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ON THE ANALYTICAL THEORY FOR STATIONARY SELF-FOCUSING OF RADIATION

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On the Analytical Theory for Stationary Self-Focusing of Radiation

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ABSTRACT

Investigation of dynamics of a laser beam in a medium, dielectric permittivity of which depends on the beam power, represents one of the classical problems of nonlinear physics. In the present paper this problem is solved analytically in the framework of the two-dimensional Schroedinger equation with the focusing cubic nonlinearity and small defocusing nonlinearity of the fifth order. The model is shown to describe the whole process of stationary self-focusing, that spreads along the direction of the beam propagation and consists in decreasing oscillations of the waveguide width. The shape and also the upper and lower envelopes of these oscillations are studied in detail. Parameters of the final state, that is a homogeneous waveguide, are determined.

INTRODUCTION

Strong flow of electromagnetic radiation propagating trough a medium exerts influence on its parameters. If the dielectric permittivity of the medium in the central region of the flow is larger than at the periphery, then the speed of light decreases to the center and the wave surfaces are deformed in such a way that the focusing or waveguide propagation of the flow can occur. This phenomenon was predicted in 1962 by G. Askar'yan [1] and is named the self-focusing of radiation.

An existence of the self-focusing was confirmed experimentally. Depending on the conditions of experiment, a local destruction of the medium or formation of bright thin threads were observed. Some authors interpreted the threads as running focuses, others preferred the idea of waveguides along which the radiation propagates (see, for instance, reviews [2,3]). Pulsation of the thread's width, observed also, was treated as a multifocus structure or transitional process of a homogeneous waveguide formation, respectively.

Mathematical models for the self-focusing of radiation were suggested in papers of 60's (see [4-7] and also reviews [8,9]). Since then numerous attempts have been undertaken to solve the self-focusing equations or, at least, to clear up the most essential properties of their solutions for interpretation of the experimental data. However, investigation of the mathematical models appeared to be a difficult task. Even the simplest model, in which the nonlinear addition to the dielectric permittivity of the medium is proportional to the local energy density of the radiation, has called difficulties. This model has led to the two-dimensional Schroedinger equation with attracting cubic nonlinearity.

In order to clarify the problem, it is useful to note that dynamics of the self-focusing is determined by competition of the focusing nonlinearity (which contracts the flow) and transverse dispersion of the waves (which spreads the flow). At a given power of the radiation, both of the nonlinearity and dispersion are inversely proportional to the cross section of the flow, i.e., to the square of its width. The ratio of the above opposing factors depends on the power of radiation and does not depend on the width of flow. If the power exceeds some critical value, then the nonlinearity overbalances the dispersion, that results in an unlimited transverse compression of the flow. This process is named the critical collapse. Attempts to study the dynamics of the critical collapse were giving contradictory results for a long time. An asymptotic law for a decrease of the flow width along the direction of the beam propagation has been determined carefully only last years [10-13] (for more references on the critical collapse problem see review [14] and paper [12]). However, this asymptotics was not reached in the numerical simulations, even when the maximum field amplitude has increased by many millions times.

As the energy density of radiation increases infinitely in the framework of the simplest model, the applicability conditions of the model have to be broken. First of all, the quadratic term becomes essential in the dielectric permittivity expansion in a power series of the radiation energy density. The corresponding term in the nonlinear Schroedinger equation is of the fifth order in the electromagnetic field amplitude. Under conditions when this term also reduces the phase velocity of the waves, focusing ones, an explosive increase of the field amplitude, generally speaking, is going on up to the local destruction of the medium. For such media, the final stage of the process is hardly accessible for universal macroscopic description.

In the present paper the case of a "soft" saturation of nonlinearity is investigated. This implies that the quadratic term in the dielectric permittivity expansion is defocusing, in contrast to the linear term. It seems, for the first glance, that for such media an explosive increase of the field amplitude will continue until all terms in the dielectric permittivity expansion become values of the same order. Maybe because of this, many models for the saturation of nonlinearity were suggested and the two-dimensional Schroedinger equation with the focusing cubic nonlinearity and weak defocusing nonlinearity of the fifth order was considered only as a one of those

(see, for instance, [14-17]). However, this equation can be shown to have a more general quantitative sense. The point is that the contracting action of the main (focusing) nonlinearity is almost exactly counterbalanced by the dispersion, while the surplus power of the trapped waves has irradiated, basically, at an early stage of the critical collapse. Because of the major factors balance, even a small defocusing nonlinearity stops the compression of the flow, and all higher terms of the dielectric permittivity expansion remain negligible. This makes possible a universal macroscopic description for all the process of stationary self-focusing.

2. QUASI-STATIONARY SOLITONS

Stationary propagation of a coherent quasi-monochromatic linearly polarized electromagnetic wave is described in numerous media by equation of the form (see, for instance, [1,2])

$$i \frac{\partial \psi}{\partial z} = \left(-\Delta + V\right) \psi \quad . \qquad (2.1)$$

Here ψ is an envelope of the wave, z is the distance along the direction of the wave propagation, Δ is the two-dimensional Laplace operator acting in the transverse plane, $-V$ is the nonlinear addition to the dielectric permittivity of the medium.

When the spatial dispersions of nonlinearity is negligible, the addition V is a function of $|\psi|^2$. Assuming the electromagnetic field of the radiation is small in comparison with internal microscopic field in the medium, one can expand the function $V(|\psi|^2)$ in the Taylor series and hold the lowest terms of the expansion:

$$V = -|\psi|^2 + \delta^2\ |\psi|^4 \quad . \qquad (2.2)$$

The signs of coefficients in (2.2) correspond to a medium with a "soft" saturation of nonlinearity. The first term describes the focusing nonlinearity and the second term describes the defocusing nonlinearity. The modulus of the first coefficient is standardized by a proper choice of units for physical values. Remaining freedom in the choice allow one to make the typical value of the electromagnetic field $|\psi|$ in the initial state ($z = 0$) to be of the order of unity. Parameter δ, that has the meaning of the ratio of typical initial field of the radiation and internal microscopic field in the medium, is assumed below to be a very small value, $\delta \lll 1$. The expansion of the dielectric permittivity (2.2) is applicable until the field of radiation remains small in comparison with the internal field,

$$\delta\ |\psi| \ll 1\ . \tag{2.3}$$

At soft saturation of the nonlinearity, this condition is satisfied along the whole axis z, that makes possible a complete macroscopic description of the stationary self-focusing of radiation.

An essential role in the posterior analysis belongs to the well-known integrals of equations (2.1) - (2.2), the "number of quanta",

$$N = \frac{1}{2\pi} \int d^2\vec{r}\ |\psi|^2\ , \tag{2.4}$$

and the Hamiltonian,

$$H = \frac{1}{2\pi} \int d^2\vec{r} \left(|\nabla\psi|^2 - \frac{1}{2}|\psi|^4 + \frac{\delta^2}{3}|\psi|^6 \right)\ . \tag{2.5}$$

The generalized Talanov transformation, brought into use under the investigation of the critical collapse,[9] is also relevant. It looks as

$$\psi(\vec{r},z) = \frac{1}{a(z)} f\left(\frac{\vec{r}}{a(z)},\ \zeta(z)\right) exp\left\{i\left[\zeta(z) + \frac{a'(z)}{a(z)}\frac{r^2}{4}\right]\right\}, \tag{2.6}$$

$$\zeta(z) = arg\ \psi(0,\ z), \qquad a^{-2}(z) = \zeta'(z),$$

where the primes signify z-derivatives.

Studying a single channel of self-focusing, one can place at its axis the origin of transverse coordinates and treat $\zeta' \equiv \omega$ as a "bound energy" of the trapped quanta in the "potential" (2.2) and a as a width of the channel.

Talanov transformation leads to the following equation for the function f:

$$\left(i\frac{\partial}{\partial\zeta} - 1\right) f = \left(-\Delta + U\right) f\ , \tag{2.7}$$

$$U = -|f|^2 + \frac{\delta^2}{a^2}|f|^4 - \frac{1}{4}\beta\rho^2, \tag{2.8}$$

$$arg\ f(0,\ \zeta) = 0, \qquad \beta = -\ a^3\ a''.$$

Here, Δ is already the Laplacian in new variables, $\vec{\rho} = \vec{r}/a$. The field f, real at the axis of the channel, has there the value of the order of unity. The term $-\beta\rho^2/4$, has no analogous in formula (2.2), can be interpreted as a potential of inertia forces acting on quanta in the intrinsic reference frame, contracting or expanding together with the channel, whose size varies with the acceleration $|a''|$. The coefficient β characterizes also the degree of adiabaticity of the "potential" (2.2) variation. Indeed, the typical scale of the value V variation along the coordinate z is of the order of $(|a''|/a)^{-1/2} \equiv \gamma^{-1}$, while the ratio

$$\gamma/\omega = (a^3|a''|)^{1/2} = |\beta|^{1/2}$$

is just the parameter of adiabaticity.

If the channel arises as a result of the modulation instability of a powerful radiation flow, rather than it is prepared artificially by some special way, then ω and γ are originally the values of the same order. At $\beta \sim 1$ the quanta leave the channel easily and predominance of the focusing nonlinearity over the dispersion decreases quickly. This implies also decreasing of the parameter β, which turns into zero when the above opposing factors are exactly balanced. An approximate balance of those comes at an early stage of the critical collapse and improves later. At $\beta \ll 1$ the losses of trapped quanta are exponentially small in terms of the inverse parameter of adiabaticity. In conformity with a general adiabatic theory, the regions $\rho \lesssim 1$ and $\rho > 2\beta^{-1/2} \gg 1$, accessible for a classical particle of unit bound energy in the potential U, are separated at $\beta \ll 1$ by the barrier hardly penetrable for the quanta.

As it was already mentioned, the defocusing term $\frac{\delta^2}{a^2}|f|^4$ in (2.8) remains small along the waveguide. Thus, at $\beta \ll 1$ one has two small parameters. In the limit of $\beta = \delta = 0$ the equation (2.7) has localized stationary solution, $f = R$:

$$(\Delta + R^2 - 1)R = 0, \qquad \lim_{\rho\to\infty} R(\rho)\ \rho^{1/2}e^{\rho} = A \cong 3.52, \tag{2.9}$$

which was found in the well-known paper [6] and is named the Townes soliton*. The constant A can be expressed in terms of the Townes soliton by the formula

$$A = (\pi/2)^{1/2} \int_0^\infty d\rho\, \rho\, R^3(\rho)\, I_0(\rho).$$

According to (2.6)-(2.8), the localized state of the field ψ, corresponding in new variables to the Townes soliton, is not necessarily a stationary state; it may be contracting or expanding with a constant velocity, as $a'' = 0$ entails $\beta = 0$. Such a solution, found in [21,22], is referred below as the Talanov soliton. The number of trapped quanta in the Talanov (and Townes) soliton,

$$N_c = \int_0^\infty d\rho\, \rho\, R^2 \cong 1.86 ,$$

is "critical" in the sense that under the condition $N < N_c$ the nonlinearity is overbalanced by the dispersion and the collapse is impossible even in the framework of cubic nonlinear Schroedinger equation. The Hamiltonian of the Townes soliton equals to zero,

$$H_c = \int_0^\infty d\rho\, \rho \left[\left(\frac{dR}{d\rho} \right)^2 - \frac{1}{2} R^4 \right] = 0,$$

that implies exact balance between the nonlinearity and dispersion. Because of the same balance, the Hamiltonian of the Talanov soliton equals to the "kinetic energy" of the trapped quanta directed motion:

$$H_c(a') = M\, a'^2 , \qquad M = \frac{1}{4} \int_0^\infty d\rho\, \rho^3 R^2 \cong 0.55.$$

At small (but finite) values of the parameters β and δ, equation (2.7) has quasi-stationary solutions close to the Talanov soliton in the range $\rho \ll |\beta|^{-1/2}$, where the term $\beta\rho^2$ is small. The contributions of the quasi-stationary soliton to the number of quanta and Hamiltonian, N_s and H_s, can be calculated by the perturbation theory using the Talanov soliton as a zero approximation:

$$N_s - N_c = \tilde{N} \cong \beta M + 2 \frac{\delta^2}{a^2} N_c , \qquad (2.10)$$

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* Apart from the Townes soliton (decreasing monotonously in terms of ρ) a numerable set of nonmonotone solutions decreasing at $\rho \to \infty$ exists for the equation (2.9) (see [16,19,20]), but those are exponentially unstable with respect to small perturbations and are not realizable under initial conditions of general kind.

$$H_s \cong M\, a'^2 + \frac{\delta^2}{a^4} N_c - \frac{\tilde{N}}{a^2} . \qquad (2.11)$$

Formula (2.10) can be obtained also by differentiation of (2.11) with respect to z or ζ, at fixed values of $\tilde{N}$ and H_s, if one uses the value's β definition (see (2.8)). For the further, this definition is convenient to replace by a more simple relationship (2.10).

Conditions for the self-focusing, $\tilde{N} > 0$ and $H_s < 0$, together with the formulae (2.10) and (2.11) entail the inequality

$$|\beta| \le \tilde{N}/M,$$

which shows that the adiabaticity parameter is indeed small at $\tilde{N} \ll 1$. In the range $a^2 \gg \delta^2 N_c/\tilde{N}$, where the defocusing term is negligible, formula (2.10) turns into direct relationship between the parameters β and $\tilde{N}$:

$$\beta \cong \tilde{N}/M.$$

3. THE SHAPE OF THE SOLITON OSCILLATIONS

Equation (2.11) enables one to determine a variation law for the soliton size, a, at given values of $\tilde{N}$ (the surplus number of the trapped quanta) and H_s (the soliton contribution to the Hamiltonian). In terms of the variable ζ (see (2.6)), the above equation looks as:

$$\left(\frac{da}{d\zeta}\right)^2 \cong \left(\tilde{N}a^2 - N_c\delta^2 + H_s a^4\right)/M \cong -\frac{H_s}{M}\left(a_M^2 - a^2\right)\left(a^2 - a_m^2\right) . \qquad (3.1)$$

The extreme sizes of the soliton, introduced here, satisfy the relationships

$$a_m^2\, a_M^2 = -\frac{\delta^2 N_c}{H_s} , \qquad a_m^2 + a_M^2 = -\frac{\tilde{N}}{H_s} . \qquad (3.2)$$

Defining new variable, φ, according to the formula

$$\frac{d\varphi}{d\zeta} = \left(\frac{-H_s}{M}\right)^{1/2} a_M = \left(\frac{\tilde{N}}{M}\right)^{1/2} \left(1 + q^2\right)^{-1/2} , \qquad q \equiv \frac{a_m}{a_M} , \qquad (3.3)$$

one can rewrite the equation (3.1) in a simpler form:

$$\left(\frac{da}{d\varphi}\right)^2 \cong \left(1 - \frac{a^2}{a_M^2}\right)\left(a^2 - a_m^2\right) . \qquad (3.4)$$

At slowly varying envelopes a_m and a_M, i.e. for quasi-periodic oscillations of the soliton, equation (3.4) can be solved in terms of the Jacobi elliptic functions of modulus $q' = \sqrt{1 - q^2}$, complementary to $q = \frac{a_m}{a_M}$:

$$a \cong a_M \, dn(\varphi, q'). \tag{3.5}$$

The real periods of the function (3.5) over the variables φ and ζ, computed formally at fixed meanings of q and $\tilde{N}$, equals to

$$\Delta\varphi \cong 2K(q') \quad \text{and} \quad \Delta\zeta \cong 2\left(\frac{M}{\tilde{N}}\right)^{1/2}\left(1 + q^2\right)^{1/2} K(q') \tag{3.6}$$

respectively, where K is a complete elliptic integral of the first kind. The imaginary half-period over ζ, computed under the same convention, is signified below as $i\Lambda$ and is given by the formula

$$\Lambda \cong 2\left(\frac{M}{\tilde{N}}\right)^{1/2}\left(1 + q^2\right)^{1/2} K(q). \tag{3.7}$$

At $q \ll 1$, the above formulae can be simplified and, apart from this, extended to some first soliton oscillations, which aren't quasi-periodic ones. Thus, (3.2) reduces to

$$a_M^2 \cong -\frac{\tilde{N}}{H_S}, \qquad a_m^2 \cong \frac{\delta^2 N_c}{\tilde{N}}, \tag{3.8}$$

while (3.4) comes into two equations:

$$\left(\frac{da}{d\varphi}\right)^2 \cong a^2 - a_m, \qquad a^2 \ll a_M^2, \tag{3.9}$$

and

$$\left(\frac{da}{d\varphi}\right)^2 \cong \left(1 - \frac{a^2}{a_M^2}\right) a^2, \qquad a^2 \gg a_m^2. \tag{3.10}$$

Noteworthy that at $a^2 \ll a_M^2$ the behavior of value a do not depend on H_S. The parameter drops out also of the variable φ definition, as it takes the form

$$\frac{d\varphi}{d\zeta} \cong \left(\frac{\tilde{N}}{M}\right)^{1/2}.$$

Hence, only the global variation of H_S in the range $a^2 \ll a_M^2$ appear to be

essential. In particular, the slowness of H_S variation in that range is not required for the equation (3.9) applicability.

If both of the conditions $a_m^2 \ll a^2 \ll a_M^2$ are satisfied, equation (3.4) takes especially simple form, independent of both values H_S and $\tilde{N}$:

$$\left(\frac{da}{d\varphi}\right)^2 \cong a^2. \tag{3.11}$$

As seen, in this range the ζ-dependence of a is exponential one:

$$a \propto exp(\mp\varphi), \tag{3.12}$$

The signs "-" and "+" in (3.12) correspond to the compression and expansion of the soliton respectively.

The solutions of equations (3.9) and (3.10) in the vicinities of n-th minimum and maximum of the soliton size look as:

$$a \cong a_{mn} \, ch(\varphi - \varphi_{mn}), \tag{3.13}$$

$$a \cong \frac{a_{Mn}}{ch(\varphi - \varphi_{Mn})}. \tag{3.14}$$

The subscript "mn" or "Mn" signifies that the value relates to the point of n-th minimum or maximum of the soliton size. The coincidence conditions for formulae (3.12), (3.13) and (3.14) inside their mutual applicability regions imply that:

$$\varphi_{Mn} - \varphi_{mn} \cong ln\,\frac{4a_{Mn}}{a_{mn}}, \qquad \varphi_{m,n+1} - \varphi_{Mn} \cong ln\,\frac{4a_{Mn}}{a_{m,n+1}}. \tag{3.15}$$

For large ($q \ll 1$) quasi-periodic oscillations, formulae (3.5) and (3.6) agree with (3.12)-(3.15).

4. THE POWER OF TRAPPED QUANTA LOSSES

Formula (2.11) links together three parameters of the quasi-stationary soliton: a, $\tilde{N}$ and H_S. Two more relationships between these values are required to get a complete set and forget about the original equation in partial derivatives. As the shape of soliton oscillations at given values of $\tilde{N}$ and H_S is already known, the powers of the trapped quanta and Hamiltonian losses, J and J_H, can be computed principally. Then, the pair of equations required will be gotten in the form

$$\frac{d\tilde{N}}{d\zeta} = - J, \qquad \frac{dH_S}{d\zeta} = - J_H. \tag{4.1}$$

The values J and J_H are positive, as the back process of quanta trapping from the free "background" is negligible for a single soliton. At adiabatic variation of the soliton, the value J is exponentially small in terms of the inverse parameter of adiabaticity:

$$-\ln J \sim \omega/\gamma. \tag{4.2}$$

The value J_H is linked with J by the estimation

$$J_H \sim J\,\omega, \tag{4.3}$$

as the soliton spends the "bound energy" ω to rid a trapped quantum.

The quantitative formulae for the values J and J_H are convenient to derive step by step. First, the range $a^2 \gg \Lambda a_m^2$, where the defocusing nonlinearity do not influence the trapped quanta losing, has to be studied. "Potential" U for the Schroedinger equation (2.7) looks there as

$$U \cong |f|^2 - \frac{1}{4}\beta\rho^2, \qquad \beta \cong \frac{\tilde{N}}{M}\,. \tag{4.4}$$

and varies with the same typical "time" as the value $\tilde{N}$, i.e., much slower than a. The trapped quanta losses, caused by such a variation of U, are negligible in comparison with the losses via a tunneling in this potential. Therefore, computing the losses, one can treat potential U as a stationary one. Use of a stationary version of the WKB-method gives:

$$J \cong 2A^2 exp(-\Lambda) \equiv J_0(\Lambda), \tag{4.5}$$

where A is the same constant as in (2.9) and Λ is the same value as in (3.7) at $\Lambda q^2 \ll 1$. Formula (4.5) makes more accurate the estimation (4.2) conformably to the region $a^2 \gg \Lambda a_m^2$ of the soliton size variation. Note, that the possibility of reduction of the non-adiabatic quanta losses, existing in terms of the original Schroedinger equation for the field ψ, to the tunnel losses in a quasi-stationary potential is caused by the additional symmetries arising at $\delta = 0$.

In order to cover a broader range $a^2 \gg a_m^2$ (including the subrange $a^2 \le \Lambda a_m^2$), a small nonstationary correction to the value β in the

potential U is to be taken into account. According to the formulae (2.10) and (3.8),

$$\beta \cong \frac{\tilde{N}}{M}\left(1 - \frac{2a_m^2}{a^2}\right). \tag{4.6}$$

Then, use of nonstationary WKB-method gives

$$J \cong 2A^2 exp\left\{-\left(\frac{M}{\tilde{N}}\right)^{1/2}\left[\pi + 4a_m^2\int_0^{\pi/2} ds\,\cos^2 s\,Re\,a^{-2}(\varphi + is)\right]\right\}. \tag{4.7}$$

Formula (3.12) for $a(\varphi)$ enable one to replace (4.7) by a local expression,

$$J \cong J_0(\Lambda)\,exp\left(-\frac{\Lambda a_m^2}{2a^2}\right), \tag{4.8}$$

applicable at $a_H^2 \gg \Lambda a_m^2$. At $a^2 \gg \Lambda a_m^2$, this formula reduces to (4.5). In the range $a^2 \le \Lambda a_m^2$ the function (4.8) decreases quickly, as the value a decreases, so that at $a^2 \ll \Lambda a_m^2$ the quanta losses are practically absent.

The true losses at $a^2 \ll \Lambda a_m^2$ aren't as small as formula (4.8) predicts, for it takes into account only tunneling. Apart from the tunneling, the quanta losses caused by nonadiabatic transitions in slowly varying potential U exist. The total value of those, reached during the soliton passage through the region, $a^2 \le \Lambda a_m^2$ appear to be about $\Lambda^{-4}exp(-\Lambda)$. This value is also small and can be neglected below.

Comparing the flux of quanta with the flux of Hamiltonian slightly right the classical turning point $\rho = 2\beta^{-1/2}$, but inside the applicability region of WKB-method already, one can get the following relationship:

$$J_H \cong J\left(\frac{d}{d\varphi}\frac{1}{a}\right)^2. \tag{4.9}$$

While the formula (3.12) valid, this relationship takes exactly the same form, as the estimation (4.3), namely,

$$J_H \cong J a^{-2}. \tag{4.10}$$

As seen from (4.8) and (4.10), the basic contribution to the soliton Hamiltonian losses comes from the region $a^2 \sim \Lambda a_m^2$. For some first passages through the region, the soliton Hamiltonian relative variations are not small. This do not affect the shape of the oscillations, if condition $a_M^2 \gg \Lambda a_m^2$ is satisfied (see the remark on the equation (3.9) applicability).

As the soliton loses the quanta, the sweep of the oscillations decreases and the above condition is broken. To secure the unaffected shape farther, the oscillations with $a_M^2 \sim \Lambda a_m^2$ should be nearly periodic already, i.e. the soliton Hamiltonian relative variations per period should already be small. Under such a condition, the formula (3.14) for the shape of the oscillations in the vicinity of n-th maximum remains valid till $a_M^2 \gg a_m^2$. Substituting (3.14) to (4.9) and (4.7), one gets:

$$J_H \cong J\, a_M^{-2}\, sh^2(\varphi - \varphi_M), \tag{4.11}$$

$$J \cong J_0(\Lambda) exp\left[-\frac{\Lambda a_m^2}{2a_M^2} sh^2(\varphi - \varphi_M)\right], \tag{4.12}$$

where Λ is given by the formula (3.7) for $q \ll 1$, namely,

$$\Lambda = \pi\left(\frac{M}{\tilde{N}}\right)^{1/2}\left(1 + \frac{3a_m^2}{4a_M^2}\right). \tag{4.13}$$

These expressions are applicable until $\Lambda q^4 \ll 1$.

In order to overcome the latter limitation and to get a complete description of stationary self-focusing, a slightly different approach is to be used. It rests on that for the quasi-periodic oscillations of the soliton, the true distribution of the losses along a period is not essential. As any distribution providing the right meaning of the total losses per period is suitable, one can use the flux of the irradiated quanta far away from the soliton. This brings some advantages in comparison with the calculation of the flux inside the so-called close zone of radiation, where the field structure becomes more complicated at $a_M \sim a_m$. As a result, a simple description of the quasi-periodic oscillations with arbitrary q is possible to construct.

Analysis, exploiting the quasi-classical theory of multiquantum transitions (see, for instance, [23]), shows that the basic channel of the trapped quanta losses is associated with the tunneling again, while nonadiabatic transitions caused by slow variation of potential U remain negligible.

The main contribution to the losses comes from the closest to the real axis settle points of the classical action in the complex plane ζ.

Positions of the settle points, responsible for the released quanta with the wave number k, are determined by the equation

$$1 + k^2a^2(\zeta_s) = 0. \tag{4.15}$$

For the function $a(\zeta)$ satisfying (3.4) and (3.3), the points ζ_s have removed from the real axis by the quarter of the imaginary period at any meaning of k, while the real part of ζ_s depends on k:

$$Im\,\zeta_s = \Lambda/2, \qquad k^2a_M^2 = \frac{sn(\varphi(Re\zeta_s),q')}{cn(\varphi(Re\zeta_s),q')}. \tag{4.16}$$

Here φ is the function (3.3) of variable $\zeta = Re\zeta_s$ and Λ is given by the formula (3.7) again.

In the complex plane z, the settle points have removed from the real axis by the distance

$$Im\,z_s \equiv \frac{1}{2}\Lambda a_M^2\left(1 - \frac{E(q)}{K(q)}\right), \tag{4.17}$$

where $E(q)$ is a complete elliptic integral of the second kind.

The probability for quantum transition from the soliton bound state to the state with a wave vector $\vec{k}$ is proportional to the exponent

$$exp\,(-2Im\zeta_s - 2Imz_s k^2). \tag{4.18}$$

At $q \ll 1$, this agrees with formula (4.12), as expressions (4.17) and (4.16) are reduced for such q to:

$$Im\,z_s \cong \frac{1}{4}\Lambda a_m^2, \tag{4.19}$$

$$k^2 \cong a_M^{-2} sh^2(\varphi - \varphi_M). \tag{4.20}$$

At $a_M^2 \ll \Lambda a_m^2$ (i.e. $\Lambda q^2 \gg 1$), the "energies" of irradiated quanta are, basically, much smaller than a_M^{-2}:

$$k^2 \sim Imz_s^{-1} \sim (\Lambda a_m^2)^{-1} \ll a_M^{-2}.$$

According to (4.16), in this range $k^2 \cong a_M^{-2}\,(\varphi - \varphi_M)^2$, that can be gotten from (4.20) also. Thus, the the formula (4.20) appear to be applicable for computation of the quanta losses at all meanings of q.

The total quanta losses per period are gotten by integration of the spectral density over all $\vec{k}$. Transformation of the integral from the variable k to the variable $\zeta = Re\zeta_S$, treated conventionally as "the moment of the quanta irradiation", leads to the following expression for the power of quanta losses:

$$J \cong J_0(\Lambda)exp\left[-\Lambda\left(1 - \frac{E(q)}{K(q)}\right)sh^2(\varphi - \varphi_M)\right]. \qquad (4.21)$$

At $q \ll 1$, this formula reduces to (4.12).

The effective power of the soliton Hamiltonian losses, J_H, is linked with J by the relationship $J_H = J\ k^2$. In view of the formula (4.20) validity, this relationship takes the form (4.11).

Thus, the formulae (4.11) and (4.21) appear to be applicable at all q. The only condition required for such a universality consists in overlapping of the range $a_M^2 \gg \Lambda a_m^2$ with the range of quasi-periodic oscillations. If the oscillations aren't quasi-periodic yet at $a_M^2 \sim \Lambda a_m^2$, then the soliton Hamiltonian variation perturbs the shape of oscillations at $a \sim a_M$ and the description is broken there. However, at sufficiently small δ, the above ranges are indeed well overlap one another.

The formulae (4.1), (4.11) and (4.21), together with (3.3) and (3.4), constitute a complete set.

5. THE FIRST CYCLE OF THE SOLITON COMPRESSION

The first cycle of the soliton compression requires special consideration. Starting with a more or less arbitrary initial condition, it results in the establishment of a universal dynamic state, that provides the universality of the farther picture of stationary self-focusing. The consideration is given below in the framework of adiabatic approximation. This implies that the surplus number of trapped quanta,

$$\tilde{N} \cong \beta M \cong \pi^2 M\ \Lambda^{-2}, \qquad (5.1)$$

varies much slowly than the phase of the soliton, ζ, oscillates, i.e.

$$\zeta_N \equiv \left|\frac{d\zeta}{d\ln\tilde{N}}\right| \cong \frac{\tilde{N}}{J} \gg 1. \qquad (5.2)$$

At the soliton sizes satisfying the conditions

$$a_M^2 \gg a^2 \gg \Lambda a_m^2, \qquad (5.3)$$

where a_M^2 and a_m^2 are defined by the formulae (3.8), the flux J is given by the formula (4.5) and requirement (5.2) looks as

$$\zeta_N \cong \frac{\pi^2 M}{2A^2\Lambda^2} exp\ \Lambda \gg 1. \qquad (5.4)$$

Taking into account formula (3.12), one can show that the value $\tilde{N}$ varies much slowly than the soliton size also under the condition

$$1 \ll \left|\frac{d\ln\tilde{N}}{d\ln a}\right|^{-1} = \left|\frac{d\ln\tilde{N}}{d\varphi}\right|^{-1} \cong \frac{\pi}{\Lambda}\zeta_N, \qquad (5.5)$$

slightly more restrictive than (5.4). While (5.5) is satisfied, the soliton Hamiltonian, H_S, can be computed easily by the formula (4.10):

$$-H_S = \int J_H d\zeta \cong \frac{\Lambda A^2}{\pi a^2} exp(-\Lambda). \qquad (5.6)$$

Then, the first of inequalities (5.3), equivalent to $-H_S a^2 \ll \tilde{N}$, reduces to (5.5) exactly. The second one is satisfied till the final stage of the cycle, when the defocusing nonlinearity stops the compression.

Substituting the expressions for J and $\tilde{N}$ in terms of Λ to the equation (4.1) for $\tilde{N}$, one can easily get the formula

$$\zeta \cong \zeta_0 + B\int_{\Lambda_0}^{\Lambda} dl\ l^{-3} exp\ l, \qquad B \equiv \frac{\pi^2 M}{A^2} \cong 0.438. \qquad (5.7)$$

This formula, together with the expression (5.1) for $\tilde{N}$ and the relationships following from (3.3) and (3.12),

$$ln\frac{a_0}{a} \cong \varphi - \varphi_0 \cong \pi\ B\int_{\Lambda_0}^{\Lambda} dl\ l^{-4} exp\ l, \qquad (5.8)$$

determines ζ-dependence of the relevant values in the parametric form.

All these formulae were derived for the adiabatic regime of the soliton compression. If the initial state were close to the Townes soliton, the evolution would be adiabatic from the very beginning. Under initial conditions of general kind, first the formation of the soliton goes on, that is non-adiabatic process. The idea of quanta trapping gets some sense at $\zeta_N > 1$, i.e., $\Lambda > 5$. Taking into account that "initial" size and phase of the soliton were not defined quantitatively yet, one can choose the constants of integration in formulae (5.7) and (5.8) more or less arbitrarily. In particular, the following choice is suitable:

$$\Lambda_0 = 5, \qquad \zeta_0 = \varphi_0 = 0, \qquad a_0 = 1. \tag{5.9}$$

As the value of Λ growth, the accuracy of the adiabatic approximation improves quickly, so that the meaning $\zeta_N = 10$ is reached at $\Lambda \cong 8$. At large Λ, formulae (5.7) and (5.8) may be replaced by simpler ones:

$$\zeta \cong \frac{B \exp \Lambda}{\Lambda^2(\Lambda - 3)}, \qquad ln\,\frac{1}{a} \cong \frac{\pi B \exp \Lambda}{\Lambda^3(\Lambda - 4)}. \tag{5.10}$$

A relative accuracy of such an approximation is about 0.1 at $\Lambda = 8$ and better at larger Λ. Noteworthy that the value of a, corresponding to the meaning $\Lambda = 8$, is not very small, $a \cong 0.2$. This implies that the adiabatic approximation and approximation (5.10) become valid at an early stage of the compression, after the soliton size has decreased about five times.

In order to restore the dependence on z, one must use the definition of the soliton size (see (2.6)). This results in:

$$z = \int d\zeta\, a^2 \cong -\frac{\Lambda\, a^2}{2\,\pi} + const\,, \qquad a \cong \left(2\pi\,(z_* - z)/\,\Lambda\right)^{1/2}. \tag{5.11}$$

Here z_* is the point where singularity of the field ψ would arise if the defocusing nonlinearity were absent (i.e., if δ were equal to zero).

Tending, formally, a to zero, one can get from (5.10) and (5.11) the well-known asymptotic law for the critical collapse:

$$a \cong \left(2\pi\,\frac{z_* - z}{ln\,ln\,\dfrac{1}{z_* - z}}\right)^{1/2}. \tag{5.12}$$

This asymptotics was not reached in numerical simulation even when the soliton size has decreased by many millions times. More precise formulae for the factor $\Lambda \to \infty$ in (5.11),

$$\Lambda \cong ln\,ln\,\frac{1}{a} + 4\,ln\,\Lambda, \tag{5.13}$$

clearly shows why that has happen. As the item $4 ln\Lambda$ (which was neglected in (5.12)) is larger than the item $ln ln(1/a)$ (which was taken into

account there) until $ln(1/a) > 10^5$, the asymptotics (5.12) is practically unaccessible. At any realistic values of $a \ll 1$, more precise formulae (5.7) and (5.8) (or, at least, (5.10)) must be used for comparison with results of physical or numerical experiments.

6. LARGE OSCILLATIONS OF THE SOLITON

At a sufficiently small δ, the sweep of some first oscillations of the soliton is large, so that the condition $a_M^2 \gg \Lambda a_m^2$ is satisfied. If $a^2 \gg \Lambda a_m^2$ also, then the simplest formulae (4.5) and (5.1) for J and $\tilde{N}$ are valid, that leads to the parametric representation similar to (5.7), (5.8). In the vicinity of n-th maximum of the soliton size it looks as:

$$\zeta \cong \zeta_{Mn} + B \int_{\Lambda_{Mn}}^{\Lambda} dl\, l^{-3} exp\, l, \qquad \varphi \cong \varphi_{Mn} + \pi B \int_{\Lambda_{Mn}}^{\Lambda} dl\, l^{-4} exp\, l. \tag{6.1}$$

The soliton size, a, is given there by the formula (3.14).

The variation of the soliton contribution to Hamiltonian, H_s, in the range $a^2 \gg \Lambda a_m^2$ can be neglected,

$$-H_s \cong H_{Mn}, \qquad a^2 \gg \Lambda a_m^2. \tag{6.2}$$

The value H_s varies, basically, at $a^2 \sim \Lambda a_m^2$. At $a^2 \ll \Lambda a_m^2$, both of the values $\tilde{N}$ and H_s, are practically invariable (see, (4.8) and (4.10)),

$$\tilde{N} \cong N_{mn}, \qquad \varphi - \varphi_{mn} \cong \frac{\pi}{\Lambda_{mn}}\,(\zeta - \zeta_{mn}), \tag{6.3}$$

$$-H_s \cong H_{mn}, \qquad a^2 \ll \Lambda a_m^2. \tag{6.4}$$

As above, the subscript "Mn" or "mn" signifies the meaning of the relevant value reached at the n-th maximum or minimum of the soliton size.

The relationships (6.3) are, in fact, applicable in the region $a^2 \sim \Lambda a_m^2$ also, as $\tilde{N}$ varies there by the value exponentially small in terms of Λ. Integration of equations (4.1), (4.8) and (4.10) through this region at given value of the exponent, $\Lambda \cong \Lambda_{mn}$, leads to the following formulae:

$$\tilde{N} - \tilde{N}_{mn} \cong J_0(\Lambda_{mn}) \left(\varphi - \varphi_{mn} - sign(\varphi - \varphi_{mn})\, \delta\varphi(\Lambda_{mn})\right), \tag{6.5}$$

$$H_{mn} - H_{M,n-1} \cong H_{Mn} - H_{mn} \cong \frac{J_0(\Lambda_{mn})}{\pi a_m^2(\Lambda_{mn})} . \qquad (6.6)$$

Here

$$\delta\varphi(\Lambda) \equiv \frac{1}{2}\left(ln(2\Lambda) + \gamma\right), \qquad a_m(\Lambda) \equiv \delta\,\frac{\Lambda}{\pi}\left(\frac{N_c}{M}\right)^{1/2} , \qquad (6.7)$$

$\gamma \cong 0.577$ is the constant of Euler - Mascheroni and the range $a^2 \gg \Lambda a_m^2$, but $|\Lambda - \Lambda_{mn}| \ll 1$, is meant for the formula (6.5). To get rid of the latter limitation, (6.5) can be replaced by the formula

$$|\varphi - \varphi_{mn}| \cong \delta\varphi(\Lambda_{mn}) + \pi B\left|\int_{\Lambda_{mn}}^{\Lambda} dl\, l^{-4} exp\, l\right| . \qquad (6.8)$$

The coincidence condition of the formulae (6.8) and (6.1) inside their mutual applicability regions implies the following recurrent relationships:

$$\varphi_{Mn} - \varphi_{mn} \cong \delta\varphi(\Lambda_{mn}) + G(\Lambda_{mn}, \Lambda_{Mn}),$$

$$\varphi_{m,n+1} - \varphi_{Mn} \cong \delta\varphi(\Lambda_{m,n+1}) + G(\Lambda_{Mn}, \Lambda_{m,n+1}), \qquad (6.9)$$

$$G(x, y) \equiv \pi B \int_x^y dl\, l^{-4} exp\, l . \qquad (6.10)$$

Taking into account (3.15) and (3.13), one can present (6.9) in the form

$$G(\Lambda_{mn}, \Lambda_{Mn}) \cong ln\,\frac{2a_{Mn}}{a_{bn}}, \qquad G(\Lambda_{Mn}, \Lambda_{m,n+1}) \cong ln\,\frac{2a_{Mn}}{a_{b,n+1}} , \qquad (6.11)$$

where a_{bn} is the meaning of the soliton size at $|\varphi - \varphi_{mn}| = \delta\varphi(\Lambda_{mn})$:

$$a_{bn} = a_b(\Lambda_{mn}), \qquad a_b(\Lambda) = a_m(\Lambda)\left(\frac{1}{2}\,\Lambda\, exp\, \gamma\right)^{1/2} . \qquad (6.12)$$

Note, that just the same result would be obtained if the soliton weren't losing the quanta at all in the region $|\varphi - \varphi_{mn}| < \delta\varphi(\Lambda_{mn})$ and were losing the quanta according to the simplest formula $J = J_0(\Lambda)$ everywhere in the region $|\varphi - \varphi_{mn}| > \delta\varphi(\Lambda_{mn})$. Thus, the power of the trapped quanta losses (4.8) can be replaced by the stepped function

$$J = J_0(\Lambda)\,\theta(a - a_b(\Lambda)), \qquad \theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} . \qquad (6.13)$$

The ratios $a_{Mn}/a_b(\Lambda)$ for the right hand sides of equations (6.11) can be expressed in terms of Λ, Λ_{Mn}, Λ_{mj} ($j \le n$) by the formulae (3.8), (6.6), (6.7) and (6.12). The result reads as

$$\frac{a_b^2(\Lambda)}{a_{Mn}^2} \cong \frac{2\Lambda_{Mn}^2\Lambda^3}{\pi B}\, e^{\gamma} \sum_{j=1}^{j=n} \Lambda_{mj}^{-2}\, exp\left(-\Lambda_{mj}\right). \qquad (6.14)$$

Equations (6.11), (6.12) and (6.14) determine successively the values Λ_{mn} and Λ_{Mn} as the functions of a single value Λ_{m1}. The latter is determined by the equation

$$G(\Lambda_0, \Lambda_{m1}) \cong ln\,\frac{a_0}{a_{b1}} , \qquad (6.15)$$

which secures the coincidence of the formulae (6.8) and (5.8) inside their mutual applicability region. At known values Λ_{mn} and Λ_{Mn}, all other parameters of the soliton oscillations are easily computable by the above formulae.

The recurrent relationships (6.11) can be unified by the following way. Let Λ_j be the meaning of parameter Λ at j-th extremum of the soliton size, so that

$$\Lambda_{2n-1} = \Lambda_{mn}, \qquad \Lambda_{2n} = \Lambda_{Mn}. \qquad (6.16)$$

Then, (6.11) can be written as

$$G(\Lambda_j, \Lambda_{j+1}) \cong \frac{1}{2}\left(ln\,\frac{2}{p_j} - \gamma\right), \qquad (6.17)$$

where

$$p_{2n-1} = \frac{\Lambda_{mn}\, a_{mn}^2}{4a_{Mn}^2}, \qquad p_{2n} = \frac{\Lambda_{m,n+1}\, a_{m,n+1}^2}{4a_{Mn}^2}. \qquad (6.18)$$

The latter definitions can also be unified:

$$p_j = (\pi B)^{-1}\, \Lambda_j^2\, \Lambda_{j+1}^2\, \Lambda_{2[j/2]+1}\, S_j , \qquad (6.19)$$

$$S_j = \sum_{2l \le j+1} \Lambda_{2l-1}^{-2}\, exp(-\Lambda_{2l-1}). \qquad (6.20)$$

These formulae are valid until $p \sim \Lambda q^2 \ll 1$. A simple extension to the range $p \ge 1$ is possible, if at $p \sim 1$ the oscillations are nearly periodic already. For such oscillations the true distribution of losses along a period is not essential, while the global losses per period are described by the formulae (4.11) and (4.12) right at $q \ll 1$. In order to get universal description of the large oscillations, (4.12) can be replaced by a stepped function looking like (6.13). The boundary meaning of soliton size, $a_b(\Lambda,\ p)$, for this function is determined to get the right value of the global quanta losses between each neighboring extrema of the oscillations. The soliton Hamiltonian losses can be thought as ones localized at the boundary points $a = a_b(\Lambda,\ p)$. As a result, the formula (6.19) does not change, while the recurrent relationship (6.17) and formula (6.20) turn to:

$$G(\Lambda_j,\ \Lambda_{j+1}) \cong F_0(p_j), \qquad (6.21)$$

$$S_j = \sum_{2l \le j+1} \Lambda_{2l-1}^{-2}\, exp(-\Lambda_{2l-1}) \left[F_1(p_{2l-2}) + F_1(p_{2l-1})\right]. \qquad (6.22)$$

Here F_σ (σ = 0; 1) signify integrals, expressible in terms of the modified Bessel functions of the second kind:

$$F_0(p) \equiv \int_0^\infty d\varphi\, exp\left[-2p\ sh^2\varphi\right] = \frac{1}{2}\, K_0(p)\, e^p, \qquad (6.23)$$

$$F_1(p) \equiv -\ p\frac{dF_0(p)}{dp} = \frac{p}{2}\left[K_1(p) - K_0(p)\right] e^p. \qquad (6.24)$$

At $p \ll 1$, when

$$K_0(p) \cong ln\,\frac{2}{p} - \gamma, \qquad K_1(p) \cong \frac{1}{p}\,,$$

the formulae (6.21) and (6.22) come back to (6.17) and (6.20).

The values Λ_j and p_j, found by the formulae (6.19), (6.21)-(6.24), for the first sixteen osçillations at several meanings of the parameter δ are given in the tables 1 and 2 respectively. At δ = 0.01, parameter p_j is not very small even for the first oscillation, that indicates the accuracy of the above approximation is not very good. At smaller δ, the

approaches of large and quasi-periodic oscillations have well overlapped applicability ranges.

At $\delta \to 0$, that is $\Lambda_1 \to \infty$, the values Λ_j and $a_m(\Lambda_j) \propto \Lambda_j$) remain practically invariable during an exponentially large, in terms of Λ_1, number of the soliton oscillations, $n = 2j$, while the sum S_j and value $a_M^{-2} = 4p_j\Lambda_j^{-1}a_m^{-2} \propto p_j \propto S_j$ increases proportionally to n. The features of such a behavior are seen in the tables 1 and 2, though the accuracy of the asymptotic approximation at realistic meanings of δ is not very good. Therefore, the use of more precise formulae given above is advisable.

Λ_j: $j \backslash \delta$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
1	8.311	9.559	10.3	10.82	11.22	11.53
2	9.197	10.16	10.76	11.19	11.53	11.8
3	9.71	10.57	11.1	11.48	11.78	12.03
4	10.06	10.86	11.35	11.7	11.97	12.2
5	10.33	11.1	11.56	11.88	12.14	12.35
6	10.55	11.29	11.73	12.04	12.28	12.49
7	10.73	11.46	11.88	12.18	12.41	12.6
8	10.89	11.6	12.01	12.3	12.53	12.71
9	11.03	11.73	12.13	12.41	12.63	12.81
10	11.15	11.85	12.24	12.51	12.72	12.9
11	11.27	11.95	12.34	12.6	12.81	12.98
12	11.37	12.05	12.43	12.69	12.89	13.05
13	11.47	12.14	12.51	12.77	12.96	13.13
14	11.55	12.22	12.59	12.84	13.04	13.19
15	11.63	12.3	12.66	12.91	13.1	13.26
16	11.71	12.37	12.73	12.98	13.16	13.32
17	11.78	12.44	12.8	13.04	13.22	13.37
18	11.85	12.5	12.86	13.1	13.28	13.43
19	11.91	12.57	12.92	13.15	13.33	13.48
20	11.97	12.62	12.97	13.21	13.38	13.53
21	12.03	12.68	13.03	13.26	13.43	13.57
22	12.08	12.73	13.08	13.31	13.48	13.62
23	12.14	12.78	13.12	13.35	13.52	13.66
24	12.19	12.83	13.17	13.4	13.57	13.7
25	12.23	12.88	13.22	13.44	13.61	13.74
26	12.28	12.92	13.26	13.48	13.65	13.78
27	12.32	12.96	13.3	13.52	13.69	13.82
28	12.37	13	13.34	13.56	13.73	13.86
29	12.41	13.04	13.38	13.6	13.76	13.89
30	12.45	13.08	13.42	13.63	13.8	13.93
31	12.49	13.12	13.45	13.67	13.83	13.96
32	12.52	13.16	13.49	13.7	13.86	13.99

Table 1. The meanings of the value Λ_j at the first thirty two extrema of the soliton size oscillations for several meanings of the parameter δ according to the formulae (6.19), (6.21) - (6.24).

$j \backslash \delta$	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
1	0.1025	0.0448	0.02662	0.01836	0.01379	0.01093
2	0.1702	0.06077	0.03307	0.02172	0.01582	0.01228
3	0.2539	0.09316	0.05229	0.03526	0.02623	0.02072
4	0.3026	0.1067	0.05842	0.03873	0.02846	0.02227
5	0.3635	0.1311	0.07333	0.04949	0.0369	0.02922
6	0.4035	0.1429	0.07897	0.05282	0.03911	0.0308
7	0.4526	0.1629	0.09133	0.06186	0.04628	0.03676
8	0.4871	0.1734	0.09653	0.06501	0.04842	0.03833
9	0.5289	0.1905	0.1072	0.07287	0.0547	0.04359
10	0.5595	0.2001	0.112	0.07585	0.05676	0.04511
11	0.5962	0.2151	0.1215	0.08284	0.06238	0.04984
12	0.624	0.2239	0.126	0.08567	0.06436	0.05131
13	0.6569	0.2373	0.1345	0.09199	0.06946	0.05562
14	0.6823	0.2455	0.1387	0.09468	0.07135	0.05705
15	0.7123	0.2578	0.1464	0.1005	0.07604	0.06102
p_j 16	0.7358	0.2655	0.1505	0.103	0.07786	0.06241
17	0.7635	0.2767	0.1576	0.1084	0.0822	0.06609
18	0.7855	0.284	0.1614	0.1108	0.08395	0.06743
19	0.8111	0.2944	0.1681	0.1158	0.088	0.07088
20	0.8318	0.3013	0.1717	0.1182	0.0897	0.07218
21	0.8558	0.311	0.1779	0.1228	0.0935	0.07542
22	0.8754	0.3175	0.1814	0.1251	0.09514	0.07668
23	0.898	0.3267	0.1872	0.1295	0.09872	0.07974
24	0.9166	0.3329	0.1905	0.1317	0.1003	0.08097
25	0.938	0.3416	0.1961	0.1358	0.1037	0.08387
26	0.9557	0.3476	0.1993	0.1379	0.1052	0.08506
27	0.976	0.3558	0.2045	0.1419	0.1085	0.08783
28	0.993	0.3615	0.2076	0.1439	0.11	0.08899
29	1.012	0.3694	0.2126	0.1477	0.113	0.09163
30	1.029	0.3748	0.2156	0.1497	0.1145	0.09276
31	1.047	0.3823	0.2204	0.1533	0.1174	0.09528
32	1.063	0.3876	0.2232	0.1552	0.1188	0.09638

Table 2. The meanings of the value p_j for the first sixteen oscillations of the soliton size at several meanings of the parameter δ according to the formulae (6.19), (6.21) - (6.24).

7. QUASI-PERIODIC OSCILLATIONS OF THE SOLUTION

At $|\Lambda_{j+1} - \Lambda_j| \ll 1$ (that does not exclude the above formulae applicability) the relative variation of the soliton parameters during a period of the oscillations is small and continuous description for the envelopes is possible. In particular, the recurrent relationships (6.21) and (6.22) can be replaced by the differential equations:

$$\frac{d\Lambda}{dn} \cong 2(\pi B)^{-1} \Lambda^4 \exp(-\Lambda) F_0(p), \tag{7.1}$$

$$\frac{dS}{dn} \cong 2\Lambda^{-2} \exp(-\Lambda) F_1(p), \qquad S \cong \pi B \Lambda^{-5} p. \tag{7.2}$$

Here $n = 2j$ is the number of the soliton oscillation. The ζ-dependence is restored, if necessary, by the equation (3.6), which leads at $q \ll 1$ to:

$$\Delta\zeta \equiv \frac{d\zeta}{dn} \cong 2\frac{\Lambda}{\pi} \ln\frac{4}{q} \cong \frac{\Lambda}{\pi} \ln\frac{4\Lambda}{p}. \tag{7.3}$$

The ratio of equations (7.1) and (7.2) do not contain n and can be presented in the form

$$\frac{d\ln S}{d\ln\Lambda} \cong p^{-1} F(p), \qquad F(p) = \frac{F_1(p)}{F_0(p)}. \tag{7.4}$$

The solution of this equations,

$$\Lambda \cong \Lambda_* \lambda(p), \qquad \lambda(p) \equiv \exp\int_0^p dp' \left[5p' + F(p')\right]^{-1}, \tag{7.5}$$

is a universal function. Only the constant of integration, Λ_*, depends on the parameter δ. The meaning of Λ_* is determined to secure the agreement of the formula (3.5) with results of the previous section inside the mutual applicability region. For several δ, the meanings of Λ_* are presented in the table 3:

δ	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
Λ_*	6.792	8.532	9.551	10.23	10.73	11.12
Λ_f	16.08	21.38	24.62	26.83	28.48	29.78
a_*/δ	3.976	4.994	5.591	5.988	6.281	6.509
a_f/δ	9.412	12.52	14.41	15.71	16.67	17.43

Table 3. The basic parameters of quasi-periodic oscillations.

At given Λ_*, the formulae (7.5) and

$$\Lambda_j \cong \Lambda_* \lambda(\bar{p}_j), \qquad \bar{p}_j \equiv \frac{1}{2}(p_j + p_{j-1}) \tag{7.6}$$

restore the table 1 by the table 2 with absolute error less than 0.1 for all j, if $\delta \le 10^{-4}$, for $j \ge 2$ at $\delta = 0.001$ and for $j \ge 5$ at $\delta = 0.01$.

The extreme sizes of the soliton are also universal functions of p:

$$a_m \cong a_* \lambda(p), \qquad a_* \equiv \delta \frac{\Lambda_*}{\pi} \left(\frac{N_c}{M}\right)^{1/2}, \tag{7.7}$$

$$a_M \cong \frac{1}{2}\left(\frac{\Lambda}{p}\right)^{1/2} a_m \cong \frac{1}{2}\Lambda_*^{1/2} a_* \lambda(p)^{3/2} p^{-1/2}.$$

The dependence on n is restored by the integration of (7.1):

$$n \cong \frac{1}{2}\pi B \int_0^p dp'\left[5p' F_0(p') + F_1(p')\right]^{-1} \Lambda(p')^{-3} exp\left[\Lambda(p')\right]. \qquad (7.8)$$

At $p \gg 1$, when

$$F_0(p) \cong 2F_1(p) \cong \left(\frac{\pi}{8p}\right)^{1/2}, \qquad F(p) \cong \frac{1}{2}, \qquad (7.9)$$

the formula (7.5) is reduced to

$$\lambda(p) \cong C\ (10\ p + 1)^{1/5}, \qquad (7.10)$$

where

$$C \equiv exp \int_0^\infty dp \left\{\left[5p + F(p)\right]^{-1} - \left(5p + \frac{1}{2}\right)^{-1}\right\} \cong 1.131. \qquad (7.11)$$

The relative error of such a reduction appear to be less than 3% at $p \geq 0.1$ and quickly decreases at larger p, reaching the meanings 0.5% at $p = 0.5$ and 0.1% at $p = 1$. The accuracy slightly worsen when the unity in the right hand side of (7.10) is neglected. Then the formula (7.10) turns to

$$p \cong 0.1\ C^{-5}\lambda^5. \qquad (7.12)$$

As $p \cong \Lambda q^2/4$, the relationship (7.12) can be rewritten in the form

$$\Lambda \cong \Lambda_f\ q^{1/2}, \qquad \Lambda_f = \left(\frac{5}{2} C^5 \Lambda_*^5\right)^{1/4}. \qquad (7.13)$$

Substitution of (7.13) to (7.7) gives:

$$a_m \cong a_f\ q^{1/2}, \qquad a_M \cong a_f\ q^{-1/2}, \qquad a_f \equiv \delta\,\frac{\Lambda_f}{\pi}\left(\frac{N_c}{M}\right)^{1/2}. \qquad (7.14)$$

The expression (7.8) for n is also simplified:

$$n \cong \left(\frac{\pi}{2}\right)^{1/2} B\ \Lambda_f^{-2} \int_{\Lambda_*}^{\Lambda} dl\ l^{-3/2} exp\ l. \qquad (7.15)$$

The relationship (7.12) implies that the value S (see (7.1)) and the soliton Hamiltonian, H_S, (which differs from S by a constant factor

only) are invariable at $p \gg 1$. The conclusion on the soliton Hamiltonian invariance remains valid even when the condition $q^2 \ll 1$, meant above, is broken. In fact, general formulae (4.21) and (4.11) give the following values for the quanta and Hamiltonian losses by the soliton per period:

$$-\frac{d\tilde{N}}{dn} \cong \Lambda J_0(\Lambda) F_0(p)/K(q), \qquad -\frac{dH_S}{dn} \cong \frac{\Lambda J_0(\Lambda) F_1(p)}{2\ K(q)\ p\ a_M^2}, \qquad (7.16)$$

$$p \equiv \frac{1}{2}\Lambda\left[1 - \frac{E(q)}{K(q)}\right]. \qquad (7.17)$$

The ratio of equations (7.16) can be presented in the form

$$-\frac{d ln|H_S|}{d ln\tilde{N}} \cong \frac{(1 + q^2)F(p)}{2\ p}. \qquad (7.18)$$

At $p \gg 1$ the right hand side of (7.18) is much smaller than unity. This implies that H_S varies much slower than $\tilde{N}$ and can be treated as invariable value. At $q^2 \ll 1$ equations (7.16) and (7.18) are reduced to (7.1), (7.2) and (7.4).

As the soliton contribution to the Hamiltonian remains at $p \gg 1$ invariable and equal to its final value $-H_f$,

$$-H_S \cong H_f \equiv \frac{\pi^2 M}{\Lambda_f^2 a_f^2} = \frac{\delta^2 N_c}{a_f^4}, \qquad (7.19)$$

all other parameters of the soliton oscillations can be expressed in terms of one of them, say q. The relevant expressions for extreme soliton sizes are given by the the formulae (7.14) again, while formula (7.13) for Λ is replaced by

$$\Lambda \cong \frac{2}{\pi} K(q)\ q^{1/2}\ \Lambda_f. \qquad (7.20)$$

The expression for the value $\tilde{N}$ looks as:

$$\tilde{N} \cong \tilde{N}_f\left(q + q^{-1}\right), \qquad \tilde{N}_f \equiv \frac{\delta^2 N_c}{a_f^2} = \frac{\pi^2 M}{\Lambda_f^2}. \qquad (7.21)$$

The n-dependence is restored by the integration of the first equation (7.16), that results in:

$$n \cong \left(\frac{\pi}{2}\right)^{1/2} \frac{B}{\Lambda_f^3} \int_0^q dq_1 \left(q_1^{-2} - 1\right) p(q_1)^{1/2}\ q_1^{-1/2} exp\left[\Lambda(q_1)\right]. \qquad (7.22)$$

At $q \ll 1$ this formula is reduced to (7.15).

At $q' \ll 1$, i.e. for nearly homogeneous waveguide of mean width a_f, another simplification of the above formulae is possible:

$$p \cong \frac{1}{2}\Lambda \cong \frac{\Lambda_f}{\pi} \ln\frac{4}{q'}, \qquad n \cong \frac{64\pi^{3/2}B}{\Lambda_f^4}\int^{\Lambda} d\Lambda\, \Lambda^{1/2} exp\left[\Lambda\left(1 - \frac{2\pi}{\Lambda_f}\right)\right]. \qquad (7.23)$$

8. CONCLUDING REMARKS

The picture of stationary self-focusing of radiation in the medium with soft saturation of nonlinearity can be summarized by the following way.

In the vicinity of the medium boundary strong flow breaks up into a number of channels. Each of the channels traps the power some higher the critical one. After compression about five times in linear size a channel becomes axisymmetric and nearly the same in shape as the Townes soliton. Its behavior becomes adiabatic. An excess of the trapped power over the critical one is irradiated basically. Thus a universal dynamic state is established. All farther behavior of a channel is described quantitatively by the above theory. Conformably to the first cycle of the compression, this description improves the asymptotic theory of the critical collapse developed earlier, which is not applicable quantitatively even after the channel compression by many millions times.

Since the defocusing nonlinearity has stopped the compression, the width of the waveguide oscillates. Initially, a sweep of the oscillations is large. It decreases as far as the channel loses trapped power. Because of the adiabaticity of the oscillations, the losses and decrease are exponentially small in terms of the trapped power excess. First, the ratio of extreme widths of the channel behaves, roughly speaking, as the square root of the number of the oscillation. Then, the law is changed for the logarithmic one. Finally, the homogeneous waveguide is established.

The above description meant implicitly that the Townes soliton has no excited bound states, existence of which could drastically change the dynamics of power losses. This property of the soliton was proved in the paper [24].

Stability of the given solution in the framework the two-dimensional nonlinear Schroedinger equation follows from the analytical results of papers [10,24] and numerical results of paper [25]. More exactly, one should speak here about stability of the whole set of the above solutions, since even homogeneous waveguides may differ in width and move in transverse directions with arbitrary constant velocities. The problem of structural stability for stationary self-focusing of radiation is studied in [26].

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