14

ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ
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PREPRINT 92-17

НОВОСИБИРСК

WAKE POTENTIALS OF DIELECTRIC CANAL

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ABSTRACT

The longitudinal and transverse wake fields induced by a charge moving inside cylindrical dielectric canal are obtained analytically. Simple formulas for wake potentials are presented.

1. INTRODUCTION

A charge moving along a dielectric tube awakes Cherenkov fields. The problem of calculation of such fields and connected with them forces is reduced to obtaining the longitudinal electrical field E. The transverse force $\vec{F}_\perp$ is determined by well known Panofski-Wentzel formula:

$$\vec{F}_\perp = - \frac{d}{d\vec{r}_\perp} \int E(z')\, dz' . \qquad (1)$$

The fields are calculated further by means of Fourier transformation method.

2. LONGITUDINAL WAKE POTENTIAL

For the beginning let us derive the fields induced by a point charge travelling along the axis of cylindrically symmetric dielectric canal, presented in Fig. 1. The outer surface of the canal is covered by ideal metal. We assume that the bunch velocity is equal to that of light.

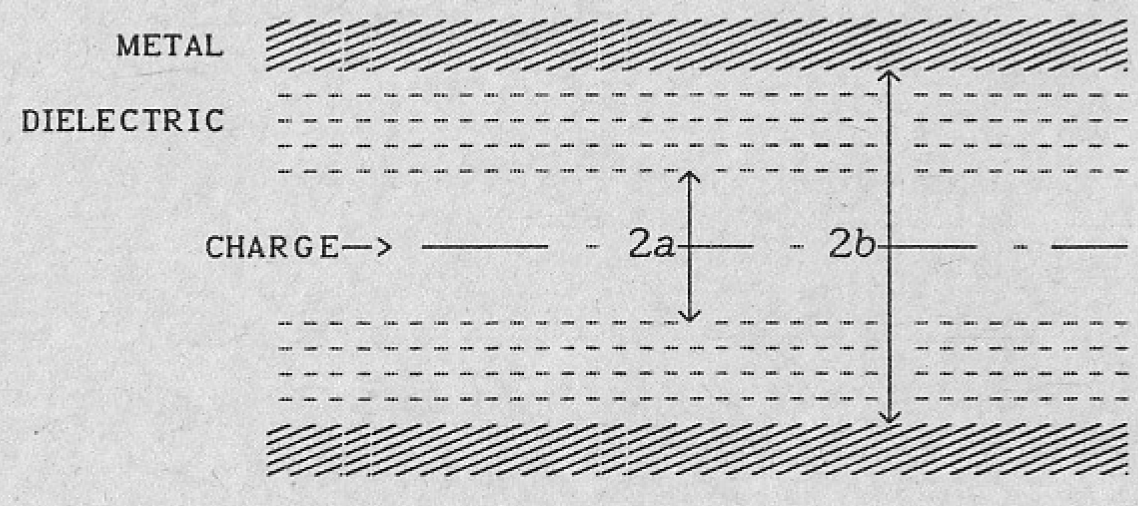

Figure 1. Dielectric canal.

Using Maxwell equations and boundary conditions one can find Fourier component for longitudinal electric field on the axis of a tube (in vacuum) [1]:

$$E(k) = \frac{2i\sqrt{\varepsilon - 1}}{a\ \varepsilon} \cdot \frac{1}{\frac{Z_1}{Z_0} - \frac{\text{æ}a}{2\varepsilon}}\ , \qquad (2)$$

where $\text{æ} = k\sqrt{\varepsilon - 1}$.

Functions Z_0 and Z_1 can be described by Neiman (N_0, N_1) and Bessel (J_0, J_1) functions, so that

$$\frac{Z_1}{Z_0} = \frac{N_1(\text{æ}a)J_0(\text{æ}b) - N_0(\text{æ}b)J_1(\text{æ}a)}{N_0(\text{æ}a)J_0(\text{æ}b) - N_0(\text{æ}b)J_0(\text{æ}a)}\ . \qquad (3)$$

When $\text{æ}a \gg 1$ the asymptotic expression for (3) is given by

$$\frac{Z_1}{Z_0} = \mathrm{ctg}(\text{æ}(b-a))\ . \qquad (4)$$

The field $E(z)$ is calculated by reverse Fourier transformation of (2):

$$E(z) = \int E(k)\, e^{-ikz} \frac{dk}{2\pi}\ .$$

The sign of the distance z is determined so that behind the charge $z > 0$.

The contour of integration is lying above all singularities of $E(k)$, hence, the causality principle is provided:

at $z < 0$

$$E(z) = 0\ .$$

When moving the contour of integration to lower half-plane (at $z > 0$) it catches the poles, lying on the real axis, where

$$\frac{Z_1}{Z_0} = \frac{\text{æ}a}{2\varepsilon}\ .$$

The number of poles is infinite and all of them are simple and located on the real axis. The poles that give dominated contribution into integral are concentrated in the region

$$\frac{|\text{æ}a|}{2\varepsilon} \lesssim 1\ .$$

Suppose that

$$2\varepsilon \gg 1\ , \qquad (5)$$

one can use asymptotic expression for Bessel functions (4). The derivative of the denominator in the pole is

$$\frac{d}{dæ}\left(\operatorname{ctg}(æ(b-a)) - \frac{æa}{2\varepsilon} \right) = -(b-a)\left(1+\left(\frac{æa}{2\varepsilon}\right)^2\right) - \frac{a}{2\varepsilon} . \quad (6)$$

At the distances small enough, when

$$\Delta kz = \frac{\pi z}{(b-a)\sqrt{\varepsilon - 1}} \ll 1 , \quad (7)$$

the phases of neighboring addenda differ not too much and the sum of series can be changed by integral. If the dielectric layer is sufficiently thick

$$b - a \gg \frac{a}{2\varepsilon} , \quad (8)$$

the result of integration does not depend on outer radius b and can be presented by the next formula [1]:

$$E(z) = - \frac{4}{a^2} e^{-z/s} , \quad (9)$$

where s is the effective length

$$s = \frac{a\sqrt{\varepsilon - 1}}{2\varepsilon} . \quad (10)$$

One can see that there is some contradiction between last obtained formula (9) and initial expression for the field Fourier component (2). Namely, the formula (9) gives nonzero integral above the infinite z-axis, that does not correspond to condition

$$\int_0^\infty dz\, E(z) = E(k)\big|_{k=0} = 0 . \quad (11)$$

Consequently, calculation of wakefield for distances more than effective length s demands taking into account second approximation in the expansion of (2) by means of small parameter $1/(2\varepsilon)$.

It is possible, however, to come to required result from another side. For the sufficiently thick dielectric layer (7, 8) the value of its outer radius b does not matter, hence, in such case the field Fourier component may be taken from calculations when b is infinite. This Fourier component differs from (2) only by change of Bessel functions: instead of vanishing on outer radius linear combination of converging and diverging cylindrical waves (function Z) one have to take the diverging wave only (Hankel function $H^{(1)}$). Therefore, in such formulation of the problem

$$E(k) = \frac{2i\sqrt{\varepsilon - 1}}{2\varepsilon} \; \frac{1}{\dfrac{H_1^{(1)}(æa)}{H_0^{(1)}(æa)} - \dfrac{æa}{2\varepsilon}} . \quad (12)$$

$$E(z) = \int E_k\, e^{-ikz} \frac{dk}{2\pi} = \frac{2i}{\varepsilon a^2} \int \frac{e^{-iq\zeta}}{\dfrac{H_1(q)}{H_0(q)} - \dfrac{q}{2\varepsilon}} \frac{dq}{2\pi} . \quad (13)$$

$$\zeta = \frac{z}{a\sqrt{\varepsilon - 1}} . \tag{14}$$

When contour of integration (13) is shifting down to the lower half-plane of complex q it catches there not the infinite sequence of poles, but the cut along negative part of imaginary axis and the only pole q_p which is situated there too (Fig. 2):

$$q_p \approx - 2i\varepsilon . \tag{15}$$

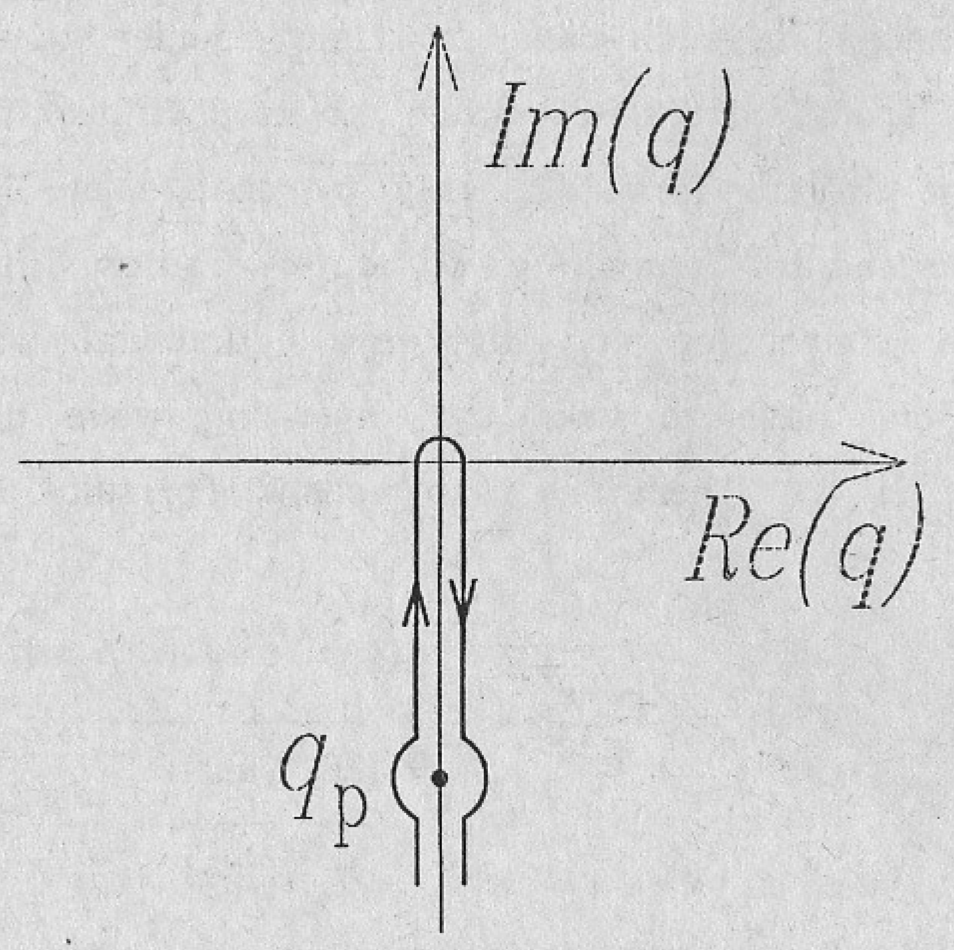

Fig. 2. Contour of integration.

Thus the integral (13) is presented as a sum of two contributions

$$E(z) = E_p(z) + E_c(z) .$$

Let us consider the pointed contributions to integral (13) separately.

As concerning pole contribution one can see that an asymptotic expansion of Hankel functions for the large value of argument here permits to calculate it to an arbitrary precision. The first approximation gives result presented above (9). Taking into account the second member of the expansion permits to obtain more precise value for the pole contribution in the integral (13) $E_p(z)$:

$$E_p(z) = - \frac{4}{a^2} (1 + \frac{1}{4\varepsilon}) e^{-z/s_0} . \tag{16}$$

$$s_0 = s (1 + \frac{1}{4\varepsilon}) = \frac{a\sqrt{\varepsilon - 1}}{2\varepsilon} (1 + \frac{1}{4\ \varepsilon}) . \tag{17}$$

The cut contribution, as it can be seen from (13), has in the comparison with the pole one an additional multiplier $1/2\varepsilon$. This is the reason for the sufficiency of the cut contribution calculation only in the first approximation in terms of this small parameter. In order to do it one can take into account that the values of integrated expression at opposite edges of the cut differ significantly only at $q \lesssim 1$ - in another case the difference becomes exponentially small. The first consequence from this circumstance is the sufficiency of taking into account only Hankel function quotient in the integrated expression denominator, without small term $q/2\varepsilon$. The second one is the cut contribution independence on the distance ζ at $\zeta \ll 1$. This contribution can be noticed to be connected with the second approximation of the pole contribution: the sum of them gives the corresponding correction to the wake field jump at $z = 0$. This jump may be easily obtained with absolute precision from the field Fourier component $E(k)$ behavior at $k \to \infty$. The result (9) as one can be convinced is in the exact accordance with this asymptotic behavior, therefore the cut contribution E_c at

$$\zeta \ll 1 ,$$

is:

$$E_c = \frac{1}{\varepsilon a^2} . \qquad (18)$$

In the opposite case, when $\zeta \gg 1$, the integral along the cut is mainly concentrated **in** the region of more smaller arguments $|q| \lesssim \frac{1}{\zeta} \ll 1$ what allows one to use Bessel function expansion at small values of argument. After this procedure the integration becomes quite simple and gives the following result at:

$$z \gg a\sqrt{\varepsilon - 1} \qquad (\text{ i.e. } \zeta \gg 1) ,$$

$$E_c(z) = \frac{2\ (\varepsilon - 1)}{\varepsilon z^2} , \qquad (19)$$

Limiting cases expressions (18, 19) permits one to describe the cut contribution by some approximate formula which is valid with rather good accuracy for all distances and exactly gives obtained expressions (18, 19) in corresponding limiting cases. The simplest formula seems to be:

$$E_c(z) = \frac{1}{a^2\varepsilon\ (1 + \frac{\zeta^2}{2})} \qquad (20)$$

with the corresponding result for the whole longitudinal wake field (13):

$$E(z) = -\frac{4}{a^2}\left\{\left(1+\frac{1}{4\varepsilon}\right) e^{-z/s_0} - \frac{1}{4\varepsilon\left[1 + \frac{z^2}{2a^2\ (\varepsilon-1)}\right]}\right\} . \qquad (21)$$

For obtaining the accuracy of the approximation (20, 21) in the intermediate case $\zeta \simeq 1$ one can use the known value of the total integral (11), what means the mutual annihilation of integrals from pole and cut contributions. The precision of such annihilation is the measure of the obtained formula (21) accuracy at $\zeta \simeq 1$ where the main part of the integral over distance from the cut contribution is concentrated. Denoting the corresponding contributions in (11) as $\int_p$ and $\int_c$ one can obtain:

$$\frac{\int_c - \int_p}{\int_p} = \frac{\pi}{2\sqrt{2}} - 1 \approx 0.11 .$$

Therefore the accuracy is about 11%.

The analytic expression for the longitudinal wake field potential in a dielectric canal (21) was compared with the results of the numerical simulations for this electrodynamical problem. The accuracy of the correspondence is illustrated in Fig. 3. The solid line demonstrates the numerical result, dotted - analytical one. The narrow bell corresponds to the bunch density.

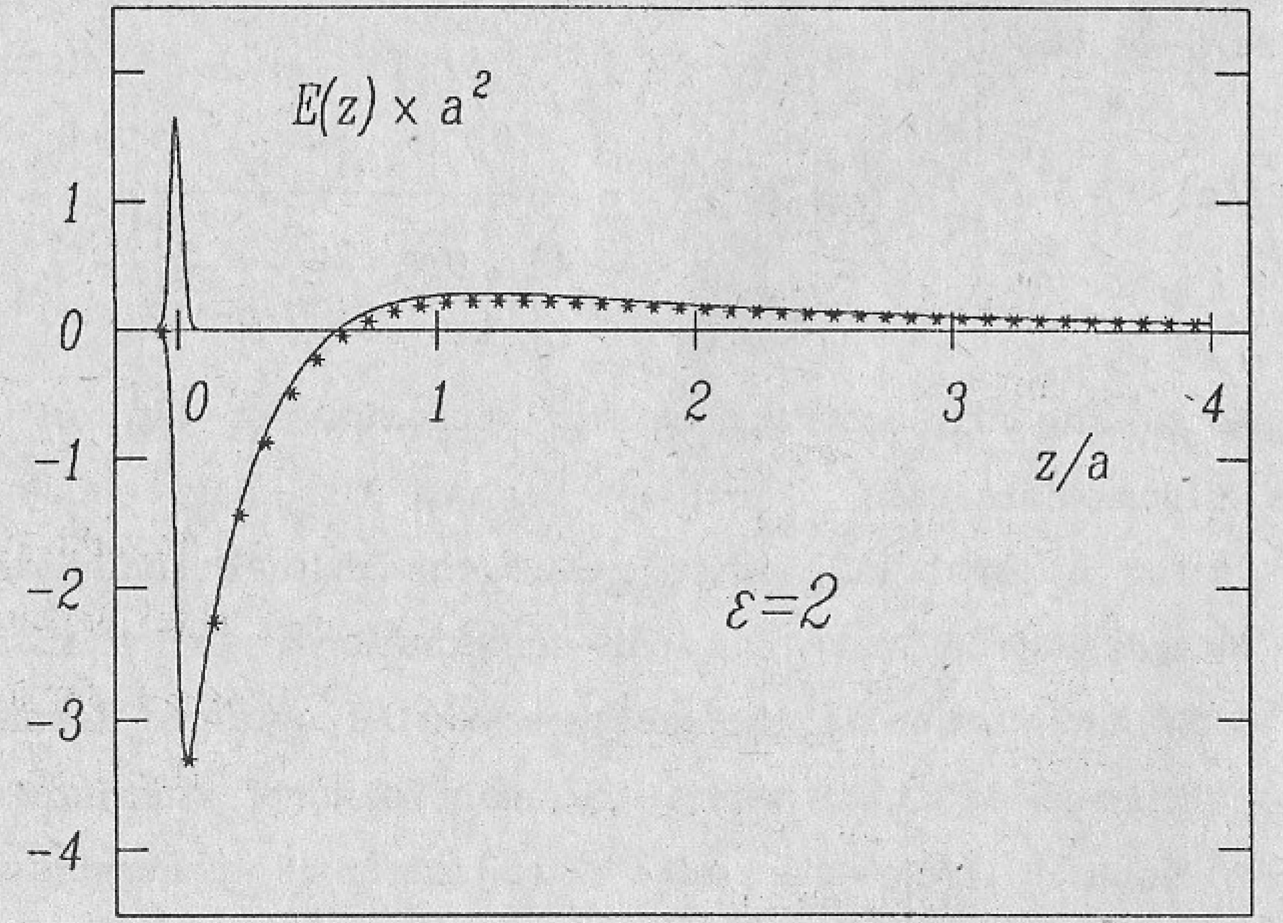

Fig. 3. Longitudinal wake field.

The energy losses of a bunch with density $\rho(z)$ after passing the dielectric canal of length L is

$$\Delta E = \int dz\, dz' \rho(z)\rho(z')\, W(z-z') ,$$

where

$$W(z) = E(z)\, L ,$$

is the canal wake potential. Supposing a bunch density to be Gaussian:

$$\rho(z) = N \frac{\exp\left(-\frac{z^2}{2\sigma^2}\right)}{\sqrt{2\pi}\,\sigma} ,$$

one may calculate analytically its energy losses in some limiting cases.

If the bunch is shorter then characteristic length s,

$$\sigma \ll s ,$$

then

$$\Delta E = \frac{2LN^2e^2}{a^2} .$$

In another case with

$$s \ll \sigma \ll a\sqrt{\varepsilon - 1} ,$$

one can obtain

$$\Delta E = \frac{2LN^2e^2}{a^2} \frac{s}{\sqrt{\pi}\sigma} .$$

In the conclusion of this chapter let us say some words about the field behavior at very long distances

$$z > (b - a)\sqrt{\varepsilon} .$$

For such consideration one have to return to the initial field presentation as a sum of contributions from infinite series of poles. In our case when $2\varepsilon \gg 1$ poles are situated almost equidistantly: the contribution of high harmonics with $\ae a \gg 2\varepsilon$, where the equidistance is broken, is suppressed as $\left(\frac{\ae a}{2\varepsilon} \right)^2$, that can be seen from (6). Such approximate equidistance means the approximate periodicity of the field with the period λ_0 equal to wavelength of the lowest harmonic

$$\lambda_0 \approx 4\ (b - a)\sqrt{\varepsilon}\ .$$

The violation of equidistance corresponds to super slow oscillations in the element of field periodicity; the more the harmonic number, the faster oscillations. The field behavior is illustrated in Fig. 4, where electric field lines are pictured for two values of dielectric constants.

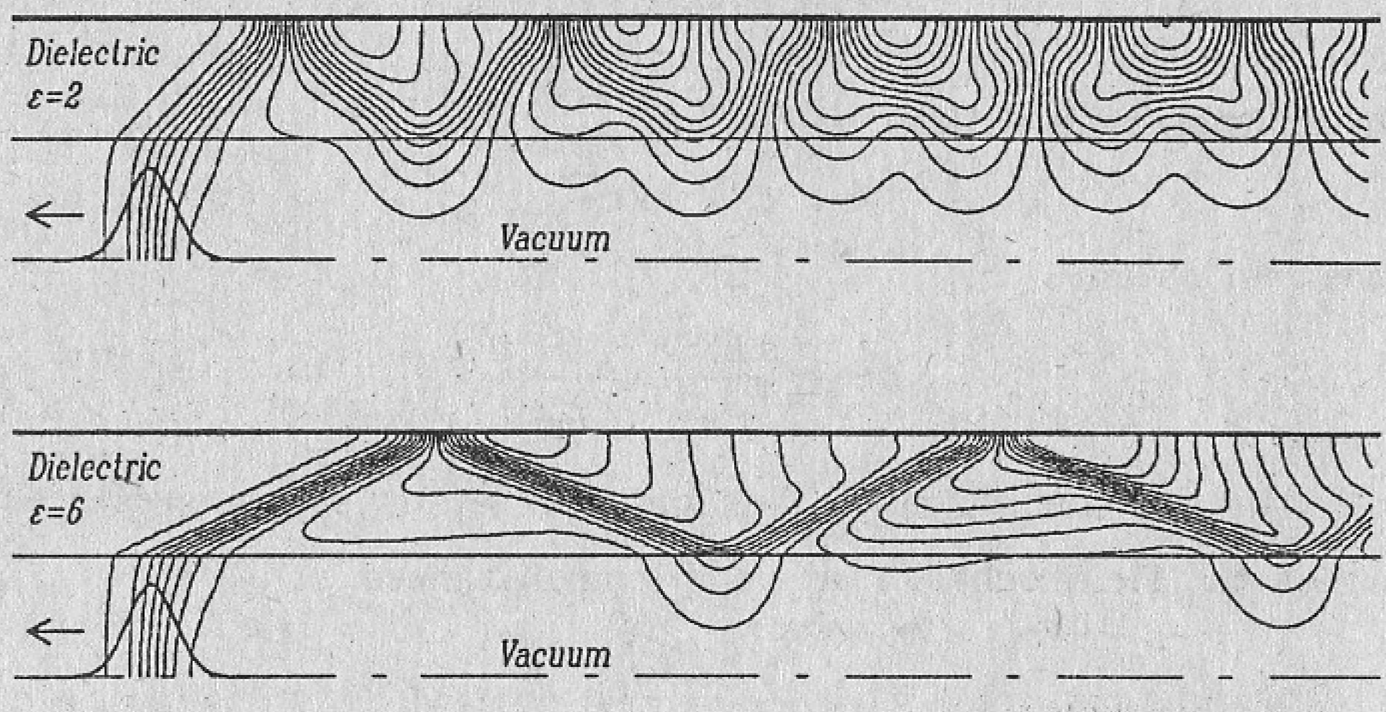

Fig. 4 Electric field lines for different dielectric constants.

3. TRANSVERSE WAKE POTENTIAL

In agreement with Panofski-Wentzel formula (1) the problem of transverse wake force calculation is reduced to obtaining the longitudinal electric field E_m. The calculations for some multipolarity m are carried out in the close analogue with the axially symmetrical case considered above and the results are:

Fourier component of longitudinal field E is

$$E_m(k) = \frac{4iI_m}{a^{m+1}} \frac{\sqrt{\varepsilon-1}}{\varepsilon+1} \frac{\left(\frac{r}{a}\right)^m}{\frac{H_{m+1}(\ae a)}{H_m(\ae a)} - \frac{\ae a}{(m+1)(\varepsilon+1)}}, \qquad (22)$$

where I_m is the multipole moment of the charge

$$I_m = \int \rho(\vec{r})\ r^m \cos(m\vartheta)\ d\vec{r}\ .$$

Pole contribution to longitudinal wake field is

$$E_{mp}(z) = -\frac{4I_m(m+1)}{a^{m+2}} \left[1 + \frac{2m+1}{2(m+1)(\varepsilon+1)}\right] e^{-z/s_m} \left(\frac{r}{a}\right)^m . \qquad (23)$$

$$s_m = \frac{a\sqrt{\varepsilon - 1}}{(m+1)(\varepsilon+1)} \left[1 + \frac{2m+1}{2(m+1)(\varepsilon+1)}\right] . \qquad (24)$$

Cut contribution for $\zeta = \frac{z}{a\sqrt{\varepsilon - 1}} \ll 1$ is

$$E_{mC}(z) = -\frac{4I_m(m+1)}{a^{m+2}} \frac{2m+1}{2(m+1)(\varepsilon+1)} \left(\frac{r}{a}\right)^m , \qquad (25)$$

and for $\zeta \gg 1$

$$E_{mC}(\zeta) = (-1)^m \frac{4I_m}{a^{m+2}\,\zeta^{2m+2}\,(\varepsilon+1)} \frac{(2m+1)!}{2^{2m}\, m!\, m!} \left(\frac{r}{a}\right)^m . \qquad (26)$$

Complete wake field (like for m=0) is the sum of contributions:

$$E_m = E_{mp} + E_{mC} .$$

Using (1) one can obtain, in the particular, the expression for the dipole transverse wake force F_1 (m=1):

for $\zeta = \frac{z}{a\sqrt{\varepsilon - 1}} \ll 1$

$$F_1(z) = \frac{8I_1}{a^4} \left[\left(1 + \frac{3}{4(\varepsilon+1)}\right) \left(1 - e^{-z/s_1}\right) s_1 - \frac{3z}{4(\varepsilon+1)}\right], \qquad (27)$$

and if $z \ll s_1$

$$F_1(z) = \frac{8I_1 z}{a^4} , \qquad (28)$$

and for $\zeta \gg 1$

$$F_1 = -\frac{2I_1(\varepsilon-1)^2}{z^3(\varepsilon+1)} . \qquad (29)$$

The comparison between analytical solution (27) and numerical results for effective gradient of dipole wake potential

$$G = F_1 / I_1 ,$$

is presented in Fig. 5, where solid lines demonstrate numerical results and dotted lines demostrate analytical ones for two values of dielectric constants: ε=6 (upper line) and ε=2 (lower line). The narrow bell corresponds to the bunch density.

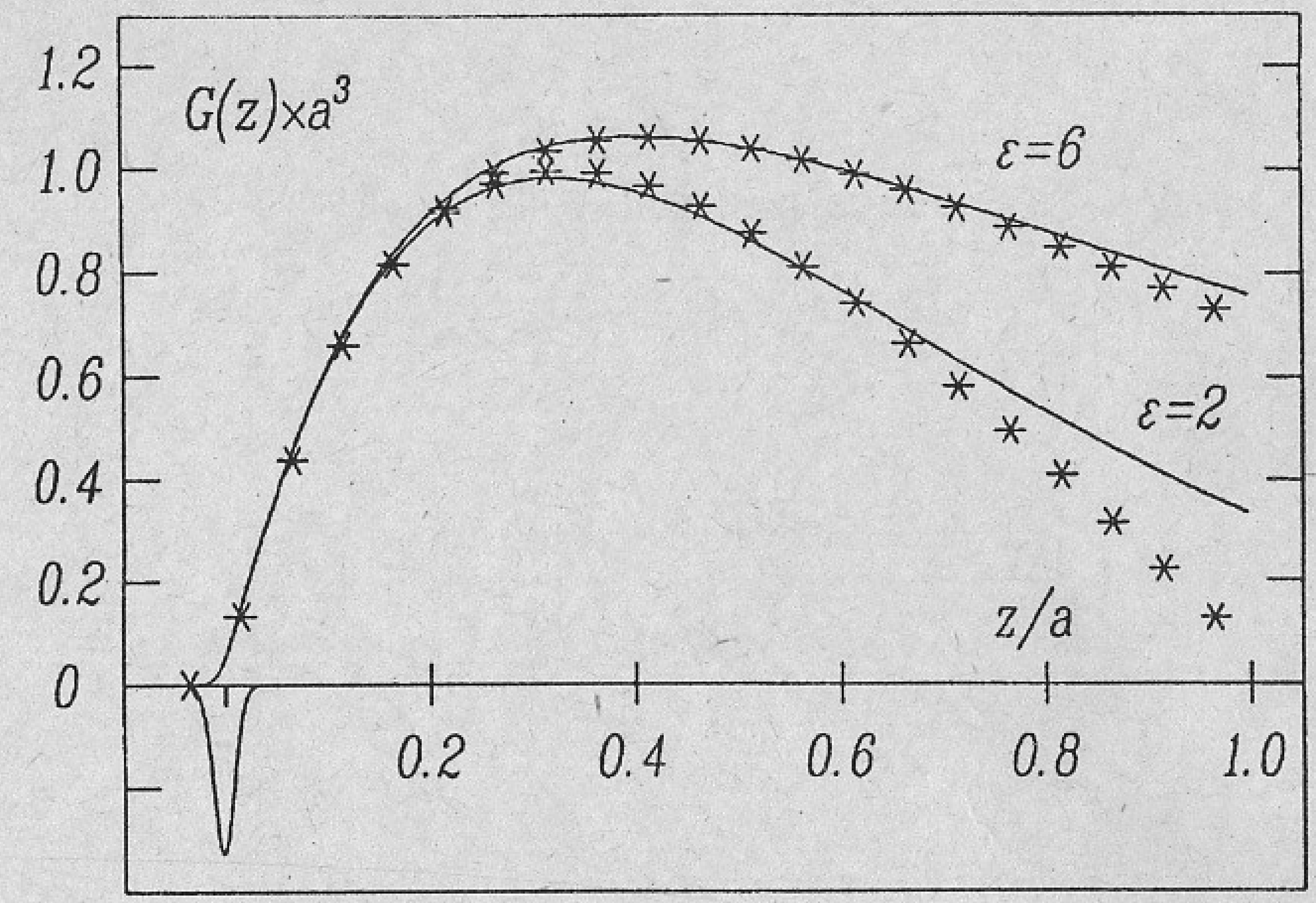

Fig. 5. Transverse wake field.

Acknowledgements. The authors would like to thank N.S. Dikansky for the interest to this work and also A.A. Kulakov for the help in computer simulations.

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Кильватерные потенциалы диэлектрического канала

Ответственный за выпуск С.Г. Попов

Работа поступила 10 марта 1992 г.
Подписано в печать 10.03.1992 г.
Формат бумаги 60×90 1/16 Объем 1,0 печ.л., 0,9 уч.-изд.л.
Тираж 200 экз. Бесплатно. Заказ N 17

Обработано на IBM PC и отпечатано на
ротапринте ИЯФ им. Г.И. Будкера СО РАН,
Новосибирск, 630090, пр. академика Лаврентьева, 11.