^1}\right)^2,$$

$$g_{12} = -(\psi^{(k)}\psi^{(k)}) \frac{\partial \psi^{(e)}}{\partial u^1} \frac{\partial \psi^{(e)}}{\partial u^2} + \left(\psi^{(k)} \frac{\partial \psi^{(k)}}{\partial u^1}\right)\left(\psi^{(e)} \frac{\partial \psi^{(e)}}{\partial u^2}\right), \qquad (5.3)$$

$$g_{22} = (\psi^{(k)}\psi^{(k)}) \frac{\partial \psi^{(e)}}{\partial u^2} \frac{\partial \psi^{(e)}}{\partial u^2} - \left(\psi^{(k)} \frac{\partial \psi^{(k)}}{\partial u^2}\right)^2.$$

In modern terminology equation (5.1) with the real u^1, u^2 is referred usually as the perturbed string equations (if $p \to 0$ as $u^{1^2} + u^{2^2} \to \infty$) or as the perturbed telegraph equation (if $p \to \text{const} \neq 0$ as $u^{1^2} + u^{2^2} \to \infty$). In the case $u^2 = \bar{u}^1$ it is the two-dimensional stationary Schrödinger equation with the potential p. Equation (5.1) is well studied in both cases by different

methods (see e.g. [43–53]). Exact solutions of (5.1) obtained by all these methods induce via (5.2) the corresponding surfaces in R^3.

Evolutions of the potential p and wavefunction ψ in time t which preserve the LP (5.1) are defined as the compatibility condition of (5.1) with the LPs

$$\psi_t + A_n \psi = 0\,, \qquad (5.4)$$

where the operators A_n are of the form

$$A_n = \sum_{l=0}^{n} (q_n(u,t)\partial_{u^1}^{2n+1} + r_n(u,t)\partial_{u^2}^{2n+1}) \qquad (5.5)$$

and q_n, r_n are scalar functions.

The simplest case $n = 1$ corresponds to the equation

$$p_t + \alpha p_{u^1u^1u^1} + \beta p_{u^2u^2u^2} + 3\alpha(pW_{u^1})_{u^1} + 3\beta(pW_{u^2})_{u^2} = 0,\ W_{u^1u^2} = p, \qquad (5.6)$$

where α and β are arbitrary constants and $p_{u^i} \equiv \frac{\partial p}{\partial u^i}$. The corresponding problem (5.4) is of the form

$$\psi_t + (\alpha\partial_{u^1}^3 + \beta\partial_{u^2}^3 + 3\beta W_{u^1u^1}\partial_{u^1} + 3\alpha W_{u^2u^2}\partial_{u^2})\psi = 0\,. \qquad (5.7)$$

In the case of real u^1, u^2 equation (5.6) has been derived for the first time in [54]. In the case $u^2 = \bar{u}^1$ it was discovered independently in [46]. We shall refer to equation (5.6) as the Nizhnik–Veselov–Novikov (NVN) equation: NVN-I and NVN-II equations respectively for real and complex ($u^2 = \bar{u}^1$) coordinates.

So the NVN-I equation induces the integrable dynamics of the surfaces with negative Gaussian curvature referred to their asymptotic lines while integrable dynamics of induced surfaces with positive Gaussian curvature is induced by the NVN-II equation.

Eliminating the potential p from the system (5.1), (5.7), one gets the following equation for the wavefunction ψ

$$\psi_t + \alpha\psi_{u^1u^1u^1} + \beta\psi_{u^2u^2u^2} + 3\beta W_{u^1u^1}\psi_{u^1} + 3\alpha W_{u^2u^2}\psi_{u^2} = 0\,,$$

$$W_{u^1u^2} = -\psi^{-1}\psi_{u^1u^2}\,. \qquad (5.8)$$

Since the Lelieuvre formulas (5.2) contain only the wavefunctions $\psi^{(k)}$ the wavefunction equation (5.8) is of the principal importance in the whole method of induced dynamics of surfaces.

The IST method allows us to find wide classes of exact explicit solutions of the NVN equation and solve (linearize) the initial-value problem $p(u,t=0) \to p(u,t)$ [54,43,52] and correspondingly $\psi(u,t=0) \to \psi(u,t)$. Consequently, the formula (5.2) induces the solution of the initial-value problem for the induced dynamics of surfaces:

$$X^i(u,t=0) \to X^i(u,t)\,,$$
$$g_{\alpha\beta}(u,t=0) \to g_{\alpha\beta}(u,t)\,. \qquad (5.9)$$

The compatibility conditions of (5.1) and the problems (5.4) with $n = 2,3,4,\ldots$ give rise to the NVN hierarchy of nonlinear PDEs. All of them are integrable by the IST method. They induce the infinite hierarchy of the integrable dynamics of the surfaces referred to their asymptotic lines.

6 Generalized Weierstrass–Enneper inducing

Our second example is concerned to the integrable dynamics of surfaces referred to their minimal lines. The generating LP is of the form

$$L\psi \equiv \begin{pmatrix} \partial_z & 0 \\ 0 & \partial_{\bar z} \end{pmatrix} \psi + \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix} \psi = 0\,, \qquad (6.1)$$

where $p(z,\bar z)$ is a real scalar function and ψ is 2×2 matrix.

It is not difficult to check that ψ^* obeys the same equation as ψ^T (script T denotes the transposition of matrix). So one can identify $\psi^* = \psi^T$. Second, the LP (6.1) admits the constraint (involution) $\sigma_2 \psi \sigma_2^{-1} = \bar\psi$ where $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus a solution ψ of (6.1) can be choosen of the form $\psi = \begin{pmatrix} \psi_1 & -\bar\psi_2 \\ \psi_2 & \bar\psi_1 \end{pmatrix}$. With the use of these properties it is not difficult to show that the requirements of reality of X^i ($i = 1,2,3$) and off-diagonality of $g_{\alpha\beta}$ ($g_{\alpha\beta} = 0, \alpha = \beta$) give

$$\frac{\partial X^1}{\partial z} = i(\psi_2^2 + \bar\psi_1^2)\,, \quad \frac{\partial X^1}{\partial \bar z} = -i(\psi_1^2 + \bar\psi_2^2)\,;$$
$$\frac{\partial X^2}{\partial z} = \bar\psi_1^2 - \psi_2^2\,, \quad \frac{\partial X^2}{\partial \bar z} = \psi_1^2 - \bar\psi_2^2\,; \qquad (6.2)$$
$$\frac{\partial X^3}{\partial z} = -2\psi_2\bar\psi_1\,, \quad \frac{\partial X^3}{\partial \bar z} = -2\psi_1\bar\psi_2\,.$$

Equations (6.2) are compatible due to (6.1).

One has

$$g_{12} = \frac{\partial X^i}{\partial z}\frac{\partial X^i}{\partial \bar z} = 2(\psi_1\bar\psi_1 + \psi_2\bar\psi_2)^2 = 2\det{}^2\psi\,. \qquad (6.3)$$

and

$$d_{12} = 2p\det\psi\,. \qquad (6.4)$$

Thus, the LP (6.1) with the real $p(z,\bar z)$ induces the surfaces referred to its minimal lines via

$$X^1 + iX^2 = 2i\int_\Gamma^z (\bar\psi_1^2 dz' - \bar\psi_2^2 d\bar z')\,,$$

$$X^1 - iX^2 = 2i\int_\Gamma^z (\psi_2^2 dz' - \psi_1^2 d\bar z')\,, \qquad (6.5)$$

$$X^3 = -2\int_\Gamma^z (\psi_2\bar\psi_1 dz' + \psi_1\bar\psi_2 d\bar z')\,,$$

where Γ is an arbitrary curve on the complex plane ending at the point z. The corresponding first fundamental form is

$$\Omega_1 = 4\det{}^2\psi dz d\bar z \qquad (6.6)$$

and the Gaussian and mean curvatures are given by

$$K = -2\det{}^{-2}\psi \cdot (\log\det\psi)_{z\bar z}\,,$$
$$K_m = 2p(\det\psi)^{-1}. \qquad (6.7)$$

The integral curvature χ is

$$\chi = \frac{1}{2\pi}\iint_C K\sqrt{\det g}\, dz \wedge d\bar z = -\frac{2i}{\pi}\iint_C dz \wedge d\bar z (\log\det\psi)_{z\bar z}\,. \qquad (6.8)$$

Using the Gauss–Green formula, one gets

$$\chi = \frac{2i}{\pi}\iint_{\partial C} dz (\log\det\psi)_z\,. \qquad (6.9)$$

Thus, the integral curvature of the induced surface is defined by the asymptotics of ψ_1 and ψ_2.

In terms of ψ_1 and ψ_2 the LP (6.1) is

$$\psi_{1z} = p\psi_2\,,$$
$$\psi_{2\bar z} = -p\psi_1\,. \qquad (6.10)$$

Let $p \to 0$ as $|z| \to \infty$. So $\psi_1 \to a(\bar z)$ $\psi_2 \to b(z)$ as $|z| \to \infty$ where a and b are arbitrary functions. For the solutions ψ_1, ψ_2 of (6.10) defined by

$$|\psi_1|^2 \to |z|^n, \quad \psi_2 \to 0 \qquad (6.11)$$

as $|z| \to \infty$, one gets

$$\chi = -2n\,. \qquad (6.12)$$

Minimal surfaces ($K_m = 0$) correspond to the case $p = 0$. At $p = 0$ one has

$$\bar\psi_{1\bar z} = 0\,, \quad \psi_{2\bar z} = 0\,. \qquad (6.13)$$

With the identification $\psi = \frac{1}{\sqrt{2}}\psi_2$, $\phi = \frac{1}{\sqrt{2}}\bar\psi_1$ the formulas (6.5) at $p = 0$ coincide with the Weierstrass–Enneper formulas (32). So the formulas (6.5) represent the generalization of the Weierstrass–Enneper formulas to the case of nonminimal surfaces.

Surfaces of constant mean curvature K_m is an another interesting particular case. They are induced by the LP (6.1) under the constraint

$$p = \frac{1}{2}\,K_m(\psi_1\bar\psi_1 + \psi_2\bar\psi_2)\,, \qquad (6.14)$$

where K_m is a constant. The LP (6.1) or (6.10) in this case is equivalent to the nonlinear system

$$\psi_{1z} - \frac{K_m}{2}\,(\psi_1\bar\psi_1 + \psi_2\bar\psi_2)\psi_2 = 0\,,$$
$$\psi_{2\bar z} + \frac{K_m}{2}\,(\psi_1\bar\psi_1 + \psi_2\bar\psi_2)\psi_1 = 0\,. \qquad (6.15)$$

Minimal surfaces and surfaces of constant mean curvature have a number of interesting applications in physics (see [1–3]).

The LP (6.1) is amenable to the dressing method (see e.g. [21,22]. It provides the infinite class of solvable cases for (6.1). They induce via (6.5) the infinite class of surfaces referred to their minimal lines.

The integrable dynamics of the constructed surfaces is induced by the integrable evolutions of the potential p and wavefunctions ψ in (6.1). The latter are equivalent to the compatibility condition of (6.1) with the LPs of the type (4.7) with $u^1 = z$, $u^2 = \bar z$. The simplest nontrivial example corresponds to the second linear problem of the form

$$\left(\partial_t + \partial_z^3 + \partial_{\bar z}^3 + 3\begin{pmatrix}0 & p_z\\ 0 & \omega\end{pmatrix}\partial_z + 3\begin{pmatrix}\bar\omega & 0\\ p_{\bar z} & 0\end{pmatrix}\partial_{\bar z} + \frac{3}{2}\begin{pmatrix}\bar\omega_{\bar z}\,, & 2p\omega\\ -2p\bar\omega\,, & \omega_z\end{pmatrix}\right)\psi = 0\,. \qquad (6.16)$$

The associated nonlinear integrable equation for p is of the form

$$p_t + p_{zzz} + p_{\bar z\bar z\bar z} + 3p_z\omega + 3p_{\bar z}\bar\omega + \frac{3}{2}\,p\bar\omega_{\bar z} + \frac{3}{2}\,p\omega_z = 0, \;\; \omega_{\bar z} = (p^2)_z\,. \quad (6.17)$$

Equation (6.17) is the first higher equation associated with the Davey–Stewartson (DS) system for two functions p, q under the constraint $q = -p$ (see e.g. [8–22]). It was shown in [55] that equation (6.17) is connected via (degenerated) Miura type transformation with the Veselov–Novikov (NVN-II) equation. So one can refer to equation (6.17) as the modified VN (mVN) equation. The hierarchy of integrable PDEs associated with the LP (6.1) arises as the compatibility condition of (6.1) with the LPs of the form (5.4) with odd n. All members of this mVN hierarchy commute to each other and are integrable by the IST method. The time evolutions of the wavefunction ψ also is governed by the hierarchy of nonlinear integrable equations.

Thus the integrable dynamics of surfaces referred to their minimal lines is induced by the mVN hierarchy via (6.5). For such dynamics one is able to solve the initial value problem for the surface $(g_{\alpha\beta}(z,\bar z,t=0), d_{\alpha\beta}(z,\bar z,t=0)) \to (g_{\alpha\beta}(z,\bar z,t)), d_{\alpha\beta}(z,\bar z,t))$ and construct the infinite family of explicit exact evolutions, using the corresponding results for the equations from mVN hierarchy. This integrable dynamics of surfaces inherits all properties of the mVN hierarchy. Note that the minimal surfaces ($p = 0$) are invariant under such dynamics.

7 General matrix 2D LP as the inducing problem

In the previous two examples we considered the particular 2D LPs which induce the surfaces referred to the certain particular local coordinate systems.

Here we shall discuss the general 2D LP which shall induce generic surfaces in R^3.

So let we have the matrix LP

$$A_2\partial_{u^1}\psi - A_1\partial_{u^2}\psi + P(u^1,u^2)\psi = 0\,. \tag{7.1}$$

where $\partial_{u^i} \equiv \frac{\partial}{\partial u^i}$, ψ, P are the matrix-valued functions on u^1, u^2 and A_1, A_2 are the constant diagonal matrices. LP formally adjoint to (7.1) is of the form

$$\partial_{u^1}\psi^* A_2 - \partial_{u^2}\psi^* \cdot A_1 - \psi^* P = 0\,. \tag{7.2}$$

It follows from (7.1) and (7.2) that

$$\partial_{u^1}(\psi^* A_2\psi) - \partial_{u^2}(\psi^* A_1\psi) = 0\,. \tag{7.3}$$

Equation (7.3) implies that

$$\psi^* A_2\psi = \partial_{u^2}\omega\;, \psi^* A_1\psi = \partial_{u^1}\omega\,, \tag{7.4}$$

where $\omega(u^1,u^2)$ is a matrix-valued function. Introducing three linearly independent matrices H^i $(i=1,2,3)$ and denoting $X^i = \mathrm{tr}\,(H^i\omega)$, one rewrites (7.4) in the form

$$\frac{\partial X^i}{\partial u^\alpha} = \mathrm{tr}\,(H^i\psi^* A_\alpha\psi)\,,\quad i=1,2,3\,;\quad \alpha = 1,2\,. \tag{7.5}$$

A next step of the approach is to treat the three functions X^i as the coordinates of the variable point of the surface. So, starting with the LP (7.1), we define the surface in R^3 by the formulas

$$X^i = \int_\Gamma^{(u^1,u^2)} d\tilde{u}^\alpha\, \mathrm{tr}\,(H^i\psi^* A_\alpha\psi)\,,\quad (i=1,2,3)\,, \tag{7.6}$$

where Γ is a contour on the plane u^1, u^2. In virtue of (7.5), the integral (7.6) does not depend on the contour Γ. Using (2.3), (2.4) and (7.5), one finds the tensors $g_{\alpha\gamma}$ and $d_{\alpha\beta}$. In particular

$$g_{\alpha\beta} = \mathrm{tr}\,(H^i\psi^* A_\alpha\psi)\,\mathrm{tr}\,(H^i\psi^* A_\beta\psi)\,,\quad (\alpha,\beta = 1,2)\,. \tag{7.7}$$

To construct the surface explicitly one has to have explicit solutions of (7.1) and (7.2). It is sufficient to take real A_1, A_2, p and ψ to get real surface. The simplest choice $p=0$ give rises apparently to the metric $g_{\alpha\beta}$, in general,

with the exponential dependence on u^1 and u^2. Less trivial explicit solutions of (7.1) and (7.2) can be constructed by the IST method. In particular, using the so-called $\bar\partial$-dressing method (see e.g. reviews in [18,20–22]), one gets the infinite classes of exact explicit solutions of (7.1) and (7.2) with functional parameters, rational solutions and so on. All these solutions induce the corresponding surfaces. If the order of the matrices in (7.1) is higher than two then the procedure described above give rises to the surfaces with functionally independent $g_{\alpha\beta}$.

Now let us consider the dynamics of the surfaces constructed. The integrable dynamics of surface is induced obviously by the integrable evolutions of the LP (7.1). Integrable evolutions of the potential P arise as compatibility condition of the LP (7.1) with another LP the type

$$\psi_{t_n} = \sum_{e,k=0}^{k+e=n} B_{ke}(u,t)\partial^k_{u^1}\partial^e_{u^2}\psi\,, \tag{7.8}$$

which defines the time evolution of ψ in time. Here B_{ke} are matrix-valued functions. In the simplest case $n=1$ the compatibility condition of (7.1) and (7.8) is equivalent to the nonlinear equation ($A_2 \equiv 1$, $A_1 \equiv A$, $B_{10} = 0$, $B_{01} \equiv B$, $B_{00} \equiv Q$, $D_{ii} = 0$)

$$P_{t_1} + Q_{u^1} - AQ_{u^2} - BP_{u^2} + [P,Q] = 0\,, \tag{7.9}$$

where $[B,P] = [Q,A]$ and $f_{u^i} \equiv \frac{\partial f}{\partial u^i}$. Equation (7.9) describes the resonantly interacting waves on the plane. It is integrable by the IST method ([56–57] and e.g. [16–22]). The time evolution of ψ^* is described by the equation adjoint to (7.9) the compatibility condition of which with (7.2) give rises to the same equation (7.9).

Equation (7.9) is the first member of infinite family of integrable equations associated with the LP (7.1). Considering the second LPs (7.8) for $n = 2,3,4,\ldots$, one constructs so-called higher equations (7.9). All the members of such hierarchy commute to each other. All of them are amenable to the IST method [16–22]. If ψ evolves in time according to (7.8) and (7.9) then the formulas (7.6) induce the time evolution of the surface. So we have the infinite hierarchy of integrable evolutions of surfaces. The IST method allows us to solve (linearise) the initial value problem $(P(u^\alpha, t=0), \psi(u^\alpha, t=0)) \to (P(u^\alpha,t), \psi(u^\alpha,t))$ for equation (14) and all higher equations via certain set of linear problems. This procedure apparently generates the solution of the initial value problem $(g_{\alpha\beta}(u^1,u^2,t=0), d_{\alpha\beta}(u^1,u^2,t=0)) \to (g_{\alpha\beta}(u^1,u^2,t), d_{\alpha\beta}(u^1,u^2,t))$ for the induced surfaces. The IST method provides also the infinite class of exact explicit solutions with functional parameters, multisoliton solutions, rational solutions for equation (7.9) and its higher equations. These solutions give rise to the corresponding explicit exact dynamics of the surfaces.

8 Particular inducings

The general LP (7.1) contains several interesting special cases. First, we shall consider the 2×2 LPs ($N = 2$).

1. First example $\underline{N=2}$: $A_2 = 1$, $A_1 = -\sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where $\sigma^2 = \pm 1$. One has the LP

$$\frac{\partial \psi}{\partial u^1} + \sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial \psi}{\partial u^2} + \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \psi = 0\,, \tag{8.1}$$

where p and q are scalar functions. LP problem (8.1) is known as the two-dimensional ZS–AKNS or Davey–Stewartson (DS) LP. Following the general approach of section 5 one can induce a surface in R^3, using LP (8.1). Such surface is not a generic one since LP (8.1) contains only the two functional parameters p and q.

The dynamics of the induced surfaces is generated by the nonlinear PDEs which are equivalent to the compatibility conditions of the LP (8.1) and LPs (7.8). This is the well-known DS hierarchy (see e.g. [16–22]). Thus we have the dynamics of surfaces induced by the DS hierarchy.

2. $\underline{N=2}$. $A_2 = 1$, $A_1 = -\sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $q = -p$, i.e. the LP

$$\frac{\partial \psi}{\partial u^1} + \sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial \psi}{\partial u^2} + \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix} \psi = 0\,. \tag{8.2}$$

Introducing the characteristic variables ξ and η via $\partial_\xi = \partial_{u^1} + \sigma \partial_{u^2}$, $\partial_\eta = \partial_{u^1} - \sigma \partial_{u^2}$, one gets

$$\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\eta \end{pmatrix} \psi + \begin{pmatrix} 0 & -p \\ p & 0 \end{pmatrix} \psi = 0\,. \tag{8.3}$$

In the case $\sigma^2 = -1$, ($\sigma = i$) one has the LP (6.1). Hence, one has the induced surfaces referred to their minimal lines and their dynamics induced by the mVN-hierarchy. At $\sigma = 1$ the variables ξ and η are real. The LP (8.3) again induces a surface referred to its minimal lines. But now such a surface is a complex one.

3. $\underline{N=2}$. $A_2 = 1$, $A_1 = -\sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $q = -1$, the LP is

$$\frac{\partial \psi}{\partial u^1} + \sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial \psi}{\partial u^2} + \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \psi = 0\,. \tag{8.4}$$

In the characteristic variables ξ and η, one has for ψ_2 the LP (5.1)

$$\frac{\partial^2 \psi_2}{\partial \xi \partial \eta} - p\psi_2 = 0\,. \tag{8.5}$$

So in this case one has the NVN induced surfaces and the dynamics induced by the NVN hierarchy (section 5).

4. $\underline{N=2}$. $A_2 = 1$, $A_1 = -\sigma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $q = -1$. Under this reduction the LP (7.1) is equivalent to the following scalar LP for ψ_1

$$\sigma \frac{\partial \psi_1}{\partial u^2} + \frac{\partial^2 \psi_1}{\partial u^{1^2}} + p\psi_1 = 0\,. \tag{8.6}$$

It is the well-known LP associated with the Kadomtsev–Petviashvili (KP) equation [16–22]. So one can construct surfaces induced by the KP linear problem (8.6). Their integrable dynamics is induced by the KP hierarchy of equations. These surfaces are not generic: their metric $g_{\alpha\beta}$ is parametrized by the single function $p(u^1, u^2, t)$.

5. The simplest nontrivial generic case corresponds to the LP (7.1) with $N = 3$ and the skew-symmetric potential P, i.e to the problem

$$\frac{\partial \psi}{\partial u^1} + A \frac{\partial \psi}{\partial u^2} + P\psi = 0\,, \tag{8.7}$$

where $A_{ik} = a_i \delta_{ik}$, $P_{ii} = 0$, $P_{ik} = -P_{ki}$ $(i, k = 1, 2, 3)$. For the problem (8.7) ψ^T obeys the same equation as ψ^*. So one can put $\psi^* = \psi^T$. The surfaces which are induced by the formulas (7.6) are now generic one.

The nonlinear PDE (7.9) in this case is the system of the three nonlinear PDEs for the components of P_{ik} and they describe the resonant interaction of the three ways. Thus, the simplest integrable dynamics of the generic surfaces is induced by the three resonantly interacting waves equations.

Considering other 2+1-dimensional integrable PDEs one can construct induced surfaces and their integrable dynamics of the other types.

9 Integrable dynamics of curves as the one-dimensional limit of the integrable dynamics of surfaces

Now let us study the one-dimensional limit of our construction. For definetiness we shall consider the inducing described in section 6. The one-dimensional limit means that the function p is the LP (6.1) depends only on the one variable. The constraints on p and ψ which corresponds to the one-dimensional limit and which are typical for the IST method are of the form

$$(\partial_z - \partial_{\bar z})p = 0, \ \ (\partial_{\bar z} - \partial_z)\psi = 2i\lambda\psi \, , \tag{9.1}$$

where λ is a real parameter. The matrix $\bar\psi$ obeys the same constraint as ψ. As a result, the constraint for the coordinates X^k is of the form

$$(\partial_{\bar z} - \partial_z)X^k = 4i\lambda X^k \quad (k = 1,2,3)\, . \tag{9.2}$$

In the terms of real isometric coordinates s and σ defined via $z = \frac{1}{2}(s - i\sigma)$ the above constraints imply $p = p(s,t)$, $\psi = \exp(\lambda\sigma)\cdot\chi(s,t)$ and

$$X^i = \exp(2\lambda\sigma)\tilde X^i(s,t) \quad (i = 1,2,3)\, . \tag{9.3}$$

Then the formulae (6.7) gives $K = 0$ and $K_m = 2p\exp(-2\lambda\sigma)$. These formulae show us that in the one-dimensional limit (9.1) our surface is the cone type surface generated by the curve with coordinates $\tilde X^i(s,t)$ $(i = 1,2,3)$, i.e. it is reduced, effectively, to a curve with the curvature $p(s,t)$. Under the constraint (9.1) the LP (6.1) is reduced to the one-dimensional AKNS type LP for χ with the spectral parameter λ (see e.g. [16–22])

$$\partial_s\chi = \begin{pmatrix} i\lambda & p \\ -p & -i\lambda \end{pmatrix}\chi \tag{9.4}$$

and equation (6.17) is converted into the modified Korteweg-de Vries (mKdV) equation

$$p_t + 2p_{sss} + 12p^2 p_s = 0\, . \tag{9.5}$$

The higher mVN equations are reduced to the higher mKdV equations.

Thus, in the one-dimensional limit (6.1) our approach provides the induced curves in R^3 and their integrable dynamics induced by the mKdV hierarchy. As far as concerned to the integrable dynamics of curves via the mKdV hierarchy we reproduce here the results of the papers [58–62].

Further the formula (9.2) implies $(\partial_{\bar z} - \partial_z)X^i\cdot(\partial_{\bar z} - \partial_z)X^i = -16\lambda^2 X^i X^i$. Taking into account that $\partial_z X^i \cdot \partial_z X^i = \partial_{\bar z} X^i \cdot \partial_{\bar z} X^i = 0$ and using (6.5), one gets $(2\lambda)^2 X^i X^i = \det^2\psi$. In virtue of (9.3) and $\psi = \exp(\lambda\sigma)\chi(s,t)$, one has $(2\lambda)^2 \tilde X^i \tilde X^i = \det^2\chi$. For the one-dimensional problem (9.4) $\det\chi$=const. Let $\det\chi = 1$. So one finally gets

$$\tilde X^i \tilde X^i = (2\lambda)^{-2}. \tag{9.6}$$

Thus, the curve the points of which have coordinates $\tilde X^i(s,t)$ lies on the sphere of radius $1/2\lambda$. The entire dynamics of the curve takes place on the sphere S^2. Integrable motions of curves on spheres have been discussed recently in [62]. In our approach such motions appears naturally as the one-dimensional limit (9.1) with $\lambda \neq 0$ of integrable motions of surfaces. At $\lambda = 0$ one has the integrable motions of curves on the plane described in [58–61].

In similar manner one can consider the one-dimensional limit for the dynamics induced by the other LPs.

10 Integrable evolutions of surfaces and three-dimensional Riemann spaces

First, we note that all the above constructions can be, obviously, used to induce surfaces in the pseudo-Euclidean space with the metric $h_{ik} = \varepsilon_i\delta_{ik}$ $(i,k = 1,2,3)$ $\varepsilon_i = \pm 1$. The only difference is that the usual summation over indices i,k,e should by substituted by the summation $X^i X^i \stackrel{\text{def}}{=} X^i\varepsilon_i X^i$.

Second, since the 2D LPs have infinite number of linearly independent solutions one can introduce any number of the coordinates X^i of the type (4.6). So one can induce surfaces in the pseudo-Euclidean space E^n of any dimension. In this case the induces i,k,e take the values $1,2,3,...,n$ in all above formulas.

Of course, any surface can be immersed in E^3. But to consider the geometrical objects which are obtained by the evolutions (deformations) of surfaces it is convenient to embed surfaces in the higher dimensional spaces E^n.

The geometrical object which is the collection of the induced surfaces taken at all values of time t is a three-dimensional manifold. It is, in fact, the three-dimensional Riemann space. We shall consider three examples to demonstrate this fact.

General 2D LP (7.1) and their integrable dynamics is the first example. Let now ψ in (7.1) is the column with m components. Correspondingly ψ^*

is the row. The coordinate X^i of the surface in E^n are defined via

$$\frac{\partial X^i}{\partial u^\alpha} = \psi^{*(i)} A_\alpha \psi^{(i)}\,, \quad i = 1, ..., n; \;\; \alpha = 1, 2\,, \tag{10.1}$$

where $\psi^{(i)}$ and $\psi^{*(i)}$ are linearly independent solutions of the LPs (7.1) and (7.2).

Let the evolution of such induced surface is generated by the nonlinear PDE (7.9). The corresponding LP (7.8) is of the form

$$\psi_{u^3} = A_3 \psi_{u^2} + Q\psi\,, \tag{10.2}$$

where $[A_3, P]=[Q, A]$ and we denote $t_1 \equiv u^3$, $B = A_3$. The problem formally adjoint to (10.2) is of the form

$$\psi^*_{u^3} = \psi^*_{u^2} A_3 - \psi^* Q\,. \tag{10.3}$$

Equations (10.2) and (10.3) imply that

$$\frac{\partial}{\partial u^3}(\psi^{*(i)}\psi^{(i)}) = \frac{\partial}{\partial u^2}(\psi^{*(i)} A_3 \psi^{(i)})\,. \tag{10.4}$$

Equation (10.4) together with (7.4) shows us that one can define the quantities X^i via

$$\frac{\partial X^i}{\partial u^\alpha} = \psi^{*(i)} A_\alpha \psi^{(i)}\,, \quad i = 1, ..., n\,, \tag{10.5}$$

where now $\alpha = 1, 2, 3$. Equations (10.5) are compatible due to (7.1), (7.2), (10.2), (10.3).

Now one can treat u^1, u^2, u^3 as the local coordinates of the three-dimensional manifold and X^i given by integrals

$$X^i = \iiint\limits_\Gamma^u \sum_{\alpha=1}^{3} du^\alpha \psi^{*(i)} A_\alpha \psi^{(i)}, \quad i = 1, ..., n \tag{10.6}$$

as the coordinates of this manifold embedded into E^n. We introduce the metric into this three-dimensional manifold by the formula

$$g_{\alpha\beta} = \frac{\partial X^i}{\partial u^\alpha} \varepsilon^i \frac{\partial X^i}{\partial u^\beta} = \psi^{*(i)} A_\alpha \psi^{(i)} \varepsilon_i \psi^{*(i)} A_\beta \psi^{(i)}\,, \tag{10.7}$$

where $\varepsilon_i = \pm 1$.

Thus, the 2+1-dimensional resonantly interacting waves equations (7.9) induce the three-dimensional Riemann space via (10.6). Since any three-dimensional Riemann space can be embeded into E^6 (see e.g. [63]) then it is sufficient to take $n = 6$ in the formulas (10.6).

Second example is given by the Lelieuvre's inducing (section 5). Let the evolution of the induced surfaces is generated by the NVN equation (5.6) with $\alpha = \beta = 1$ and $t = u^3$. It is not difficult to show that the LP (5.7) together with (3.4) implies that

$$\frac{\partial X^i}{\partial u^3} = -\varepsilon^{ike}\left(\psi^{(k)}\frac{\partial^3\psi^{(e)}}{\partial u^1} - \psi^{(k)}\frac{\partial^3\psi^{(e)}}{\partial u^2} + 2\frac{\partial^2\psi^{(k)}}{\partial u^1}\frac{\partial\psi^{(e)}}{\partial u^1}\right), \tag{10.8}$$
$$i = 1, ..., n\,,$$

where $\psi^{(k)}$ ($k = 1, ..., n$) are linearly independent solutions of the LP (3.4).

An infinite family of the induced surfaces, taken at all values of the time t form the three-dimensional manifold. It is the Riemann space with the local coordinates u^1, u^2, u^3 which is embeded into the pseudo-Euclidean space E^n by the formula

$$X^i = \varepsilon^{ike}\iiint\limits_\Gamma \Big\{\psi^{(k)}\frac{\partial\psi^{(e)}}{\partial u^1}du^1 - \psi^{(k)}\frac{\partial\psi^{(e)}}{\partial u^2}du^2 -$$
$$\left(\psi^{(k)}\frac{\partial^3\psi^{(e)}}{\partial u^1} - \psi^{(k)}\frac{\partial\psi^{(e)}}{\partial u^2} + 2\frac{\partial^2\psi^{(k)}}{\partial u^1}\frac{\partial\psi^{(e)}}{\partial u^1}\right)du^3\Big\}\,. \tag{10.9}$$

The metric $g_{\alpha\beta}$ of this Riemann space is

$$g_{\alpha\beta} = \frac{\partial X^i}{\partial u^\alpha}\varepsilon_i\frac{\partial X^i}{\partial u^\beta}\,, \tag{10.10}$$

where $\frac{\partial X^i}{\partial u^\alpha}$ are given by (3.5) and (10.8).

In similar manner one can construct the three-dimensional Riemann spaces associated with the integrable dynamics of the other induced surfaces discussed above.

Instead we shall consider the three-dimensional Riemann space which is induced by the nonlinear integrable system which contains all independent variables u^1, u^2, u^3 in a very symmetric manner. This is the Darboux system

$$\frac{\partial^2 H_\alpha}{\partial u^\beta \partial u^\gamma} = \frac{1\partial H_\beta}{H_\beta \partial u^\gamma}\frac{\partial H_\alpha}{\partial u^\beta} + \frac{1}{H_\gamma}\frac{\partial H_\gamma}{\partial u^\beta}\frac{\partial H_\alpha}{\partial u^\gamma}\,, \tag{10.11}$$

where $\alpha, \beta, \gamma = 1, 2, 3; \alpha \neq \beta \neq \gamma \neq \alpha$; $H_\alpha(u^1, u^2, u^3)$ are scalar functions and there is no summation over repeated indices in this and next formulas. The system (10.11) have appeared within the description of the triply conjugate systems of surfaces in R^3 [9]. The applicability of the IST method to the system (10.11) has been discovered in [64]. Wide classes of exact solutions of this system have been found in [36,22].

The Darboux system (10.11) can be reformulated as the system of the first order PDEs [9]. Indeed, introducing the quantities

$$p_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{H_\alpha} \frac{\partial H_\beta}{\partial u^\alpha}, \tag{10.12}$$

one rewrites (10.1) in the equivalent form [9]

$$\frac{\partial p_{\alpha\beta}}{\partial u^\gamma} = p_{\alpha\gamma} p_{\gamma\beta}, \quad \alpha, \beta, \gamma = 1, 2, 3; \quad \alpha \neq \beta \neq \gamma \neq \alpha. \tag{10.13}$$

The system (10.13) is equivalent to the compatibility condition for the LP [9]

$$\frac{\partial \psi_\alpha}{\partial u^\beta} = p_{\alpha\beta} \psi_\beta, \quad \alpha, \beta = 1, 2, 3. \tag{10.14}$$

The LP adjoint to (10.14) is of the form [9,37,65]

$$\frac{\partial \psi^*_\alpha}{\partial u^\beta} = p_{\beta\alpha} \psi^*_\beta. \tag{10.15}$$

Equations (10.14) and (10.16) imply the existence of the function M such that [9,37,65]

$$\frac{\partial M}{\partial u^\alpha} = \psi^*_\alpha \psi_\alpha \quad (\alpha = 1, 2, 3). \tag{10.16}$$

Considering the n linearly independent solutions $\psi^{(i)}_\alpha$, $\psi^{*(i)}_\alpha$ of the LPs (10.14), (10.15), one can introduce n quantities X^i via

$$\frac{\partial X^i}{\partial u^\alpha} = \psi^{*(i)}_\alpha \psi^{(i)}_\alpha, \quad i = 1, ..., n; \quad \alpha = 1, 2, 3. \tag{10.17}$$

Then the formula

$$X^i = \iiint \sum_{\alpha=1}^{3} du^\alpha \psi^{*(i)}_\alpha \psi^{(i)}_\alpha, \quad i = 1, ..., n \tag{10.18}$$

defines the embeding of the three-dimensional manifold with the 8local coordinates u^1, u^2, u^3 into the pseudo-Euclidean space E^n. The corresponding Riemann metric $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta} = \sum_{i=1}^{n} \frac{\partial X^i}{\partial u^\alpha} \varepsilon_i \frac{\partial X^i}{\partial u^\beta} = \sum_{i=1}^{n} \psi^{*(i)}_\alpha \psi^{(i)}_\alpha \varepsilon_i \psi^{*(i)}_\beta \psi^{(i)}_\beta \quad (\alpha, \beta = 1, 2, 3), \tag{10.19}$$

where $\varepsilon_i = \pm 1$. It is apparently sufficient to take $n = 6$. The metric (10.19) is not the generic one since one has, in fact, only three free functional parameters H^*_α ($\alpha = 1, 2, 3$) (see (10.12)).

The inducing the three-dimensional Riemann spaces via the 2+1-dimensional integrable PDEs described here is, to some extent, a generalization of the 1+1-dimensional "soliton surface approach" of [39].

11 Conclusion

The integrable dynamics of induced surfaces inherits all remarkable properties and features of the 2+1-dimensional integrable PDEs: special, soliton-like, exact solutions, infinite number of conserved quantities, infinite symmetry groups, bilocal and hamiltonian structures, τ-function, vertex operators, Darboux–Backlund transformations (see e.g. [16–22]). The study of the corresponding induced properties of 8surface dynamics is of a great interest.

In particular, the fact that the quantities of the type ω^{ik} appear in the theory of the Darboux transformations [65] indicates a strong relevance of the induced Darboux transformations in the theory of the induced surfaces.

The 2+1-dimensional integrable PDEs with constraints considered, for instance, in [66–68], can be used for inducing the integrable dynamics of surfaces which preserve certain geometrical structures.

The method described in this paper demonstrates also the importance of the wavefunctions ψ of the LPs. First, they are the basic quantities to induce surfaces: tangent vectors on the surfaces (see (4.5)) are the bilinear combinations of the wavefunctions ψ, ψ^*. Then, the dynamics of surfaces is induced, in fact, by the time dynamics of the wavefunctions ψ. The corresponding wavefunction equations (see e.g. (5.8)) are the IST integrable and have a number of interesting properties [69].

The method of the inducing the surfaces via 2D LPs and their integrable dynamics via the 2+1-dimensional integrable PDEs may have applications both in physics and mathematics. In physics it may lead to the invention of the exactly solvable models of the time dynamics of interfaces, fronts. It may be an effective tool to treat some old and new problems of the differential geometry of surfaces. It will be particularly useful in the theory of deformations of surfaces. There are a number of open problems connected with the method proposed in this paper. We shall discuss some of them in future publications.

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