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# DEPENDENCE OF TRANSVERSE INSTABILITY OF BEAM ON WAKE-POTENTIAL FORM

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## DEPENDENCE OF TRANSVERSE INSTABILITY OF BEAM ON WAKE-POTENTIAL FORM

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### ABSTRACT

This work is devoted to the question of the growth of transverse beam break-up instability caused by wake fields with power law wake potential. This problem is topical interest for linear colliders where particle dynamics depends significantly on fields of radiation in periodical accelerating structures. Some kinds of beam-plasma instabilities are other possible applications of the work.

Analytical calculations show that the emittance growth of advanced instability in time is smaller when the dependence of wake potential on distance has a bigger power coefficient.

### АННОТАЦИЯ

В работе рассмотрен вопрос о зависимости поведения скорости поперечной пучковой неустойчивости, вызываемой следовыми (кильватерными) полями сил от характера этих сил. Задача актуальна для линейных коллайдеров, где влияние полей излучения на динамику пучков имеет определяющее значение, а также применима к некоторым типам пучково-плазменных неустойчивостей.

Найдены решения для степенных зависимостей функций Грина (дельта потенциала) этих сил от расстояния до точки возбуждения. При большем показателе степени рост неустойчивости во времени слабее.



### INTRODUCTION

This work is devoted to frequently occurring "head-tail" or beam-break up (BBU) transverse instability. This instability originates from any interaction between a particle and other particles behind it. These interactions are well known in the physics of beams. They are usually characterized by so called wake potential (or Green's function) $W(s)$ which is equal to the force which acts from a point-like particle with a unit charge onto another unit charge at a distance $s$ behind the first.

In this article we will take into consideration transverse wake fields with power law $W(s) \propto s^{\beta}$. Such fields act in many systems. For example, in ref. [1] the instability of electron beam propagating in the plasma of ionized residual gas was investigated analytically and experimentally. This instability appears due to the interaction between the beginning of the electron beam and the end with a constant wake potential $W(s)=const$ i.e. $\beta=0$.

Another example is the transverse beam dynamics in linear accelerators. The tendency to construct of electron-positron linear colliders (LC) for high energy physics (100 - 1000 GeV) demands an increase number of interacting particles and, thus, transverse wake fields become very dangerous because it can lead to BBU. Some of existing LCs or those under design operate in a single bunch regime and their bunch length is shorter than wavelength of accelerating RF fields. The form of transverse wake field potential in this case can be described by power law $W(s) \propto s^{\beta}$ but parameter $\beta$ varies for different models. A detailed analytical investigation of transverse instability in the case of $W(s) \propto s$ ( i.e. $\beta=1$ ) was made in paper [2]. But so called "optical resonator model" [3] and estimations of work [4] point at another form $W(s) \propto \sqrt{s}$( or $\beta=1/2$ ). An example of numerical calculations of betatron oscillation amplitude growth along a bunch during BBU is shown in Fig.1. This simulations are taken from [5] where wake fields calculated in [4] were used. "BNS-damping"[6] allows to suppress the instability in the main part of the accelerator by introducing an energy spread $\delta_{BNS}$ inside the bunch but this condition is not always satisfied.

In the present work the case of transverse beam dynamics in

LC will take in mind mostly. The BBU growth rate in the case of law $W(s)\propto s^{\beta}$ with any $\beta>0$ will be estimated.

## 1. BASIC EQUATIONS AND ASSUMPTIONS

There are two main transverse forces which act on a particle inside the bunch during the acceleration in linac: forces of focusing elements of linac and forces of electromagnetic fields radiated by the head of the bunch in the accelerating structure. Since an ultrarelativistic particle doesn't change its longitudinal position inside the bunch during its movement along the accelerator we will use two independent variables everywhere below& the first one $t$ is a distance passed from the beginning of accelerator (where the energy of particles is equal to $\gamma_0 mc^2$ ), the second $s$ is a coordinate inside the bunch measured from the head. So, the head particle at the beginning of linac is characterized by coordinates $t=0$, $s=0$ and all particles behind it have $t=0$ and $s>0$. In our estimations a smooth focusing approximation will be used and the basic equation of transverse beam dynamics will look like :

$$\frac{d}{dt}\gamma(t,s)\,\frac{d}{dt}x(t,s) + \omega^2(t,s)\gamma(t,s)x(t,s)=r_0\int_0^s W(s-s')\rho(s')x(t,s')ds' \qquad (1)$$

here $x(t,s)$ is the transverse coordinate of a particle, $\omega(t,s)$ - space frequency of betatron oscillation in focusing lattice, $W(s)$ - wake potential, $\rho(s)$ - linear charge density distribution in the bunch, $r_0=e^2/mc^2$- classical radius of an electron. As it is impossible to find a solution of (1) for any functions $\gamma$, $W$, $\rho$, $\omega$ , let's use following simplifying conditions: a) the linear density of the charge is constant along a bunch $\rho(s)=\rho_0=const$ and b) the energy spread inside the bunch is very small (in compari-on with $\delta_{BNS}$).Thus,the basic equation (1) is transformed into:

$$\frac{d}{dt}\gamma(t)\frac{d}{dt}x(t,s)+\omega_0^2\gamma(t)x(t,s)=r_0\rho_0\int_0^s W(s-s')x(t,s')ds' \quad (2)$$

One wants to find a solution of (2) under the following conditions:

$$x(0,0)=a_0\ ;\ W(s)=W_0 s^{\beta}\ ,\ W_0>0 \qquad (3)$$

While $W_0 > 0$ wake force defocuses particles and leads to



instability. Even after these simplifications the equation is rather complicated and all chapters below will be devoted to a few the most important cases which often appear in beam dynamics investigations.

## 2. WEAK WAKE FORCE AND NO ACCELERATION

This conditions mean : a) $\gamma(t)=\gamma_0=const$ and b) right part of equation (2) is much smaller than $\omega_0^2 x(t,s)$ for all $s$ (in other words $\delta_{BNS}<< 1$ ). The equation of movement is :

$$\frac{d^2}{dt^2}x(t,s) + \omega_0^2 x(t,s) = A\int_0^s (s-s')^{\beta}x(t,s')ds',\ A=r_0\rho_0W_0/\gamma_0 \qquad (4)$$

Due to condition b) one can divide a solution $x(t,s)$ into two parts: "fast" part which describes betatron oscillations at a frequency $\omega_0$ of the bunch head and "slow" part which corresponds to a less relative movements of particles:

$$x(t,s)=a(t,s)\ \exp(i\omega_0 t),\ i=\sqrt{-1} \qquad (5)$$

Then the equation for amplitudes of betatron oscillations $a(t,s)$ will be :

$$2i\omega_0\frac{\partial}{\partial t}a(t,s) = A\int_0^s (s-s')^{\beta}\,a(t,s')\,ds' \qquad (6)$$

under initial condition (see(3)):

$$a(0,0) = a_0 \qquad (7)$$

As we said above in the case of $W_0 > 0$ defocusing wake fields lead to instability. Then, we will find solution of (6) in the form of:

$$a(t,s) = f(\xi)\exp(\xi) \quad \text{for } \xi > 1\ ,\ \xi = ct^a s^b\ \ Re\ \xi > 0 \qquad (8)$$

where the function $f(\xi)$ is assumed to be rather smooth in comparison with the exponent. The parameters c, a, b can be determined from (6). Let's use (8) in (6) :

$$cat^{a-1}s^b f(\xi)\exp(\xi) = \frac{A}{2i\omega_0}f(\xi)\int_0^s (s-s')^{\beta}\exp(ct^a s'^b)ds' \qquad (8)$$

One can see that the smooth character of $f(\xi)$ did allow t. simplify the integral in (6).

There are two ways to make the integration. In the case of integer $\beta = q \in I$, $q = 0, 1, 2, 3....$ the integral in (8) can be found directly because it's possible to expand power of $(s-s')$ :

$$q = 0 \quad (s-s')^q = 1$$
$$q = 0 \quad (s-s')^q = s-s'$$
$$q = 0 \quad (s-s')^q = s^2 - 2ss' + s'^2 \qquad (9)$$
$$q = 0 \quad (s-s')^q = s^3 - 3s^2s' + 3ss'^2 - s'^3$$

and make integration of exponent with any power law of $s'$. For example :

$$\exp(-x^p) \int_0^x s \exp(s^p) ds \approx \frac{1}{p} x^{2-p} - \frac{1}{p} \left(\frac{2}{p} - 1\right) x^{2-2p} \qquad (10)$$

Finally (8) can be rewritten as :

$$c a t^{a-1} s^b = \frac{A}{2i\omega_0} \frac{coeff}{b^{(q+1)}} (ct^a)^{-(q+1)/b} \xi^{(q+1)(1-b)/b} \qquad (11)$$

where the value of the coefficient found is :

$$coeff = q! \qquad (12)$$

Then, equalizing the powers of $t$ and $s$ in both parts of (11) one can find:

$$a = 1/(2+q) \; ; \; b = (1+q)/(2+q) \; ; \; c = \left[ \frac{A}{2i\omega_0} \frac{q!}{a\, b^{(q+1)}} \right]^{1/(2+q)} \qquad (13)$$

In the case when $\beta$ is not integer one can use Laplace integration technique [7]. This method allows to estimate integrals like:

$$I = \int_a^b f(s) \exp(\phi(s))\, ds \qquad (14)$$

if the function $\phi(s)$ has a maximum $\phi(s_0)$ inside $[a, b]$. In this case:

$$I \approx \sqrt{\frac{2\pi}{-\phi''(s_0)}}\, f(s_0) \exp(\phi(s_0)) \qquad (15)$$

In our case this function :

$$\phi(s') = ct^a s'^b + \beta \ln(s-s') \qquad (16)$$

achieves its maximum at point $s_0$:

$$1/(s-s_0) = c\, t^a s^b s_0^{b-1} / \beta \qquad (17).$$

The second derivation of the function at this point is equal to:

$$\phi''(s_0) \approx - \frac{b^2}{\beta} (ct^a)^2 s^{2(b-1)} \qquad (18)$$



and

$$ct^a s_0^b \approx ct^a s^b - \beta \qquad (19)$$

Finally the equation (8) will look like (10) but the value of coefficient will be equal to:

$$coeff_2 = \sqrt{2\pi\beta} \left(\frac{\beta}{e}\right)^\beta \qquad (20).$$

Due to Stierling [7] equation and feature of Gamma function :

$$\Gamma(1+\beta) = \sqrt{2\pi\beta} \left(\frac{\beta}{e}\right)^\beta \left(1 + \frac{1}{12\beta} + O\left(\frac{1}{\beta^2}\right)\right)$$
$$\Gamma(1+q) = q! \qquad (21)$$

both cases (12) and (20) can be unified :

$$coeff = \Gamma(1+\beta) \qquad (22)$$

Then the values of coefficients a, b, c for a solution (8) are :

$$a = 1/(2+\beta) \; ; \; b = (1+\beta)/(2+\beta) \; ; \; c = \left[ \frac{A}{2i\omega_0} \frac{\Gamma(1+\beta)}{a\, b^{(q+1)}} \right]^{1/(2+q)} \qquad (23)$$

and, therefore, the asymptotic of the solution (at $\xi > 1$ ) is:

$$x(t,s)/a_0 \propto \exp(\lambda(\beta) t^{1/(2+\beta)} s^{(1+\beta)/(2+\beta)}) \cos(\omega_0 t + \varphi(t,s))$$

$$\text{here} \quad \lambda(\beta) = (2+\beta) \left[ \frac{r_0\rho_0W_0}{2\gamma_0\omega_0} \frac{\Gamma(1+\beta)}{(1+\beta)^{1+\beta}} \cos\left(\frac{\pi}{2\,(2+\beta)}\right) \right]^{1/(2+\beta)} \qquad (24)$$

$$\varphi(t,s) = (2+\beta) \left[ \frac{r_0\rho_0W_0}{2\gamma_0\omega_0} \frac{\Gamma(1+\beta)}{(1+\beta)^{1+\beta}} \cos\left(\frac{\pi}{2\,(2+\beta)}\right) \right]^{1/(2+\beta)} t^{1/(2+\beta)} s^{(1+\beta)/(2+\beta)}$$

Some particular cases :

$\beta = 0$ $W(s) = const$ $\ln|x/a_0| \propto \lambda(0) t^{1/2} s^{1/2}$ as in ref. [1]

$\beta = 0.5$ $W(s) \propto s^{1/2}$ (optical resonator [3]) $\ln|x/a_0| \propto \lambda(0.5) t^{2/5} s^{3/5}$

$\beta = 1$ $W(s) \propto s$ $\ln|x/a_0| \propto \lambda(1) t^{1/3} s^{2/3}$ -in full coincidence with Chao, Richter & Yao [2]

(25 a, b, c)

### 3. STRONG WAKE FORCE AND NO ACCELERATION

In this case one can neglect the forces of focusing elements in comparison with strong wake force. Then, the equation of movement will have the form :

$$\frac{d^2}{dt^2} x(t,s) = A \int_0^s (s-s')^\beta x(t,s')\, ds', \quad A = r_0\rho_0W_0/\gamma_0 \qquad (26)$$

One can use the method applied above here too for a solution in form $f(\xi)\ exp(\xi)$ (see (8)) for $\xi > 1$ and after similar calculations :

$$x(t,s)/a_0 \propto exp(\tilde{\lambda}(\beta) t^{2/(3+\beta)}\ s^{(1+\beta)/(3+\beta)}\ )\ cos(\tilde{\varphi}(t,s))$$

$$\text{here}\quad \tilde{\lambda}(\beta) = (3+\beta)\ [\ \frac{r_0\rho_0W_0}{8\gamma_0\omega_0}\ \frac{\Gamma(1+\beta)}{(1+\beta)^{1+\beta}}\ cos(\ \frac{\pi}{2\ (3+\beta)})\ ]^{1/(3+\beta)} \qquad (27)$$

$$\tilde{\varphi}(t,s) = t^{2/(3+\beta)} s^{(1+\beta)/(3+\beta)} (3+\beta) [\frac{r_0\rho_0W_0}{8\gamma_0\omega_0}\ \frac{\Gamma(1+\beta)}{(1+\beta)^{1+\beta}} sin(\frac{\pi}{2\ (3+\beta)})]^{1/(3+\beta)}$$

Some particular cases :

$$\beta = 0 \quad W(s)=const \qquad ln|x/a| \propto \tilde{\lambda}(0)\ t^{2/3}s^{1/3}$$

$$\beta = 0.5 \quad W(s) \propto s^{1/2} \qquad ln|x/a| \propto \tilde{\lambda}(0.5)\ t^{4/7}s^{3/7} \qquad (28)$$

$$\beta = 1 \quad W(s) \propto s \qquad ln|x/a| \propto \tilde{\lambda}(1)\ t^{1/2}\ s^{1/2}$$

### 4. ADIABATIC ACCELERATION

Let's assume that an energy of particles of the bunch grows linearly and slowly along the linac ( a usual case for linear accelerators ) :

$$\gamma(t) = \gamma_0(1+Gt)\ ,\ G/\gamma << 1. \qquad (29)$$

Then, after transformation $\tilde{x}\ (t,s) = x(t,s)\ \sqrt{\gamma(t)}$ the equation for motion in the focusing lattice ( with comparably small wake fields in the bunch $\delta_{BNS} << 1$) will have the form similar to (4):

$$\frac{d^2}{dt^2}\tilde{x}(t,s)+\omega_0^2\ \tilde{x}(t,s)=\frac{A}{1+Gt}\ \int_0^s (s-s')^{\beta}\tilde{x}(t,s')ds',\ A=r_0\rho_0W_0/\gamma_0 \qquad (30)$$

As in Chapter 2 one can obtain the equation for amplitude $\tilde{a}(t,s)$ ( see (5) ) :

$$2i\omega_0 \frac{\partial}{\partial t}\tilde{a}(t,s) = \frac{A}{1+Gt} \int_0^s (s-s')^{\beta}\tilde{a}(t,s')\ ds' \quad \text{or}$$

$$2i\omega_0 \frac{\partial}{\partial \tau}\tilde{a}(t,s) = A \int_0^s (s-s')^{\beta}\tilde{a}(t,s')\ ds',\ \tau = \frac{1}{G}\ ln(\gamma(t)/\gamma_0) \qquad (31)$$

One can see that (32) has the same form as (6). Therefore the solutions in this case can be taken from (24) with three rules of changing :

$$\text{I.}\ a_0 \longrightarrow a_0\ (\gamma_0/\gamma(t))^{1/2}$$

$$\text{II.}\ t \longrightarrow \frac{1}{G}\ ln(\gamma(t)/\gamma_0) \qquad (32)$$

*III. the Rule II doesn't touch main betatron phase advance* $\omega_0 t$



### 5. EMITTANCE GROWTH AT THE BEGINNING OF INSTABILITY

Let's investigate the equation for oscillation amplitudes of particles in the bunch in the focusing lattice with wakes in the opposite case when instability begins :

$$2i\omega_0 \frac{\partial}{\partial t}a(t,s) = \frac{r_0}{\gamma_0} \int_0^s W(s-s')\ a(t,s')\ \rho(s')ds' \qquad (33)$$

At the beginning of the instability the amplitudes of all particles slightly differ from this one for a head particle $a_0$ :

$$a(t,s) = a_0\ (1+f(t,s))\ ,\ f(t,s) << 1,\ f(t,0) = 0 \qquad (34).$$

Taking it into account :

$$\frac{\partial}{\partial t}f(t,s) = \frac{r_0}{2i\omega_0\gamma_0} \int_0^s W(s-s')\ \rho(s')\ ds' \qquad (35)$$

Therefore:

$$f(t,s) = \frac{t\ r_0}{2i\omega_0\gamma_0} \int_0^s W(s-s')\ \rho(s')\ ds' \qquad (36)$$

One can estimate a transverse emittance growth $\delta\epsilon$ as :

$$\delta\epsilon/\ \omega_0 a_0^2 \approx \overline{|\ f(t\ s)\ |^2} = t^2\ \frac{r_0^2}{4\omega_0^2\gamma_0^2}\overline{[\ \int_0^s W(s-s')\ \rho(s')\ ds'\ ]^2} \propto t^2 \qquad (37)$$

here the line over expression stands for the mean value over the bunch.

Thus, the law of the emittance growth along linac during the development of instability is proportional to the squared distance passed squared $t^2$ and doesn't depend on the law of the wake field. It differs much from with the advanced instability solution (24) where the growth is basically determined by the law of $W(s)$.

## CONCLUSION

Asymptotic expressions for a transverse instability due to the "head-tail" effect are given in this paper. The solution for a comparatively small wake and strong focusing lattice shows that the logarithm of mean size of bunch $\sigma$ growth vs. distance passed as:

$$ln(\sigma/a_0) \propto t^{1/(2+\beta)} \ , \ if\ W(s) \propto s^{\beta}$$

($a_0$- initial beam displacement) i.e. the growth is smaller for bigger values of power $\beta$.

It seems possible to use this property of BBU for determination of *W(s)* from beam size growth data in linear accelerators.

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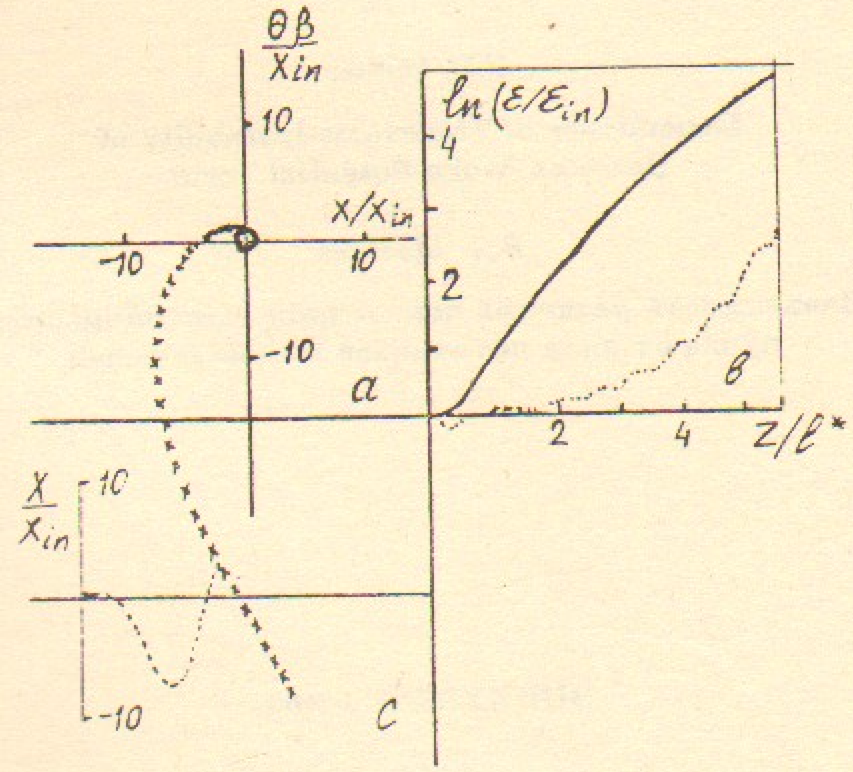

Fig. 1: Growth of r.m.s. emittance of the bunch without energy spread and with constant energy along linear accelerator (b); portrait of the bunch on the phase plate(a) and displacement of particles inside the bunch (c) at finite moment of time. Smooth focusing approximation [5].