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CONSTRAINED INSTANTON AND BARYON NUMBER NON-CONSERVATION AT HIGH ENERGIES

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Constrained Instanton and Baryon Number Non–Conservation at High Energies

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Abstract

The total cross – section for baryon number violating processes at high energies is usually parametrized as $\sigma_{total} \propto \exp(\frac{4\pi}{\alpha}F(\varepsilon))$, where $\varepsilon = \sqrt{s}/E_0$ $E_0 = \sqrt{6}\pi m_w/\alpha$. In the present paper the third nontrivial term of the expansion

$$F(\varepsilon) = -1+\frac{9}{8}\varepsilon^{4/3}-\frac{9}{16}\varepsilon^2-\frac{9}{32}\left(\frac{m_h}{m_w}\right)^2\varepsilon^{8/3}\log\left(\frac{1}{3\varepsilon}\left(\frac{2m_w}{\gamma m_h}\right)^2\right)+O(\varepsilon^{8/3})$$

is obtained. The unknown corrections to $F(\varepsilon)$ are expected to be of the order of $\varepsilon^{8/3}$, but have neither $(m_h/m_w)^2$, nor $\log(\varepsilon)$ enhancement. The total cross - section is extremely sensitive to the value of single Instanton action. The correction to Instanton action $\Delta S \sim (m\rho)^4\log(m\rho)/g^2$ is found (ρ is the Instanton radius). For sufficiently heavy Higgs boson the ρ-dependent part of the Instanton action is changed drastically. In this case even the leading contribution to $F(\varepsilon)$, responsible for a growth of cross - section due to the multiple production of classical W-bosons, is changed:

$$F(\varepsilon) = -1+\frac{9}{8}\left(\frac{2}{3}\right)^{2/3}\varepsilon^{4/3}+\ldots\ ,\ \varepsilon \ll 1 \ll \varepsilon\left(\frac{m_h}{m_w}\right)^{3/2} .$$

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1 Introduction

A few years ago it was recognized, that the total cross-section for baryon number violating processes, which is small like $\exp(-4\pi/\alpha) \approx 10^{-169}$ [1] (where $\alpha = g^2/(4\pi)$) at low energies, may become unsuppressed at TeV energies [2, 3, 4]. It was shown [5, 6] that the cross-section for the processes accompanied by the multiple emission of W and H bosons has the generic form

$$\sigma_{total} \propto \exp\left(\frac{4\pi}{\alpha}F(\varepsilon)\right)\ , \tag{1}$$

where $\varepsilon = \sqrt{s}/E_0$ and $E_0 = \sqrt{6}\pi M_w/\alpha$. Up to now the values of three nontrivial terms of $F(\varepsilon)$ expansion at small ε were claimed in the literature:

$$F(\varepsilon) = -1+\frac{9}{8}\varepsilon^{4/3}-\frac{9}{16}\varepsilon^2+\frac{3}{16}\varepsilon^{8/3}\log\left(\frac{c}{\varepsilon}\right)+\ldots . \tag{2}$$

Here the zeroth term is the old result by 't Hooft [1], the first term $\sim \varepsilon^{4/3}$ was found in refs. [5, 7, 8] and the second term $\sim \varepsilon^2$ was obtained in refs. [9, 10, 11, 12]. The last term $\sim \varepsilon^{8/3}\log(\varepsilon)$ was found in ref. [13]. Authors of [13] have calculated the 2-loop correction to multiple W-boson emission. They have not found the precise argument of log and guessed $\log c = 1$ for numerical estimates.

In the present paper some new contributions to $F(\varepsilon)$ of the order of $\varepsilon^{8/3}\log(\varepsilon)$ are found which were not taken into account in [13]. The final result reads

$$F(\varepsilon) = -1+\frac{9}{8}\varepsilon^{4/3}-\frac{9}{16}\varepsilon^2-\frac{9}{32}\left(\frac{m_h}{m_w}\right)^2\varepsilon^{8/3}\log\left(\frac{1}{3\varepsilon}\left(\frac{2m_w}{\gamma m_h}\right)^2\right)+O(\varepsilon^{8/3})\ . \tag{3}$$

Here m_w, m_h are W and Higgs boson masses, γ is the Eurler's constant ($\log\gamma = 0.5772\ldots$). In view of the current situation with H-mass the factor $(m_h/m_w)^2$ is not small and possibly is sufficiently large. The corrections to (3) are of the order of $\varepsilon^{8/3}$ but have no enhancement either $(m_h/m_w)^2$, or $\log(\varepsilon)$.

The main physical question we are faced is: can the total cross-section be large? In other words does $F(\varepsilon)$ vanish at some $\varepsilon \sim 1$? Let us see, does one can answer this question with only a few terms of $F(\varepsilon)$ expansion (3). We know only one heuristic rule which allows to check the accuracy of such a partial sum. The last term in the expansion should be smaller than the previous one. According to this "rule" the result of [13], eq. (2), still allows one to believe in the validity of the expansion up to sufficiently large values $\varepsilon \sim 1$. Unfortunately, the last term of the corrected expansion (3) becomes equal to ε^2 term (see fig.) for $m_h = 2 \div 3 m_w$ at $\varepsilon = 0.3 \div 0.12$. Moreover, in **Section 6** we argue, that formula (3) may be used only at $3\varepsilon < \left(\frac{2m_w}{\gamma m_h}\right)^2$.

The growth of total cross-section is dominated mostly by the multiple production of W-bosons (the typical multiplicity is $N \sim \varepsilon^{4/3}/g^2 \gg 1$). In the leading classical approximation the birth of new W leads to the factor

$$\frac{4\pi^2\rho^2}{g} U^{ab}\bar{\eta}^b_{\mu\nu} k_\nu \propto \rho^2 \tag{4}$$

in baryon number violating amplitude (ρ is the Instanton radius). Therefore the contribution of small size Instantons is strongly suppressed in the multiparticle cross-section. In order to perform the integration over ρ it is necessary to utilize the ρ dependence of the Instanton action, which appears in spontaneously broken Yang-Mills theory. All the authors have used for calculation of the cross-section the Instanton action of the form [1, 14]

$$S_I = \frac{8\pi^2}{g^2} + \pi^2(\rho v)^2 \ , \tag{5}$$

where v is the Higgs vacuum expectation value. It is usually forgotten that expression (5) is approximate. But it was pointed out in the paper [14], that where are the corrections to the action (5) of the form

$$\Delta S \sim (\rho v)^2 (m\rho)^2 \log(m\rho) \ , \tag{6}$$

with m standing for W or H mass. Just the calculation of corrections like (6) to the Instantonic action in the SU(2) Yang–Mills theory is the main subject of present paper.

Being only the approximate solution of the equations of motion, the Instanton in spontaneously broken Yang–Mills theory should be a subject of some additional constraint. The problem of the best choice of the constraints, which should be used in order to introduce the collective variables, is a vital problem for our consideration. In **Section 2** we examine a possibility to distinguish consistently between the collective and quantum variables for the approximate Instanton. The result is rather pessimistic. Only the negative

restrictions on the possible choice of constraint may be formulated. The direction in the functional space associated with the collective variable should not be *orthogonal* to the unperturbed Instanton zero mode, but also should not *coincide* with (or be *very close* to) the zero mode. Only a few terms of the Instanton action expansion over $(m\rho)$ are constraint independent. Only those model independent terms may be used undoubtedly for the calculation of σ_{total}.

In **Section 3** we investigate the shape of the constrained Instanton.

In **Section 4** the corrections to S_I of the type (6) are calculated. There appears two large parameters, which allow to classify various contributions to ΔS. These are $(m_h/m_w)^2$ and $\log(m\rho)$. Only the contributions to ΔS of the form $\sim (\rho v)^2 (m_w \rho)^2$ which have neither $(m_h/m_w)^2$, nor $log(m\rho)$ enhancement may be constraint dependent. Therefore we cannot calculate the contributions to $F(\varepsilon)$ (3) of the order of $O(\varepsilon^{8/3})$ which are not enhanced by either $(m_h/m_w)^2$, or $\log(\varepsilon)$.

An additional correction to $F(\varepsilon)$ of the interesting type comes from the Higgs boson production (see **Section 6**). The multiple Higgs boson production in the limit $m_h = 0$ contribute to ε^2 term of $F(\varepsilon)$ expansion. But taking into account the finite m_h leads to the correction $\sim (m_h/m_w)^2 \varepsilon^{8/3} \log(\varepsilon)$.

It seems very attractive to sum up all the corrections to $F(\varepsilon)$ enhanced by the power of (m_h/m_w) exactly. In **Section 5** the Instanton action for the case $(m_w\rho) \ll 1 \ll (m_h\rho)$ is calculated. The value of S_I in this case differs from the well known result (5) of refs. [1, 14]. As a result even the term in $F(\varepsilon)$ which appears due to the multiple massless W-boson production differs in this case from that found in refs. [5, 7, 8]

$$F(\varepsilon) = -1 + \frac{9}{8}\left(\frac{2}{3}\right)^{2/3} \varepsilon^{4/3} + \ldots , \tag{7}$$
$$\varepsilon \ll 1 \ , \ \varepsilon \left(\frac{m_h}{m_w}\right)^{3/2} \gg 1 \ .$$

In last two years some attempts were made to use the Instanton for calculation of the multiparticle cross-section in massive ϕ^4 theory [15]. The authors of [15] have used S_I – the Instanton action which was found in [14]. We tried[2] to improve (see **Section 7**) the calculation of Instanton action in

[2] this part of the work was done together with M.E. Pospelov.

massive ϕ^4 theory and to our surprise got the result different from that of [14].

2 The choice of the constraint

Consider SU(2) Yang–Mills theory coupled to a doublet of Higgs complex fields

$$S = \int \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} \overline{(D_\mu \Phi)} (D_\mu \Phi) + \frac{\lambda}{8} (\bar{\Phi}\Phi - v^2)^2 \right\} d^4 r \ ,$$

$$D_\mu = \partial_\mu - \frac{ig}{2} \sigma^a A^a_\mu \ . \tag{8}$$

If one forget about the last ($\sim \lambda$) term in the action, there may be found an Instanton

$$A^a_{\mu 0} = \frac{2}{g} U^{ab} \bar{\eta}^b_{\mu\nu} \frac{r_\nu \rho^2}{r^2(r^2+\rho^2)} \quad , \quad \Phi_0 = \frac{r}{\sqrt{r^2+\rho^2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \ . \tag{9}$$

But for the full theory (8) no nontrivial extrema of the action exist, as can be seen from the transformation

$$A^a_\mu(r) \longrightarrow a A^a_\mu(ar) \quad , \quad \Phi(r) \longrightarrow \Phi(ar) \ . \tag{10}$$

Only the zero size Instanton can be the exact extremum of (8). Nevertheless, the functional integral for the amplitudes of processes changing the baryon number should be saturated by the Instanton – like configurations very close to Instanton (9) of a very small size $m\rho \ll 1$. These configurations are only the approximate extrema of (8) since their action S_I depends slightly on ρ (the constrained Instantons [14, 16]).

Although at $m\rho \ll 1$ the action of the approximate Instanton depends on ρ only slightly, the field configuration itself changes drastically with the change of Instanton radius. It means that while ρ is not associated with the exact zero mode, the integration over the size of Instanton should be performed with the collective variables method. One introduces the constraint into the functional integral, which allows to integrate over all the short-wave quantum variables before the integration over ρ

$$\delta(\langle A \mid W \rangle - \tau) \, d(\langle A \mid W \rangle - \tau) \ , \tag{11}$$

$$\langle A \mid W \rangle \equiv \int A^a_\mu W^a_\mu(\vec{r}) d^4 r \ .$$

Here $\tau = \tau(\rho)$. The vector $W^a_\mu(r)$ may also, generally speaking, depend on τ. After we have imposed the constraint (11), the functional integration is to be done over the hyper-planes orthogonal to the constraint function $W^a_\mu(r)$. We would naturally call the constrained Instanton the configuration which minimizes the action for a given value of collective variable τ. The equation of motion for the constrained Instanton reads

$$\frac{\delta S}{\delta A^a_\mu} = \zeta W^a_\mu \ , \tag{12}$$

where ζ is a Lagrange multiplier, which allows to satisfy the constraint (11).

Of course, one can try not to find any extremum of the action, but work with the "wrong" configuration (9). The linear terms, which appear after expansion of the action near the wrong minimum, may be treated perturbatively. But doing so, one has to be convinced that the "wrong" minimum (9) actually differs very slightly from the exact solution of the constrained equation of motion (12). This is not evident even if we are dealing with simple few particle amplitudes. In this case the usual correction to Instanton action $\Delta S \sim (m_w \rho)^2/g^2 \sim 1$ is comparable with the one loop quantum corrections. But correction to the action which comes from a naive substitution of the unperturbed Instanton (9) into the last ($\sim \lambda$) term of the action (8) diverges logarithmically, thus reflecting the fact that the unperturbed Instantonic solution should be modified. In this paper we will be interested (see Sec. 6) in the processes of a huge multiplicity $N \sim \varepsilon^{4/3}/g^2$. In this case the corrections to Instanton action are large like $\sim g^{-2}$ even though the Instanton is small $(m\rho)^2 \sim \varepsilon^{4/3} \ll 1$.

Since the constrained Instanton evidently depends (see (12)) on the vector $W^a_\mu(r)$, the "best" choice of W^a_μ seems to be the vital problem for our consideration. On the general grounds one can only state, that the result of the exact calculation of the functional integral (if it is possible to perform) should be constraint independent. From the "naive" analysis of the role of zero modes in Instantonic phenomena it seems almost evident that, as our constrained Instanton looks very similar to the unperturbed Instanton , the vector $W^a_\mu(r)$ for the "best" constraint should be very close to the Instanton zero mode. The main aim of this Section is to show that such a "naive" solution of the best constraint problem is *wrong*.

We introduce the constraint (11) in order to prohibit the integration along some dangerous directions in the functional space. While doing so we suppose, that all the rest integrals are well convergent and may be treated as quantum corrections in one–loop approximation. Therefore the only *a priori* requirement to our constraint is that it should effectively suppress the integration along all "dangerous" directions in the functional space, thus providing us with reasonable quantum corrections.

The problem of an appropriate choice of the constraint first appears for the functional integral calculation around the exact Instanton of pure Yang–Mills theory, which we overview before considering the approximate Instanton. The action of SU(2) theory does not change under dilatation, translation and isotopic spin rotation of Instanton. Therefore in order to calculate the functional integral (though to one loop) one should prohibit in some way the integration along a few directions in the functional space. To this end we introduce eight δ-functions in the integral. After that the Euclidean functional integral takes the form

$$Z = \int \prod_{i=1}^{8} \delta\Big(\langle W_i \mid A - A_I\rangle\Big) d\langle W_i \mid A - A_I\rangle \times$$

$$\times \exp\left\{-S_{YM} - S_{fix} - S_{ghost}\right\} DAD\phi_{ghost} \,, \tag{13}$$

where S_{fix} may be used e.g. [1]

$$S_{fix} = \int \frac{1}{2}(D^{cl}_{\mu}(A^a_{\mu} - A^a_{\mu I}))^2 d^4r \,.$$

We suppose that $D^{cl}_{\mu} W_{\mu} = 0$, otherwise the preexponential factor in (13) should be modified slightly. It is easy to transform the integration over $d\langle W_i \mid A - A_I\rangle$ to that over the usual collective variables ρ, $\vec{r_0}$ and angles of the isotopic spin rotation matrix U_{ab}. At least until we consider the pure Yang–Mills theory there is a great freedom in the choice of the "constraints" $W^a_{\mu i}$. The explicit form of the Instantonic solution (9) does not depend on the constraint. The quantum correction to the two point Green function, for example, depends on $W^a_{\mu i}$. But in the final result for any observable value the constraint dependence should disappear to any order of the coupling constant.

Nevertheless some important restrictions may be formulated which the allowed constraints should satisfy, if one wants to have reasonable quantum corrections.

I. The functions $W^a_{\mu i}$ should not be orthogonal to the Instanton zero modes $W^{a(0)}_{\mu i}$. More precisely the matrix $J_{ij} = \langle W_i \mid W^{(0)}_j\rangle$ should not be degenerate $\left(det J \neq 0\right)$.

II. The functions $W^a_{\mu i}$ should not coincide with (and should not be very close to) the exact zero modes $W^{a(0)}_{\mu i}$. This requirement, not so obvious as **I.**, needs additional comments.

Let us consider for example the Green function $G^{ab}_{\mu\nu}$. Suppose, we do use the exact zero modes $W^{a(0)}_{\mu\nu}$ as the constraint functions. Then Green function should satisfy the equation [17]:

$$\begin{aligned} M^{ab}_{\mu\sigma} G^{bc}_{\sigma\nu} &= \delta^{ac}\delta_{\mu\nu}\delta(x-y) - \sum_{i=1}^{8} W^{a(0)}_{\mu i}(x) W^{c(0)}_{\nu i}(y) \\ M^{ab}_{\mu\nu} &= -\delta_{\mu\nu} D^{ac}_{\sigma} D^{cb}_{\sigma} - 2g\varepsilon^{acb} F^{c}_{\mu\nu} \,. \end{aligned} \tag{14}$$

The normalized zero modes $W^{a(0)}_{\mu i}$ are themselves the eigenfunctions of $M^{ab}_{\mu\nu}$ having zero eigenvalues. Let $\Psi^a_{\mu k}$ be the set of eigenfunctions of the operator $M^{ab}_{\mu\nu}$ having eigenvalue ε_k. The formal solution to the equation (14) is

$$G^{ab}_{\mu\nu} = \sum_{k=9}^{\infty} \frac{\Psi^a_{\mu k}(x)\Psi^b_{\nu k}(y)}{\varepsilon_k} \,. \tag{15}$$

Formula (13) with our choice of constraint $W_i \equiv W^{(0)}_i$ implies that eight zero modes $W^{a(0)}_{\mu i}$ (the eight terms $k = 1, ..., 8$) should be omitted in the sum (15). But in massless theories like pure Yang-Mills there exists an infinite number of operator $M^{ab}_{\mu\nu}$ eigenfunctions having arbitrary small eigenvalues. As we will see below, the summation over these "soft" fluctuations leads to a divergence of the Green function (15). The "soft" eigenmodes Ψ_k of $M^{ab}_{\mu\nu}$ (those of $\varepsilon_k \ll \rho^{-2}$) can be found explicitly. But we can prove that the Green function (15) is infinite without an explicit summation. At large x, y the eq. (14) is simplified drastically

$$\begin{aligned} -\Delta G^{ab}_{\mu\nu} &= \delta^{ab}\delta_{\mu\nu}\delta(x-y) \,, \\ G^{ab}_{\mu\nu} &= \delta^{ab}\delta_{\mu\nu}\frac{1}{4\pi^2(x-y)^2} \,, \quad x, y \gg \rho \,. \end{aligned} \tag{16}$$

Because we have used the exact zero modes $W_i^{(0)}$ as constraint functions in the functional integral (13), the Green function (15) should be orthogonal to zero modes.

Throughout the paper we are only interested in the explicit expression for dilatational zero mode

$$W_{\mu D}^{a(0)} \sim \frac{\partial A_{\mu I}^a}{\partial \rho} \sim U^{ab} \frac{\bar{\eta}_{\mu\nu}^b r_\nu}{(r^2+\rho^2)^2} . \tag{17}$$

It is seen immediately from eqs. (16),(17) that $G_{\mu\nu}^{ab}$ averaged over dilatational zero mode $W_{\mu D}^{a(0)}$ is

$$\int W_{\mu D}^{a(0)}(x) G_{\mu\nu}^{ab}(x,y) W_{\nu D}^{b(0)}(y) d^4x d^4y = \infty . \tag{18}$$

Any finite solution of eq. (14) has asymptotic (16) and thus is not orthogonal to the dilatational zero mode. The solution of (14) can be made orthogonal to zero modes by a transformation

$$\tilde{G}_{\mu\nu}^{ab} = G_{\mu\nu}^{ab} + \alpha W_{\mu D}^{a(0)}(x) W_{\nu D}^{b(0)}(y) \tag{19}$$

with suitable α. But due to (18) one has to add the zero modes with infinitely large weight α to the finite solution of (14) in order to get the orthogonalized Green function.

We have used the δ-functions in the functional integral (13) in order to avoid the integration along the most dangerous directions in the functional space. We call "dangerous" those directions in the functional space along which the Gaussian integration cannot be used. But there turns out to be an infinite number of such directions among the eigenmodes of the operator M (eq. (14)). These are not only the eight zero modes, but also the infinite number of soft, longwave excitations. If we chose zero modes as the constraint functions in (13), then we leave unaffected (due to orthogonality of Hermitian operator M eigenvalues) the integral over all the rest dangerous modes. In order to affect the integral along infinite number of modes with only a few constraints one should choose the functions $W_{\mu i}^a$ to be different from any eigenmode of the operator $M_{\mu\nu}^{ab}$ (eq. (14)).

In the paper [17] the formal expression for the Green function (14) was found. But if one calculate explicitly the integral in this expression, the Green

function become infinite. In ref. [18] the finite well-defined Green function for Yang–Mills theory gauge fields with constraints decreasing much faster than the exact zero modes was found.

Now we can turn back to the Instanton in spontaneously broken Yang–Mills theory (8). As we have said before only a small size Instanton ($m_w\rho \ll 1, m_h\rho \ll 1$) may be treated semiclassically. Like in pure Yang–Mills theory, one need eight δ–functions in the functional integral (13) in order to describe the Instanton collective variables. At $m\rho \ll 1$ not only Instanton itself, but also the quantum fluctuations around it should differ very slightly from that in the pure Yang-Mills case. Therefore all our arguments concerning a possible choice of constraints are still valid. The constrained Instanton is a solution of eq. (12), where the vector W_μ^a is taken from the constraint associated with dilatational zero mode. The solution of (12) differs slightly from the Instanton of pure Yang-Mills theory and depends, though also slightly, on the explicit choice of the constraint. Like there is no the best choice of constraint, there is also no the best choice of the Instanton in spontaneously broken Yang–Mills theory. Of course, if we really discuss a very small Instanton, all the constraint dependence should disappear in the final results to any order of perturbation theory (see for example the proof of constraint indepedence of the quantum corrections to total cross-section for baryon number violating processes [10, 13]).

The problem we have discussed looks very similar to the problem of definition of the Instanton –Antiinstanton (I–A) configuration. The I–A pair is only an approximate minimum of the action. In order to describe such a configuration one should solve the constrained equation of the type (12). Thus if we want to find exactly the I–A configuration, we are again faced with the problem of the best constraint. There exist various approaches to the problem of the "best" choice of the I–A configuration [19, 20]. But for pseudoparticles far separated from each other both approaches quoted above turn out to be very close. In this case the only particular choice of constraint functions which appears in the right hand side of the eq. (12) is the linear combination of the single pseudoparticle zero modes. But as we learned, just that choice of constraint leads to unreliable quantum corrections, at least for the Yang–Mills theory. Like for the Instanton in spontaneously broken Yang–Mills theory, there is no "best" constraint for I–A pair in pure Yang–Mills theory. Just a few terms of I–A interaction expansion over R^{-1} (R is the distance between pseudoparticles) are model independent, for example the R^{-4} term for Yang-Mills theory. The problem of the exact determination of

I–A interaction seems to have no sense at all.

In view of such an impossibility to define exactly the I–A potential the approach of ref. [21] looks very instructive. The author of [21] have found the parameters of singularity caused by I–A contribution to the Borel transform of $R_{e^+e^-} \to hadrons$. In order to find the type and strength of singularity one needs just a few terms of I–A interaction. Such singularities of the Borel transform are responsible for the large orders of perturbation theory behaviour. On the other hand, the smooth part of the Borel function is mostly affected by the first terms of perturbative expansion. Thus the problem of the calculation of the short range part of I–A interaction should be replaced by the problem of exact calculation of a few first terms in the perturbation theory expansion.

3 Computing the Constrained Instanton

Thus the constrained Instanton in SU(2) Yang–Mills theory is a solution of the equation (see eq. (12))

$$\begin{cases} \dfrac{\delta S}{\delta A_\mu^a} = -D_\nu^{ac} G_{\nu\mu}^c + \dfrac{ig}{4}\{\overline{\Phi}\sigma^a D_\mu \Phi - \overline{(D_\mu\Phi)}\sigma^a\Phi\} = \zeta W_\mu^a \\ 2\dfrac{\delta S}{\delta \overline{\Phi}} = -D^2\Phi + \dfrac{\lambda}{2}\Phi(\overline{\Phi}\Phi - v^2) = 0 \ . \end{cases} \quad (20)$$

Here ζ is the Lagrange multiplier and W_μ^a is the constraint function. We suppose that W_μ^a at $r > \rho$ (ρ is the Instanton size) goes to zero faster than any other function which we consider. The well known [1] approximate solution of (20) is given by (9). In order to solve the system (20) let us choose the ansatz

$$A_\mu^a = \frac{2}{g} U^{ab} \bar{\eta}_{\mu\nu}^b r_\nu f(r) \ , \ W_\mu^a = \frac{2}{g} U^{ab} \bar{\eta}_{\mu\nu}^b r_\nu w(r) \ , \ \Phi = \phi(r) \begin{pmatrix} 0 \\ v \end{pmatrix} . \quad (21)$$

It is easily seen, that $D_\mu^{cl} A_\mu \equiv 0$. Now instead of (20) one gets (see for example ref. [1] for properties of η – symbols):

$$\begin{cases} -f'' - \dfrac{5}{r} f' - 12 f^2 + 8 r^2 f^3 + m_w^2 \phi^2 f = \zeta w \\ -\phi'' - \dfrac{3}{r}\phi' + 3 r^2 f^2 \phi + \dfrac{m_h^2}{2}\phi(\phi^2 - 1) = 0 \ . \end{cases} \quad (22)$$

Here $m_w = gv/2, m_h = \sqrt{\lambda} v$. In this Section we discuss only very small Instantons $m_w\rho \ll 1, m_h\rho \ll 1$. Therefore one can explore two different asymptotic expansions for the solution of (22). At $m_w r \ll 1, m_h r \ll 1$ one has

$$\begin{cases} f = \dfrac{\rho^2}{r^2(r^2+\rho^2)} + m_w^2 f_1(r/\rho) + m_w^4 \rho^2 f_2(r/\rho) + \ldots \\ \phi = \dfrac{r}{\sqrt{r^2+\rho^2}} + \ldots \end{cases} \quad (23)$$

On the other hand at $r \gg \rho$ one can use the expansion

$$\begin{cases} f = \dfrac{(m_w\rho)^2}{r^2}\left[\dfrac{K_2(m_w r)}{2} + (m_w\rho)^2 F_1(m_w r) + \ldots\right] \\ \phi = 1 - \dfrac{m_h\rho^2}{r}\left[\dfrac{K_1(m_h r)}{2} + \ldots\right] , \end{cases} \quad (24)$$

where K_1 and K_2 are the McDonald functions, F_1 is an unknown function. It is our luck that at $\rho \ll r \ll m^{-1}$ both expansions (23) and (24) are valid. Due to that we have found explicitly the coefficients at McDonald functions in (24) from a comparison with (23) at $\rho \ll r \ll m^{-1}$. The small x asymptotic of McDonald functions reads

$$\begin{aligned} K_1(x) &= \frac{1}{x} + \frac{x}{2}\left[\log\left(\frac{\gamma x}{2}\right) - \frac{1}{2}\right] + \ldots \ , \\ K_2(x) &= \frac{2}{x^2} - \frac{1}{2} + \ldots \ , \end{aligned} \quad (25)$$

where $\log\gamma = 0.5772\ldots$ is the Euler's constant.

The system of equations (22) has an unique localized solution for any given value of the Lagrange multiplier ζ. One may vary the value of ζ in order to get the solution satisfying the given constraint (11). In other words, one can choose the Lagrange multiplier ζ in order to get the Instantonic solution of a given size ρ. Still we have not given the exact definition of ρ — the size of constrained Instanton. One can call the "Instanton of a size ρ" the solution of eq. (22), which coincides with the Instanton of pure Yang–Mills theory at $r \ll \rho$. This means $f_1(0) = 0, f_2(0) = 0, \ldots$ (see (23)). But for our purpose, to estimate the total cross-section for baryon number violating processes, it is more convenient to use another definition of constrained Instanton. We will call the Instanton of a size ρ in the spontaneously broken Yang–Mills theory the solution of eq. (22) which at large r behaves like

$$A^a_\mu(r \gg m^{-1}) \equiv \frac{2}{g} U^{ab} \bar{\eta}^b_{\mu\nu} r_\nu \frac{(m_w \rho)^2}{2r^2} K_2(m_w r) \,. \quad (26)$$

This mean, we require, that the function $F_1(m_w r)$ in eq. (24) should go to zero at large r much faster than the McDonald function $K_2(m_w r)$.

Substitution of (23) into (22) provides us with the equation for f_1

$$-\frac{d^2}{dr^2} f_1 - \frac{5}{r}\frac{d}{dr} f_1 - \frac{24\rho^2}{(r^2+\rho^2)^2} f_1 = \frac{\zeta}{m_w^2} w - \frac{\rho^2}{(r^2+\rho^2)^2} \quad (27)$$

The solution to this equation regular at small r reads

$$f_1\left(\frac{r}{\rho}\right) = -\frac{1}{(r^2+\rho^2)^2} \int_0^r \frac{(x^2+\rho^2)^4}{x^5} dx \int_0^x \frac{y^5}{(y^2+\rho^2)^2} \left[\frac{\zeta}{m_w^2} w - \frac{\rho^2}{(y^2+\rho^2)^2}\right] dy + \alpha \frac{\rho^4}{(r^2+\rho^2)^2} . \quad (28)$$

Here a solution of homogeneous equation is added with an arbitrary weight α. The α is the only parameter in (28), which may depend on $(m_w\rho)$ due to the boundary conditions at large r. For arbitrary Lagrange multiplier ζ the solution of eq. (23) at large r has the form

$$f = A \frac{K_2(m_w r)}{r^2} + B \frac{I_2(m_w r)}{r^2} \,, \quad (29)$$

where I_2 is the Veber function, which grows exponentially with $m_w r \gg 1$, and behaves like $(m_w r)^2$ at $m_w r \ll 1$. The Lagrange multiplier ζ is just chosen to obtain $B \equiv 0$. From (24), (25) and(28) one gets

$$\zeta \int_0^\infty \frac{r^5}{(r^2+\rho^2)^2} w(r) dr = \frac{m_w^2}{6} \,. \quad (30)$$

It is seen now from (28) that only short range $(r \sim \rho)$ behaviour of function f_1 depends on the choice of constraint function w.

Equation for $F_1(m_w r)$ is easy to found from eqs. (22) and (24)

$$-\frac{d^2}{dx^2} F_1 - \frac{1}{x}\frac{d}{dx} F_1 + \frac{4}{x^2} F_1 + F_1 = \frac{3}{x^2} K_2^2(x) + \frac{m_h}{m_w}\frac{1}{x} K_1\left(\frac{m_h}{m_w} x\right) K_2(x) \,. \quad (31)$$

It is only important for us, that this equation has no small parameters either $(m_w\rho)$, or $(m_h\rho)$. The function $F_1(x)$ may depend only on the ratio (m_w/m_h). The explicit expression for $F_1(x)$ may be found in terms of the integrals of the Bessel functions. But in the large logarithm approximation we do not need such the explicit formula. The first two terms of F_1 expansion at small x are

$$F_1(x) = -\frac{1}{x^4} - \frac{1}{x^2}[\log(x) + c_1] \,. \quad (32)$$

For inconvenient definition of the size ρ of constrained Instanton large logarithms $\log(m\rho)$ may appear in the constant c_1 due to the boundary condition for $F_1(x)$ at small x. But we have fixed the value of ρ by only the long range behaviour of the Instanton (26). The only demand that $F_1(x)$ should decrease at large x much faster than the McDonald function $K_2(x)$ allows to fix uniquely the function F_1. Now the large distance correction to Instanton field $F_1(x)$ cannot depend on either $(m_w\rho)$, or $(m_h\rho)$ even due to the boundary conditions. Thus the constant in eq. (32) should be $c_1 \sim 1$. With use of explicit formula for $F_1(x)$ one can find the value of c_1. But in the large logarithm approximation it is enough to know that $c_1 \sim 1$. From (23), (24), (28) and (32) one can find $\alpha = -\log(m_w\rho) \gg 1$. Thus, if at $r > m_w^{-1}$ the Instanton behaves like (26), then at small distances $r \sim \rho$

$$A^a_\mu(r \sim \rho) = \frac{2}{g} U^{ab} \bar{\eta}^b_{\mu\nu} r_\nu \frac{\tilde{\rho}^2}{r^2(r^2+\tilde{\rho}^2)} \,, \quad (33)$$

$$\tilde{\rho}^2 = \rho^2 \left(1 + (m_w\rho)^2 \log\left(\frac{1}{m_w\rho}\right)\right) \,.$$

4 The Instanton Action

Now we know much enough about the properties of constrained Instanton to find the action for the Instanton. Consider first the contribution to the action coming from the Higgs fields. To this end consider the second and the third terms of the action (8), but taking the gauge field A^a_μ in the form (9). It is convenient to divide the range of integration into two parts $|r| < L$ and $|r| > L$, where $\rho \ll L \ll m^{-1}$. The small distances contribution up to corrections $\sim (m\rho)^4$ reads

$$S_{\phi,|r|<L} = \int_0^L \left\{ \frac{1}{2} \overline{(D^0_\mu \Phi_0)} (D^0_\mu \Phi_0) d^4 r + \frac{1}{2} [\overline{(D^0_\mu \delta\Phi)} (D^0_\mu \Phi_0) + h.c.] \right\} +$$

$$+ \int_0^L \frac{m_h^2}{8v^2}(\overline{\Phi_0}\Phi_0 - v^2)^2 d^4 r\,, \qquad (34)$$

$$D^0_\mu = \partial_\mu - i\frac{g}{2}\sigma^a A^a_{\mu 0}\,.$$

Here $h.c.$ means hermitian conjugated. For unperturbed fields $\Phi_0, A^a_{\mu 0}$ see eq. (9). The last integral in (34) is easy to found explicitly. With the use of identity $D^0_\mu D^0_\mu \Phi_0 = 0$ the remaining integral in (34) can be transformed into the surface one

$$S_{\phi,|r|<L} = \int_{|r|=L} ds_\mu \left\{ \frac{1}{2}\overline{\Phi}_0 D^0_\mu \Phi_0 + \frac{1}{2}[\delta\overline{\Phi} D^0_\mu \Phi_0 + h.c.] \right\} + \frac{\pi^2 v^2 m_h^2 \rho^4}{4}\left[\log\left(\frac{L}{\rho}\right) - \frac{1}{2}\right]\,. \qquad (35)$$

It is our luck, that the correction $\delta\phi$ to unperturbed Instantonic solution (9) appears only in the surface integral. As we have seen in the previous section, at $r \sim \rho$ the corrections of the order of $(m\rho)^2$ to Instanton field depend on the choice of the constraint. On the other hand, at large distances $L \gg \rho$ the constraint independent expression (24) for ϕ in terms of the McDonald functions can be used. With the expansion (25) one easily finds

$$S_{\phi,|r|<L} = \pi^2\rho^2 v^2 \left[1 - \frac{2\rho^2}{L^2}\right] - \frac{\pi^2 v^2 m_h^2 \rho^4}{2}\left[\log\left(\frac{\gamma m_h L}{2}\right) - \frac{1}{2}\right] + \frac{\pi^2 v^2 m_h^2 \rho^4}{4}\left[\log\left(\frac{L}{\rho}\right) - \frac{1}{2}\right]\,. \qquad (36)$$

For the large distance contribution ($|r| > L$) one can use the following approximation for the field Φ covariant derivative (see eq. (24))

$$D^0_\mu \Phi = \left\{ \partial_\mu \left(-\frac{m_h \rho^2}{2r} K_1(m_h r) \right) - i\sigma^a U^{ab} \bar{\eta}^b_{\mu\nu} \frac{r_\nu \rho^2}{r^4} \right\} \begin{pmatrix} 0 \\ v \end{pmatrix}\,. \qquad (37)$$

The substitution of (37) and (24) into (8) leads to

$$S_{\phi,|r|>L} =$$

$$= \int_L^\infty \left\{ \frac{v^2\rho^4}{8}\left(\partial_\mu \frac{m_h}{r} K_1(m_h r)\right)^2 + \frac{v^2\rho^4}{8} m_h^2 \left(\frac{m_h}{r} K_1^2\right) + \frac{3}{2}\frac{v^2\rho^4}{r^6} \right\} d^4 r =$$

$$= \pi^2\rho^2 v^2 \left\{ \frac{\rho^2}{2L^2} + \frac{m_h^2\rho^2}{4}\left[\log\left(\frac{m_h \gamma L}{2}\right) - 1\right] \right\} + \frac{3}{2}\frac{\pi^2\rho^4 v^2}{L^2}\,. \qquad (38)$$

For integration of the squared derivative of the McDonald function it is again useful to transform the integral to the surface one. Adding of (38) to (36) provides us with the contribution to the action from the Higgs fields at fixed gauge field $A^a_{\mu 0}$ (9)

$$\delta S_\phi = \pi^2\rho^2 v^2 \left\{ 1 + \frac{m_h^2\rho^2}{4}\left[\log\left(\frac{2}{m_h \gamma \rho}\right) - \frac{1}{2}\right] \right\}\,. \qquad (39)$$

Here the first term $\pi^2\rho^2 v^2$ is a well known result [1, 14] which is usually used in the calculation of total cross-section. The new result is the second term $\sim m^2 v^2 \rho^4 \log(m\rho)$. As it is seen from eq. (39) this term tends to increase the Instanton action (or, in other words, tends to suppress the cross-section). This result is rather natural. We have two sources for $\sim m^2 v^2 \rho^4$ correction in the eq. (39). First is the last term of the action (8) $\sim \lambda(\overline{\Phi}\Phi - v^2)^2$, which is evidently positive. The second contribution comes from the correction to the squared covariant derivative. But the unperturbed solution Φ_0 (see eq. (9)) realizes the exact minimum of $\overline{(D^0_\mu \Phi)}(D^0_\mu \Phi)$ term in the action for given background field $A^a_{\mu 0}$ and given asymptotic $\langle \overline{\Phi}\Phi(r \to \infty)\rangle = v^2$. Thus the correction to $\overline{(D^0_\mu \Phi)}(D^0_\mu \Phi)$ should also be positive.

Now we have to calculate the corrections to action caused by the difference of the gauge field A^a_μ from the field of Instanton of pure Yang–Mills theory $A^a_{\mu 0}$ (eq. (9)). We will calculate such the corrections in the large logarithm $\log(m_w \rho)$ approximation, because any of them have no large factor m_h^2. For the beginning consider the second term in the action (8) $\sim \overline{(D_\mu \Phi)}(D_\mu \Phi)$. Two types of corrections $\sim v^2 m_w^2 \rho^4 \log(m_w \rho)$ can be found coming from this term. The first correction is dominated by small distances $|r| \sim \rho$. As we decided in the previous section, we call the Instanton of the size ρ configuration which has the asymptotic (26). As a result of such definition we have to renormalize the effective Instanton radius at small distances in accordance with (33). Consequently, the leading term in the correction to pseudoparticle action caused by the Higgs fields (39) should also be modified

$$\pi^2\rho^2 v^2 \to \pi^2\tilde{\rho}^2 v^2 = \pi^2\rho^2 v^2 \left(1 + (m_w\rho)^2 \log\left(\frac{1}{m_w \rho}\right)\right)\,. \qquad (40)$$

The second correction to the $\overline{(D_\mu \Phi)}(D_\mu \Phi)$ term caused by the change of the gauge field $A^a_{\mu 0}$ (9) comes from the large distances. At $r \gg \rho$ the unperturbed field $A^a_{\mu 0}$ should be replaced by the field A^a_μ (see eqs. (24) and (21)) proportional to the McDonald function. In the intermediate range $\rho \ll r \ll m_w^{-1}$, where all the large logarithms come from, one can use the approximation

$$A^a_\mu = \frac{2}{g} U^{ab} \bar{\eta}^b_{\mu\nu} r_\nu \left[\frac{\rho^2}{r^4} - \frac{(m_w \rho)^2}{4r^2} \right] . \quad (41)$$

The correction induced by the second term of (41) to the $\sim \overline{(D_\mu \Phi)}(D_\mu \Phi)$ term of the action (8) reads

$$\delta S_{\phi A} = -\frac{3}{2}\pi^2 v^2 m_w^2 \rho^4 \log\left(\frac{1}{m_w \rho}\right) . \quad (42)$$

Finally, there appears a correction to the first term of the action (8), $F^a_{\mu\nu} F^a_{\mu\nu}$. Since the unperturbed Instanton is an exact extremum for the action of pure Yang–Mills theory, one should consider the second variation of the action. With the use of expression (41) it is easy to find the corrections to the action and to $F^a_{\mu\nu}$

$$\delta F^a_{\mu\nu} = \frac{(m_w \rho)^2}{g} U^{ab} \left[\frac{r_\mu \bar{\eta}^b_{\nu i} r_i}{r^4} - \frac{r_\nu \bar{\eta}^b_{\mu i} r_i}{r^4} + \frac{\bar{\eta}^b_{\mu\nu}}{r^2} \right] ,$$

$$\delta S_A = \frac{1}{4} \int_{\rho \ll r \ll m_w^{-1}} \delta F^a_{\mu\nu} \delta F^a_{\mu\nu} d^4 r = 3\pi^2 \frac{(m_w \rho)^4}{g^2} \log\left(\frac{1}{m_w \rho}\right) . \quad (43)$$

Summing up the contributions (39),(40),(42) and (43) we can find the action of the constrained Instanton (note that $m_w = gv/2$)

$$S = \frac{8\pi^2}{g^2} \left\{ 1 + \frac{(m_w \rho)^2}{2} + \frac{m_w^2 m_h^2 \rho^4}{8} \left[\log\left(\frac{2}{m_h \gamma \rho}\right) - \frac{1}{2} \right] + \right.$$
$$\left. + \frac{(m_w \rho)^4}{8} \log\left(\frac{const}{m_w \rho}\right) \right\} \quad (44)$$

Where unknown $const$ is of order of 1.

5 Constrained Instanton for Heavy Higgs

Thus we see (44), that in the lowest order the ρ dependence of Instanton action is determined by the correction $\delta S \sim (m_w \rho / g)^2$ in accordance with the results of refs. [1, 14]. The following correction to this result is of the relative order $\sim (m_h \rho)^2 \log(m_h \rho)$. The modern restriction on the Higgs mass does not exclude the possibility of $m_h^2 \gg m_w^2$. Therefore it is interesting to analyze the situation when $(m_h \rho)$ is not small.

As we have seen in Sec. 2, in the spontaneously broken Yang–Mills theory (8) only very small Instantons may be treated semiclassically. The natural criterion of this smallness is the inequality $m_w \rho \ll 1$. In order to get the nontrivial solution of the equations of motion (20) we have used the constraint for the gauge fields A^a_μ. However for any given configuration $A^a_\mu(r)$ the action (8) has a well defined exact minimum with respect to the field $\Phi(r)$ variation. Therefore the condition $m_h \rho \ll 1$, which we have used before seems to be excessive.

In this Section we would consider the Instanton of the theory (8) for the case of $(m_h \rho)$ not small, while $m_w \rho \ll 1$. We concentrate our attention on the case $m_h \rho \gg 1$. There is no principal problems to consider the case $m_h \rho \sim 1$, but for explicit calculation of $\Phi(r)$ and Instanton action in this case one has to solve numerically the nonlinear differential equation.

Consider the classical Higgs field configuration $\Phi(r)$ for $(m_h \rho) \gg 1$. In the leading approximation one can neglect the difference of the gauge field configuration A^a_μ from the unperturbed Instanton $A^a_{\mu 0}$ (9). In this case the second of equations (22) takes the form

$$-\phi'' - \frac{3}{r}\phi' + \frac{3}{r^2}\frac{\rho^4}{(r^2+\rho^2)^2}\phi + \frac{m_h^2}{2}\phi(\phi^2 - 1) = 0 . \quad (45)$$

For $m_h \rho \gg 1$ two asymptotics of solution of this equation can be easily found. At large distances

$$\phi = 1 - \frac{3}{(m_h r)^2} \frac{\rho^4}{(r^2+\rho^2)^2} + O(m_h^{-4}) , \ m_h r \gg 1 , \quad (46)$$

and at small distances

$$\phi(0) = 0 , \quad (47)$$
$$\phi(r \ll m_h^{-1}) \sim m_h r .$$

In the intermediate region $m_h r \sim 1$ the function $\phi(r)$ interpolates smoothly between the solutions (46) and (47). In order to find the $\phi(r)$ in the region $m_h r \sim 1$ one should solve exactly the equation (45). But for calculation of the leading correction to Instanton action it is enough to know that ϕ differs very slightly from the vacuum value $\phi = 1$ at $r \sim \rho$ and at small distances $\phi(r \sim m_h^{-1}) \sim O(1)$. Correction to the action of Instanton comes from the term of the action (8) with squared covariant derivative of the Higgs field

$$\delta S \approx \int \frac{1}{2}\overline{(D_\mu \Phi)}(D_\mu \Phi) \approx \int \frac{g^2}{8} A_\mu^a A_\mu^a \overline{\Phi}\Phi \approx \frac{3\pi^2}{2} v^2 \rho^2 . \quad (48)$$

The corrections to this result are of the relative order $(m_h \rho)^{-1}$.

Therefore the Instanton action for the case $(m_w \rho) \ll 1, (m_h \rho) \gg 1$ reads

$$S = \frac{8\pi^2}{g^2}\left\{1 + \frac{3}{4}(m_w \rho)^2 + \ldots\right\} . \quad (49)$$

We see that for heavy Higgs even the first term of the action S expansion over the powers of ρ^2 differs from that for the light Higgs boson case (44).

6 The Baryon Number Non - Conservation

Before consideration of $\varepsilon^{8/3}$ corrections to $F(\varepsilon) \sim \log(\sigma_{total})$ we would like to remind the reader how the first nontrivial term $\sim \varepsilon^{4/3}$ can be found [5, 7, 8]. We will follow the paper [13]. The cross section for baryon number violating process accompanied by multiple emission of massless W-bosons is given by the formula

$$\sigma_{total} \propto \sum_n \frac{1}{n!} \int d\rho_I d\rho_A dU_I dU_A \mu(\rho_I)\mu(\rho_A) \cdot \quad (50)$$

$$\prod_{i=1}^{n} \int \frac{d^4 k_i}{(2\pi)^3} \delta_+(k_i^2) k_i^2 A_\mu^{aI}(k_i) k_i^2 A_\mu^{aA}(k_i) (2\pi)^4 \delta^4(P - \sum k_i) .$$

Here ρ_I, U_I, S_I and ρ_A, U_A, S_A are correspondingly size, orientation matrix and action of the Instanton and Antiinstanton, $\mu(\rho_{I,A}) \sim \exp(-S(\rho_{I,A}))$ are the pseudoparticles weights (see(44)), $P_\mu = (\sqrt{s}, 0, 0, 0)$ is the total 4-momentum of produced W-bosons. A_μ^{aI} and A_μ^{aA} are singular at $k^2 = 0$ parts of the Fourier transform of the Instanton and Antiinstanton field

$$k^2 A_\mu^{aI(A)} = +(-)\frac{4\pi^2 i \rho^2 U^{ab} \bar{\eta}^b_{\mu\nu} k_\nu}{g} \quad (51)$$

Only the residue of the Instanton (Antiinstanton) field Fourier transform at $k^2 = 0$ is needed for the calculation of cross-section (50). Now it is clear, why do we have defined the constrained Instanton by the formula (26). The field (26) corresponds to a massive particle, therefore its Fourier transform has a pole at $k^2 + m_w^2 = 0$ (k is the Euclidean momenta), not at $k^2 = 0$. But the residue at this pole is exactly the same as for the unperturbed Instanton (51).

One can introduce a distance between the Instanton and Antiinstanton R_μ in the usual way [5, 7, 8]

$$(2\pi)^4 \delta^4(P - \sum k_i) = \int d^4 R \exp\left\{i(PR) - i\sum(k_i R)\right\} , \quad (52)$$

$$\int \frac{d^4 k}{(2\pi)^3} \delta_+(k^2) e^{-i(kR)} = \frac{1}{4\pi^2[\vec{R}^2 - (R_0 - i0)^2]} . \quad (53)$$

After that the factorized integrals over 4-momentum of produced W-bosons sum up to the exponent:

$$\sigma_{total} = \int d^4 R e^{i(PR)} d\rho_I d\rho_A dU_I dU_A \mu(\rho_I)\mu(\rho_A) \times$$
$$\times \exp\left\{-\frac{8\pi^2}{g^2}\frac{4D\rho_I^2\rho_2^2}{[\vec{R}^2 - (R_0 - i0)^2]^2}\right\} , \quad (54)$$
$$D = \bar{\eta}^a_{\mu\alpha}(U_I^T U_A)^{ab} \bar{\eta}^b_{\mu\beta}\frac{R_\alpha R_\beta}{R^2} \in (-3, 1) .$$

The integration over R_μ and $U_{I,A}$ is performed by the steepest descent method with the saddle-point values given by

$$D = -3 \, , \, R_0 = -i\left(\frac{384\pi^2\rho_I^2\rho_A^2}{g^2\sqrt{s}}\right)^{1/5} \, , \, \vec{R} = 0 . \quad (55)$$

The integrals over ρ_I, ρ_A are also performed by the steepest descent method. In the leading order one should neglect the ρ^4 term in the Instanton action(44). The following saddle-point values are easy to find

$$(\rho m_w)^2 = \frac{3}{2}\varepsilon^{4/3} \, , \, (Rm_w)^2 = 6\varepsilon^{2/3} , \quad (56)$$

where $\varepsilon = \sqrt{s}/E_0$, $E_0 = \sqrt{6}\pi m_w/\alpha = 4\sqrt{6}\pi^2 m_w/g^2$. After all, in the leading semiclassical approximation the total cross-section is given by [5, 7, 8]

$$\sigma_{total} \propto \exp\left[\frac{4\pi}{\alpha}F(\varepsilon)\right] \approx \exp\left[\frac{4\pi}{\alpha}\left(-1 + \frac{9}{8}\varepsilon^{4/3}\right)\right] . \quad (57)$$

As can be seen from (56) the function $F(\varepsilon)$ is naturally expanded in powers of $\varepsilon^{2/3}$. The following ε^2 term [9, 10, 11, 12] reads

$$F_2 = -\frac{9}{16}\varepsilon^2 \ . \tag{58}$$

There are three effects which contribute to this value. They are:

1. the quantum correction to the classical cross-section (57),
2. the correction which comes from the slight modification of the classical result due to the finite W-boson mass,
3. the correction due to the multiple production of massless Higgs bosons.

Going further the authors of [13] have calculated the second order quantum correction to F in the limit of massless W-bosons

$$F^{qu}_{8/3} = +\frac{3}{16}\varepsilon^{8/3}\log\left(\frac{c}{\varepsilon}\right) \ . \tag{59}$$

In the previous section (see eq. (44)) we have calculated the correction $\sim (m\rho)^4\log(m\rho)$ to the single-Instanton action. With the saddle point values (56) it is seen immediately, that the modification of Instantonic measure due to this term provides the correction to $F(\varepsilon)$ of the same order as (59)

$$F^{\mu}_{8/3} = -\frac{3}{16}\varepsilon^{8/3}\left\{\left(\frac{m_h}{m_w}^2\right)\left[\log\left(\frac{1}{\varepsilon}\right)+\frac{3}{2}\log\left(\frac{2\sqrt{2}m_w}{\gamma\sqrt{3}m_h}\right)-\frac{3}{4}\right]+\log\left(\frac{1}{\varepsilon}\right)\right\} . \tag{60}$$

A number of effects exist which may lead to a correction of the order of $\varepsilon^{8/3}$ to the function $F(\varepsilon)$, such as further corrections due to a finite W mass, or quantum corrections to multiple Higgs emission. But none of them have either $\log(\varepsilon)$, or $(m_h/m_w)^2$ enhancement. We have found only one effect which leads to a correction comparable with (60) and (59). This is the correction caused by the finite Higgs mass m_h to the classical multiple Higgs production. The account of the multiple Higgs boson production can be done in the similar way, just like the total cross-section for baryon number violating process accompanied by the classical multiple W-s production (57) was found. One should only replace in the formula (50) the Fourier transform

of the gauge fields $A^{aI}_\mu(k)$, $A^{aA}_\mu(k)$ by that of Higgs field (24)

$$\Phi(k) = \frac{2\pi^2\rho^2}{k^2-m_h^2}\begin{pmatrix}0\\ v\end{pmatrix} . \tag{61}$$

Here we work in the Minkowski space-time. As far as we want to take into account the finite mass effect, the multipliers k_i^2 in (50) at the field Fourier transform should be also replaced by $(k_i^2 - m_h^2)$. Finally, one has to modify the expression (53)

$$\int \frac{d^4k}{(2\pi)^3} \delta_+(k^2-m^2)e^{-ikR} = \tag{62}$$

$$= \frac{1}{4\pi^2}\frac{m}{\sqrt{\vec{R}^2-R_0^2}}K_1(m\sqrt{\vec{R}^2-R_0^2}) ,$$

$$R_0^2 < \vec{R}^2 .$$

After taking into account the multiple production of massive Higgs bosons in the leading classical approximation the cross-section reads

$$\sigma_{total} = \sigma_w \exp\left\{\pi^2 v^2\rho_1^2\rho_2^2\frac{m_h}{\sqrt{\vec{R}^2-R_0^2}}K_1(m_h\sqrt{\vec{R}^2-R_0^2})\right\} , \tag{63}$$

where σ_w is the cross-section for multiple W-s production. If one leaves in eq. (63) the first term of the McDonald function asymptotic at $m_h\sqrt{\vec{R}^2-R_0^2} \ll 1$ only, then all the Higgs mass m_h dependence disappears and one gets the same $\sim \varepsilon^2$ correction to $F(\varepsilon)$, as was used in [9, 10, 11, 12]. The following term of the McDonald function K_1 in (63) expansion (see eqs. (25) and (56)) gives

$$F^{m_h}_{8/3} = -\frac{3}{32}\varepsilon^{8/3}\left(\frac{m_h}{m_w}\right)^2\left[\log\left(\frac{1}{\varepsilon}\right)+3\log\left(\sqrt{\frac{2}{3}}\frac{m_w}{\gamma m_h}\right)+\frac{3}{2}\right] . \tag{64}$$

Summing up of (57), (58), (59), (60) and (64) provides us with the result

$$F = -1+\frac{9}{8}\varepsilon^{4/3}-\frac{9}{16}\varepsilon^2-\frac{9}{32}\left(\frac{m_h}{m_w}\right)^2\varepsilon^{8/3}\log\left[\frac{1}{3\varepsilon}\left(\frac{2m_w}{\gamma m_h}\right)^2\right]+O(\varepsilon^{8/3}) . \tag{65}$$

Here the unknown correction behaves like $\varepsilon^{8/3}$, but have neither $\log(\varepsilon)$, nor $(m_h/m_w)^2$ enhancement.

The graphs of $F(\varepsilon)$ (65) for $m_h = m_w$, $m_h = 2m_w$ and $m_h = 3m_w$ are shown on the figure. The dashed lines are the $F(\varepsilon)$ graphs with only two or three terms of the expansion (65) taken into account. The result of ref. [13] is shown by dotts. The authors of [13] have considered the only correction of the order of $\varepsilon^{8/3}\log(c/\varepsilon)$ which comes from the second order quantum correction to the total cross-section $F^{qu}_{8/3}$ (59). They have chosen argument of logarithm to be $c = e = 2.718$.

It is interesting to discuss the range of applicability of the expression (65). As we have said in the Introduction, one can use the approximate expression (65) instead of the exact $F(\varepsilon)$ if at least the contribution of the last term $\sim \varepsilon^{8/3}$ is smaller than the ε^2 contribution. The result of [13] (dotted line in the fig.) still may be interpreted as the indication of applicability of the $F(\varepsilon)$ expansion in powers of ε up to $\varepsilon \sim 1$. Unfortunately, as far as the final result (65) is considered it is seen from the figure that $\varepsilon^{3/8}$ and ε^2 terms became equal at $\varepsilon = 0.3 \div 0.12$ for $m_h = 2 \div 3m_w$ (the $\varepsilon^{3/8}$ and ε^2 contributions coincide at the point, where dashed line a crosses one of the solid lines). Only for this region $\varepsilon < 0.3 \div 0.12$ one can believe in the applicability of the expansion (65).

Moreover, the more severe restriction for applicability of the expression (65) can be formulated. As we have seen, at sufficiently small ε the correction $\sim \varepsilon^{8/3}$ tends to decrease the total cross-section. As we would argue, this is not an occasion. For example, the correction $F^{m_h}_{8/3}$ (64) arises when we take into account the finite Higgs boson mass. This correction should suppress the cross-section, because a phase space for massive particle is smaller then the phase space available for massless one. The correction of the order of $m_h^2 m_w^2 \rho^4$ to the Instanton action (44) should also decrease the cross-section as we have argued in the previous section (see the discussion after eq.(39)). Thus, one can neglect the following terms in the expansion (65) for $F(\varepsilon)$ ($\sim \varepsilon^{10/3}$, $\sim \varepsilon^4$ et.c.) only if the last term $\sim \varepsilon^{8/3}(m_h/m_w)^2 \log(\varepsilon)$ is negative. This means

$$\varepsilon < \frac{1}{3}\left(\frac{2}{\gamma}\frac{m_w}{m_h}\right)^2 . \qquad (66)$$

Since of at $m_h \gg m_w$ the role of corrections dependent on the Higgs mass in (65) increase drastically, it seems very attractive to try to take into account all the $(m_h/m_w)^2$ corrections exactly. In the previous Section we have found the correction to Instanton action for the case $m_w\rho \ll 1 \ll m_h\rho$.

Repeating all the considerations, which have lead us to the result (57), with the new Instanton action (49) we get

$$F(\varepsilon) = -1 + \frac{9}{8}\left(\frac{2}{3}\right)^{2/3} \varepsilon^{4/3} + \dots , \qquad (67)$$
$$\varepsilon \ll 1 \, , \; \varepsilon\left(\frac{m_h}{m_w}\right)^{3/2} \gg 1 \, .$$

For the range of applicability of this result we have used the same inequality $m_w\rho \ll 1 \ll m_h\rho$ expressed in terms of the saddle point values (56). Unfortunately it seems a severe problem even to estimate the corrections to this result coming from multiple heavy Higgs production. We can describe satisfactorily only the Euclidean classical configurations. But in order to describe the Higgs boson production one should reach the far Minkowski point $m_h^2 + k^2 = 0$.

7 Massive ϕ^4 theory

Consider the theory with Euclidean action

$$S = \frac{1}{g}\int \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\phi^4\right] d^4r \, , \qquad (68)$$

where ϕ is a real scalar field. In the absence of mass ($m = 0$) where exist the Instanton of an arbitrary size ρ. The explicit formula for Instanton and its dilatational zero mode reads

$$\phi_0 = \frac{4\sqrt{3}\rho}{\rho^2 + r^2} \, , \quad \frac{\partial\phi_0}{\partial\rho} \sim \frac{r^2 - \rho^2}{(r^2 + \rho^2)^2} \, . \qquad (69)$$

Simple scale transformation $\phi(x) \to a\phi(ax)$ ensures one that the exact extremum of the action of the massive theory is the zero size Instanton only. We will discuss a small size Instantons $m\rho \ll 1$. Similarly to the case of spontaneously broken Yang–Mills theory one has to introduce a constraint to the functional integral in order to integrate over the Instanton radius ρ

$$\int \delta(\langle\phi - \phi_I | w\rangle) \, d\langle\phi - \phi_I | w\rangle \, . \qquad (70)$$

After that the equation of motion for Instantonic configuration takes the form

$$-\Delta\phi - \frac{1}{6}\phi^3 + m^2\phi = \zeta w \ , \tag{71}$$

where ζ is the Lagrange multiplier. As well as for the Yang–Mills theory, there is no best way to choose the constraint w, and therefore there is no the unique choice of the Instantonic configuration. One can state only that the constraint $w(r)$ should not be orthogonal to the dilatational zero mode $\partial\phi_0/\partial\rho$ and should not be very close to $\partial\phi_0/\partial\rho$. We will assume, that $w(r)$ is a spherically symmetric function, which goes to zero very fast at $r > \rho$. In this case the solution of eq. (71) at $r \ll m^{-1}$ and $r \gg \rho$ takes the form

$$\begin{aligned} \phi &\approx \phi_0 + \delta\phi \ \ , \ \ r \ll m^{-1} \ , \\ \phi &\approx 4\sqrt{3}\rho\frac{m}{r}K_1(mr) \ \ , r \gg \rho \ . \end{aligned} \tag{72}$$

Here $\delta\phi \sim m^2$ is a correction, which can be easily found in the same way as it was done for the Yang–Mills theory (see (23), (28), (30)). For us it is only important that in the intermediate region $\rho \ll r \ll m^{-1}$ (see (25))

$$\delta\phi = 2\sqrt{3}\rho m^2\left[\log\left(m\gamma\frac{r}{2}\right) - \frac{1}{2}\right] \ . \tag{73}$$

The finite mass correction to the action is of the form $\Delta S \sim (m\rho)^2 \cdot \log(m\rho)/g$ [14, 16]. The only term in the action (68), which may provide us with the large logarithm is

$$\Delta S \approx \frac{1}{g}\int\frac{m^2\phi^2}{2}d^4r \approx \frac{1}{g}\int_0^{m^{-1}}\frac{m^2\phi_0^2}{2}d^4r \approx \frac{48\pi^2(m\rho)^2}{g}\log\left(\frac{1}{m\rho}\right) \ . \tag{74}$$

To our surprise this result occurs to be twice larger than usually used (see e.g. [15]) result of the paper [14].

For more accurate calculation of the correction to the Instanton action it is useful to divide the range of integration into two parts $r < R$ and $r > R$ ($\rho \ll R \ll m^{-1}$) as we have done for the Yang–Mills theory. The short range contribution reads

$$S_{r<R} = \frac{1}{g}\left\{\int_0^R\left[\frac{(\partial_\mu\phi_0)^2}{2} + \frac{m^2}{2}\phi_0^2 - \frac{\phi_0^4}{4!}\right] + \int_{|r|=R}\delta\phi\partial_\mu\phi_0 ds_\mu\right\} =$$

$$= \frac{16\pi^2}{g}\left\{1 - 6\left(\frac{\rho}{R}\right)^2 + 3(m\rho)^2\left[2\log\left(\frac{2}{m\gamma R}\right) - \log\left(\frac{\rho}{R}\right) + \frac{1}{2}\right]\right\} \ . \tag{75}$$

The large distance contribution reads

$$\begin{aligned} S_{r>R} &= \frac{48\rho^2}{g}\int_R^\infty\left[\frac{1}{2}\left(\partial_\mu\frac{m}{r}K_1(mr)\right)^2 + \frac{m^2}{2}\left(\frac{m}{r}K_1(mr)\right)^2\right] = \\ &= -\frac{24\rho^2}{g}\int_{|r|=R}\frac{mK_1(mr)}{r}\partial_\mu\frac{mK_1(mr)}{r}ds_\mu = \\ &= 96\pi^2\left(\frac{\rho}{R}\right)^2 + 48\pi^2(m\rho)^2\left[\log\left(\frac{m\gamma R}{2}\right) - 1\right] \ . \end{aligned} \tag{76}$$

Summing up (75) and (76) we get the action of constrained Instanton in the massive ϕ^4 theory

$$S = \frac{16\pi^2}{g}\left\{1 + 3m^2\rho^2\left[\log\left(\frac{2}{\gamma m\rho}\right) - \frac{1}{2}\right]\right\} \ . \tag{77}$$

Corrections to this formula are of the order of $(m\rho)^4/g$ and generally speaking depends on the constraint.

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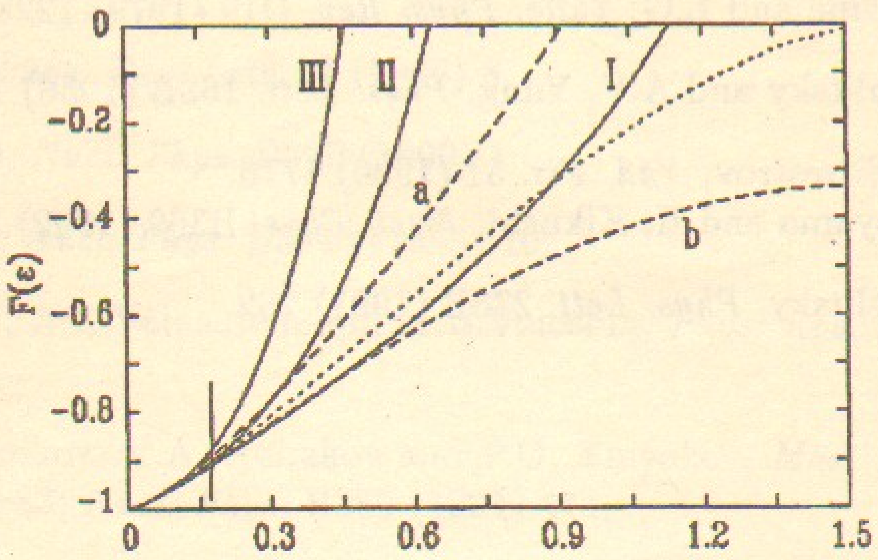

Fig. 1. The plot of the function $F(\varepsilon)$ calculated for various approximations. Dashed line a is the function $F(\varepsilon)$ with only the $\varepsilon^{4/3}$ term taken into account, dashed line b is the sum of $\varepsilon^{4/3}$ and ε^2 terms, dotted line is the result of [13]. Solid lines I, II and III are the plots of the function $F(\varepsilon)$ (65) for $m_h = m_w$, $m_h = 2m_w$ and $m_h = 3m_w$ respectively. For $m_h = 2 \div 3m_w$ one can believe in the applicability of expansion (65) only in the small part of the figure to the left of the vertical line.

P.G. Silvestrov

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П.Г. Сильвестров

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