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THE 2+1-DIMENSIONAL INTEGRABLE GENERALIZATION OF THE SINE-GORDON EQUATION.
I. $\bar{\partial}$-∂-DRESSING AND INITIAL VALUE PROBLEM

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The 2+1-Dimensional Integrable Generalization of the Sine-Gordon Equation. I. $\bar{\partial}$-∂-Dressing and Initial Value Problem

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ABSTRACT

The 2+1-dimensional integrable generalization of the sine-Gordon equation symmetric in the spatial variables is studied by the inverse spectral transform method. The solutions with functional parameters, plane solitons (kinks) and plane breathers are constructed by the dressing method based on the mixed nonlocal $\bar{\partial}$-∂-problem. The initial value problem for this equation with the constant boundaries is solved in both cases $\sigma^2=\pm1$.

1. INTRODUCTION

The sine-Gordon (SG) equation $\theta_{t\xi}= m\cdot\sin\theta$ is one of the basic example of the 1+1-dimensional nonlinear differential equations integrable by the inverse scattering transform (IST) method (see e.g. [1 - 3]). This equation has been studied in great details. The multidimensional version $\Box\theta=m\cdot\sin\,\theta$ of the sine-Gordon equation seems is not a subject of the IST method. Different 2+1-dimensional and multi-dimensional IST integrable generalizations of the SG equation have been proposed and studied in [4 - 6]. But these equations either contain the spatial variables x and y in a very unsymmetric manner [6] or include several dependent variables [4, 5].

A simple and symmetric generalization of the SG equation has been found recently in [7]. It is of the form

$$\left(e^{i(\varphi+\tilde\varphi)}\varphi_{tx}\right)_x - \sigma^2\left(e^{i(\varphi+\tilde\varphi)}\varphi_{ty}\right)_y = 0,$$
$$\left(e^{-i(\varphi+\tilde\varphi)}\tilde\varphi_{tx}\right)_x - \sigma^2\left(e^{-i(\varphi+\tilde\varphi)}\tilde\varphi_{ty}\right)_y = 0, \qquad (1.1)$$

where $\varphi(x, y, t)$, $\tilde\varphi(x, y, t)$ are scalar functions and $\sigma^2=\pm1$. In the one-dimensional limit (i.e. $\varphi_y = \tilde\varphi_y = 0$) the integration of equations (1.1) give rises to the following equation for $\theta = \varphi + \tilde\varphi$

$$\theta_{tx} = n_1(t)\cdot e^{i\theta} + n_2(t)\cdot e^{-i\theta}, \qquad (1.2)$$

which in the particular case $n_1 = -n_2 = \frac{1}{2i}m = \text{const}$ coincides with the SG equation. Equation (1.1) has a number of interesting properties. In the paper [7] equation (1.1) has been derived as the special case of the wide class of the 2+1-dimensional integrable systems. The applicability of the IST method to this general class has been discussed in [8].

The aim of the present paper is to study by the IST method the system (1.1) with constant boundary values of $(\varphi - \tilde\varphi)_{tx}$ and $(\varphi - \tilde\varphi)_{ty}$ or, equivalently, the equation

$$\theta_{t\xi\eta} + m_1\cdot\theta_\eta + m_2\,\theta_\xi + \frac{1}{4}\cdot\theta_\eta\cdot\int_{-\infty}^{\eta} d\eta'(\theta_\xi\cdot\theta_{\eta'})_t + \frac{1}{4}\theta_\xi\int_{-\infty}^{\xi} d\xi'(\theta_{\xi'}\cdot\theta_\eta)_t = 0, \qquad (1.3)$$

where $\theta = \varphi + \tilde\varphi$, $\xi = x + \sigma y$, $\eta = x - \sigma y$ ($\sigma^2=\pm1$) and m_1, m_2 are arbitrary constants. A new auxiliary linear system for equation (1.3) is found. The derivation of equation (1.3) is given within the framework of the dressing method based on the nonlocal mixed $\bar\partial$-∂-problem

$$\mathcal{D}_{\lambda,\bar\lambda}\,\chi = \iint \frac{d\lambda'\wedge d\bar\lambda'}{2\pi i}\,\chi(\lambda', \bar\lambda')\,R(\lambda',\bar\lambda'; \lambda, \bar\lambda), \qquad (1.4)$$

where

$$\mathcal{D}_{\lambda,\bar\lambda}\,\chi \equiv \begin{pmatrix} \partial\chi_{11}/\partial\bar\lambda, & \partial\chi_{12}/\partial\lambda \\ \partial\chi_{21}/\partial\bar\lambda, & \partial\chi_{22}/\partial\lambda \end{pmatrix}.$$

Using this $\bar\partial$-∂-dressing method we construct a wide class of exact solutions of the 2DGSG equation (1.3). It includes the solutions with functional parameters, plane solitons and plane breathers.

In the present paper we solve the initial value problem for the 2DGSG equation (1.3) for the class of solutions which tend to the asymptotic value $2\pi n$ (n is integer) rapidly enough and for arbitrary m_1 and m_2. The modified auxiliary linear system is found for the 2DGSG equation with arbitrary boundaries. The equations of the inverse problem are generated by the nonlocal Riemann-Hilbert problem at $\sigma^2=1$ and by the $\bar{\partial}$-equation at $\sigma^2 = -1$ respectively.

The paper is organized as follows. In section 2 the different forms of the 2DGSG equation and auxiliary linear problems are presented. The $\bar{\partial}$-∂-dressing is considered in section 3. Exact solutions of the 2DGSG equation are presented in section 4. The initial value problems in the cases $\sigma^2 = 1$ and $\sigma^2 = -1$ are considered in sections 5 and 6 respectively.

2. EQUIVALENT FORMS OF THE 2DGSG EQUATION AND LINEAR PROBLEMS

The original form of the 2+1-dimensional integrable generalization of the sine-Gordon (2DGSG) equation is the following [8]:

$$\theta_t = \Phi_1 - \Phi_2 ,$$

$$\left(e^{i\theta}\, \Phi_{1x}\right)_x - \sigma^2\left(e^{i\theta}\Phi_{1y}\right)_y = 0 , \qquad (2.1)$$

$$\left(e^{-i\theta}\, \Phi_{2x}\right)_x - \sigma^2\left(e^{-i\theta}\Phi_{2y}\right)_y = 0.$$

Introducing the variables φ and $\tilde{\varphi}$ by $\varphi_t = \Phi_1, \tilde{\varphi}_t = -\Phi_2$, one gets the system (1.1). The system (1.1) is equivalent to the compatibility condition for the linear system [7, 8]:

$$\Psi_x - \sigma\cdot\begin{pmatrix} 0, & e^{-i(\varphi+\tilde{\varphi})} \\ e^{i(\varphi+\tilde{\varphi})}, & 0 \end{pmatrix}\cdot\Psi_y = 0 ,$$

$$\Psi_{ty} + \frac{i}{2}\begin{pmatrix} \varphi_t + \tilde{\varphi}_t, & 0 \\ 0, & -\varphi_t - \tilde{\varphi}_t \end{pmatrix}\Psi_y + \frac{i}{2}\begin{pmatrix} \varphi_{ty}, & \frac{1}{\sigma}e^{-i(\varphi+\tilde{\varphi})}\cdot\tilde{\varphi}_{tx} \\ \frac{1}{\sigma}e^{-i(\varphi+\tilde{\varphi})}\varphi_{tx}, & -\tilde{\varphi}_{ty} \end{pmatrix}\Psi = 0, \qquad (2.2)$$

$$\Psi_{tx} + \frac{i}{2}\begin{pmatrix} \varphi_t + \tilde{\varphi}_t, & 0 \\ 0, & -\varphi_t - \tilde{\varphi}_t \end{pmatrix}\cdot\Psi_x + \frac{i}{2}\begin{pmatrix} \varphi_{tx}, & e^{-i(\varphi+\tilde{\varphi})}\cdot\tilde{\varphi}_{ty} \\ e^{-i(\varphi+\tilde{\varphi})}\cdot\varphi_{ty}, & -\tilde{\varphi}_{tx} \end{pmatrix}\cdot\Psi = 0.$$

One of the important feature of the 2DGSG equation (1.1) is that it is invariant under the rotation in (x, y) plane, namely, under the hyperbolic rotations in the case $\sigma^2=1$ and under usual rotations at $\sigma^2 = -1$.

In the terms of the variables $\theta \equiv \varphi + \tilde{\varphi}$, $\tilde{\theta} \equiv \varphi - \tilde{\varphi}$ and $\xi = x + \sigma y, \eta = x - \sigma y$ the 2DGSG equation looks like

$$\theta_{t\xi\eta} + \frac{1}{2}\theta_\eta\cdot\tilde{\theta}_{\xi t} + \frac{1}{2}\theta_\xi\cdot\tilde{\theta}_{\eta t} = 0 ,$$

$$\tilde{\theta}_{\xi\eta} - \frac{1}{2}\theta_\xi\cdot\theta_\eta = \alpha(\xi, \eta) ,$$

where $\alpha(\xi, \eta)$ is an arbitrary function. Introducing the variable $\rho \equiv \tilde{\theta}_t$, one gets the system

$$\theta_{t\xi\eta} + \frac{1}{2}\theta_\eta\cdot\rho_\xi + \frac{1}{2}\theta_\xi\cdot\rho_\eta = 0 ,$$

$$\rho_{\xi\eta} - \frac{1}{2}(\theta_\xi\cdot\theta_\eta)_t = 0 . \qquad (2.3)$$

The system (2.3) has a broad group of usual Lie symmetries. In addition to the obvious translational invariance it is invariant under independent scaling of each variable ξ, η, t

$$\xi \longrightarrow \xi' = \lambda_1 \xi, \quad \eta \longrightarrow \eta' = \lambda_2 \eta, \quad t \longrightarrow t' = \lambda_3 t ,$$

$$\theta \longrightarrow \theta' = \theta , \quad \rho \longrightarrow \rho' = \lambda_3^{-1} \rho , \tag{2.4}$$

where λ_1, λ_2 and λ_3 are arbitrary parameters. The group of transformations (2.4) includes the rotations in (x, y), (x, t) and (y, t) planes. Note that in the case $\sigma^2 = -1$ the variables ξ, η are complex conjugate to each other and $\lambda_2 = \bar{\lambda}_1$.

The system (2.3) is integrable with the use of the linear system (2.2) rewritten in the characteristic variables ξ and η. Combining equations (2.2) and performing the gauge transformation

$$\Psi = \begin{pmatrix} e^{-\frac{i\theta}{2}} , & e^{-\frac{i\theta}{2}} \\ -e^{\frac{i\theta}{2}} , & e^{\frac{i\theta}{2}} \end{pmatrix} \Phi ,$$

one can obtain the simplier and more convenient linear system for equation (2.3), namely, the system

$$L_1 \Phi \equiv \begin{pmatrix} \partial_\xi , & -\frac{i}{2}\theta_\xi \\ -\frac{i}{2}\theta_\eta , & \partial_\eta \end{pmatrix} \Phi = 0 , \tag{2.5a}$$

$$L_2 \Phi \equiv \begin{pmatrix} \partial^2_{t\eta} + \frac{1}{2}\cdot\rho_\eta , & -\frac{i}{2}\theta_\eta\cdot\partial_t \\ -\frac{i}{2}\theta_\xi\cdot\partial_t , & \partial^2_{t\xi} + \frac{1}{2}\cdot\rho_\xi \end{pmatrix} \Phi = 0 . \tag{2.5b}$$

The operator form of the compatibility condition for the system (2.5), i.e. the operator form of the system (2.3) is the following

$$[L_1, L_2] = A_1 L_1 + A_2 L_2 , \tag{2.6}$$

where

$$A_1 = \begin{pmatrix} 0 , & \frac{i}{2}(\theta_{\xi t} + \theta_\eta)\cdot\partial_t + \frac{i}{2}\cdot\theta_\xi\cdot\partial_t \\ \frac{i}{2}(\theta_\xi + \theta_{\eta t})\cdot\partial_t + \frac{i}{2}\cdot\theta_\eta\cdot\partial_t , & 0 \end{pmatrix} , \tag{2.7}$$

$$A_2 = -\frac{i}{2}(\theta_\xi + \theta_\eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.8}$$

The linear system (2.5) is the basic tool for the study of the 2DGSG equation (2.3).

Eliminating the variable ρ from the system (2.3), one obtains the following single equation for θ

$$\theta_{t\xi\eta} + m_1(\eta,t)\cdot\theta_\eta + m_2(\xi,t)\cdot\theta_\xi +$$

$$+ \frac{1}{4}\theta_\eta\cdot\int_{-\infty}^{\eta} d\eta' (\theta_\xi\cdot\theta_{\eta'})_t + \frac{1}{4}\theta_\xi\cdot\int_{-\infty}^{\xi} d\xi' (\theta_{\xi'}\cdot\theta_\eta)_t = 0 \tag{2.9}$$

where

$$m_1(\eta, t) = \frac{1}{2} \lim_{\xi \longrightarrow -\infty} \rho_\xi ,$$

$$m_2(\xi, t) = \frac{1}{2} \lim_{\eta \longrightarrow -\infty} \rho_\eta . \tag{2.10}$$

Note that in the case $\sigma^2 = -1$ only constant m_1 and m_2 are admissible within the class of $m_1(\eta, t)$ and $m_2(\xi, t)$ bounded at the infinity.

So the solution of equation (2.9) with the fixed functions $m_1(\eta, t)$ and $m_2(\xi, t)$ gives the solution of the 2DGSG equation (2.3) with the boundary values of ρ given by (2.10). The properties of the 2DGSG equation essentially depend on the boundaries m_1 and m_2. In the case $m_1 = m_2 = 0$ equation (2.9) is the dispersiveless one with the linear part $\theta_{t\xi\eta} = 0$. In this case equation (2.9) possesses the symmetry group (2.4).

In the case $m_2 = 0$ equation (2.9) has the solution $\theta = \theta_1(\xi, t)$, where θ_1 is an arbitrary function, while in the case $m_1 = 0$ it has the solution $\theta = \theta_2(\eta, t)$, where θ_2 is an arbitrary function. This is an obvious consequence of the general structure of the 2DGSG equation.

An important case corresponds to the constant boundary values m_1 and m_2. The dispersion law for the corresponding equation

$$\theta_{t\xi\eta} + m_1\theta_\eta + m_2\theta_\xi + \frac{1}{4}\theta_\eta \cdot \int_{-\infty}^{\eta} d\eta' (\theta_\xi \cdot \theta_{\eta'})_t + \frac{1}{4}\theta_\xi \cdot \int_{-\infty}^{\xi} d\xi' (\theta_{\xi'} \cdot \theta_\eta)_t = 0 , \tag{2.11}$$

i.e. equation (1.3) is

$$\omega(p_1, p_2) = \frac{m_1}{p_1} + \frac{m_2}{p_2} . \tag{2.12}$$

Equation (2.11) is just the main subject of our paper. We will refer to equation (2.11) as the 2DGSG-I equation in the case $\sigma^2 = 1$ and as the 2DGSG-II equation at $\sigma^2 = -1$. Equation (2.11) does not possess the full symmetry group (2.4). At $m_1 \neq 0$, $m_2 \neq 0$ it possesses the scale invariance

$$\begin{aligned} \xi &\longrightarrow \xi' = \lambda\xi , \\ \eta &\longrightarrow \eta' = \lambda\eta , \\ t &\longrightarrow t' = \lambda^{-1}t , \end{aligned} \tag{2.13}$$

where λ is an arbitrary parameter. In the case $m_1 = 0$, $m_2 \neq 0$ it is invariant under the transformation

$$\begin{aligned} \xi &\longrightarrow \xi' = \lambda_1\xi , \\ \eta &\longrightarrow \eta' = \lambda_2\eta , \\ t &\longrightarrow t' = \lambda_1^{-1}t , \end{aligned} \tag{2.14}$$

where λ_1 and λ_2 are arbitrary parameters. At $m_1 \neq 0$, $m_2 = 0$ it has the symmetry (2.14) with the exchange $\xi \longleftrightarrow \eta$. The scale symmetries (2.4), (2.13), (2.14) imply the existence of various types of the similarity solutions for equations (2.3) and (2.11).

Emphasize also that in the case $m_1 \neq 0$, $m_2 \neq 0$ equation (2.11) is not invariant under the rotations in (x, y) plane while in the case $m_1 = 0$ or $m_2 = 0$ it possesses such a symmetry group which is the subgroup of transformations (2.14) with $\lambda_1\lambda_2 = 1$.

The constant boundaries m_1 and m_2 are the 2+1-dimensional analog of the mass m (more preciesly, squared mass) in the 1+1-dimensional sine-Gordon equation. Indeed, in the 1+1-dimensional limit $\theta_\xi = \theta_\eta$ equation (2.11) looks like

$$\theta_{t\xi\xi} + (m_1 + m_2)\cdot\theta_\xi + \theta_\xi\cdot\int_{-\infty}^{\xi} d\xi' \, \theta_{\xi'}'\cdot\theta_{\xi' t}' = 0 . \qquad (2.15)$$

Assuming that

$$\theta_{t\xi} = F'(\theta) , \qquad (2.16)$$

where $F(\theta)$ is some function and $F'(\theta) = dF/d\theta$, one gets from (2.15)

$$F''(\theta) + F(\theta) = F(\theta(\xi=-\infty)) - (m_1 + m_2). \qquad (2.17)$$

The particular solution of (2.17) is

$$F(\theta) = (m_1 + m_2)\cdot\cos\theta ,$$

for which one has the sine-Gordon equation

$$\theta_{t\xi} = -(m_1 + m_2)\cdot\sin\theta , \qquad (2.18)$$

with the squared mass term $m = -(m_1 + m_2)$.

So the system (1.3) for the class of solutions with the asymptotic behaviour $\theta \longrightarrow 2\pi n$ (n is arbitrary integer) and $\rho \longrightarrow 2(m_1\xi + m_2\eta)$ at $\xi^2 + \eta^2 \longrightarrow \infty$ is an adequate 2+1-dimensional analog of the sine-Gordon equation.

In the general case of the nontrivial boundaries $m_1(\xi, t)$, $m_2(\eta, t)$ equation (2.9) is essentially distinguished from the cases discussed above. In particular, one should modify the second auxiliary linear problem. This will be done in section 5. The localized solutions of the 2DGSG equation with the nontrivial boundaries will be constructed in the part II of the paper.

Note also that the 2DGSG equation has different properties for $\sigma^2 = 1$ and $\sigma^2 = -1$. In the first case ξ and η are real variables. For real valued θ and ρ the boundaries m_1 and m_2 are arbitrary real. In the case $\sigma^2 = -1$ one has $\eta = \bar{\xi}$ where bar means the complex conjugation and the reality of θ and ρ implies $m_2 = \bar{m}_1$. In this case the linear system (2.5) possesses also the involution

$$\alpha\cdot L_i\cdot\alpha^{-1} = \overline{L_i} , \qquad i = 1,2 ,$$
$$\alpha\cdot\Phi\cdot\alpha^{-1} = \overline{\Phi} , \qquad (2.19)$$

where $\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The existence of this involution is an important property of the 2DGSG equation with $\sigma^2 = -1$.

3. $\bar{\partial}$-∂-DRESSING

So in the present paper we will study equation (2.11). We, firstly, will derive this equation within the framework of the $\bar{\partial}$-dressing method. This method has been proposed by

Zakharov and Manakov [9] and then has been developed by several authors (see e.g. [10-12]). Simultaneously with the construction of the integrable systems it allows to find a wide classes of their exact solutions.

In the usual formulation a starting point of the $\bar\partial$-dressing method is the nonlocal $\bar\partial$-problem for the matrix valued function. In the present paper we will use the other linear problem, namely, the mixed $\bar\partial$-∂-problem

$$\mathcal{D}_{\lambda,\bar\lambda}\,\chi(\lambda,\bar\lambda)=(\chi\circ R)(\lambda,\bar\lambda)\equiv\iint_C d(\lambda',\bar\lambda')\chi(\lambda',\bar\lambda')\cdot R(\lambda',\bar\lambda';\lambda,\bar\lambda), \qquad (3.1)$$

where $d(\lambda,\bar\lambda)\overset{def}{=}\frac{d\lambda\wedge d\bar\lambda}{2\pi i}$, χ and R are 2x2 matrix valued functions and

$$\mathcal{D}_{\lambda,\bar\lambda}\,\chi = \chi\begin{pmatrix} \overleftarrow{\dfrac{\partial}{\partial\bar\lambda}}, & 0 \\ 0 & \overleftarrow{\dfrac{\partial}{\partial\lambda}} \end{pmatrix} = \begin{pmatrix} \partial\chi_{11}/\partial\bar\lambda\,, & \partial\chi_{12}/\partial\lambda \\ \partial\chi_{21}/\partial\bar\lambda\,, & \partial\chi_{22}/\partial\lambda \end{pmatrix}. \qquad (3.2)$$

The problem (3.1) is, in fact, equivalent to the standard 2x2 matrix $\bar\partial$-problem. Indeed, redefining the second column by $\chi^{(2)}(\lambda,\bar\lambda)=\tilde\chi^{(2)}(\bar\lambda,\lambda)$ and correspondingly the matrix R, one can easily rewrite the system (3.1) as the usual $\bar\partial$-problem for the matrix $\begin{pmatrix}\chi_{11} & \tilde\chi_{12} \\ \chi_{21} & \tilde\chi_{22}\end{pmatrix}$. We will use the mixed $\bar\partial$-∂-problem (3.1) by the two reasons. First, in such a formulation all the equations will be compatible with the involution (2.19) in the case $\sigma^2=-1$. Second, we would like to

demonstrate that the linear problems different from the standard $\bar\partial$-problem are also possible. Note also, that for the off-diagonal matrix $R=\begin{pmatrix}0 & R_1 \\ R_2 & 0\end{pmatrix}$ the problem (2.19) in the terms of the columns $\chi^{(1)}$ and $\chi^{(2)}$ looks like

$$\frac{\partial}{\partial\bar\lambda}\chi^{(1)}(\lambda,\bar\lambda)=\iint_C d(\lambda',\bar\lambda')\,R_2(\lambda',\bar\lambda';\lambda,\bar\lambda)\,\chi^{(2)}(\lambda',\bar\lambda'),$$
$$\frac{\partial}{\partial\lambda}\chi^{(2)}(\lambda,\bar\lambda)=\iint_C d(\lambda',\bar\lambda')\,R_1(\lambda',\bar\lambda';\lambda,\lambda)\,\chi^{(1)}(\lambda',\bar\lambda'). \qquad (3.3)$$

For the local r.h.s's the system (3.3) has an obvious similiarity with the spatial auxiliary linear problem (2.5) at $\sigma^2=-1$.

So we start with the $\bar\partial$-∂-problem (3.1). We assume that the matrix χ has the canonical normalization $\chi\underset{\lambda\to\infty}{\longrightarrow}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and equation (3.1) has a unique solution. Let now the functions χ and R depend parametrically on the variables ξ, η and t. As usually for the dressing method we assume that the dependence of R on ξ, η, t is covered by the linear equations, namely

$$\frac{\partial R(\lambda',\bar\lambda';\lambda,\bar\lambda;\xi,\eta,t)}{\partial\xi} = -i\cdot\bar\lambda'\cdot\sigma_-\cdot R + i\cdot\bar\lambda\cdot R\cdot\sigma_-,$$
$$\frac{\partial R(\lambda',\bar\lambda';\lambda,\bar\lambda;\xi,\eta,t)}{\partial\eta} = i\cdot\lambda'\cdot\sigma_+\cdot R - i\cdot\lambda\cdot R\cdot\sigma_+, \qquad (3.4)$$

$$\frac{\partial R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda};\xi,\eta,t)}{\partial t} = i\cdot\begin{pmatrix} m_1/\lambda', & 0 \\ 0, & -m_2/\bar{\lambda}' \end{pmatrix}\cdot R - i\cdot R\cdot\begin{pmatrix} m_1/\lambda, & 0 \\ 0, & -m_2/\bar{\lambda} \end{pmatrix},$$

where $\sigma_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

So

$$R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}; \xi, \eta, t) =$$

$$= \exp i\left(-\bar{\lambda}'\cdot\sigma_-\cdot\xi + \lambda'\cdot\sigma_+\cdot\eta + \left(\frac{m_1}{\lambda'}\cdot\sigma_+ - \frac{m_2}{\bar{\lambda}'}\cdot\sigma_-\right)\cdot t\right) \times \tag{3.5}$$

$$\times R_0(\lambda',\bar{\lambda}';\lambda,\bar{\lambda})\cdot\exp i\left(\bar{\lambda}\cdot\sigma_-\cdot\xi - \lambda\cdot\sigma_+\cdot\eta - \left(\frac{m_1}{\lambda}\sigma_+ - \frac{m_2}{\bar{\lambda}}\sigma_-\right)t\right).$$

Then we introduce the "long" derivatives $\mathcal{D}_\xi$, $\mathcal{D}_\eta$, $\mathcal{D}_t$ by

$$\mathcal{D}_\xi\,\chi = \chi_\xi - i\bar{\lambda}\cdot\chi\cdot\sigma_-,$$

$$\mathcal{D}_\eta\,\chi = \chi_\eta + i\lambda\cdot\chi\cdot\sigma_+, \tag{3.6}$$

$$\mathcal{D}_t\,\chi = \chi_t + i\chi\cdot\left(\frac{m_1}{\lambda}\sigma_+ - \frac{m_2}{\bar{\lambda}}\sigma_-\right).$$

According to the general $\bar{\partial}$-dressing approach [9-12] we must construct the operators $\mathcal{L}$ of the form $\mathcal{L} = \sum u_{n,m,l}(\xi, \eta, t)\times \mathcal{D}^n_\xi\cdot\mathcal{D}^m_\eta\cdot\mathcal{D}^l_t$ which obey the condition

$$[\mathcal{D}_{\lambda,\bar{\lambda}}, \mathcal{L}]\,\chi = 0. \tag{3.7}$$

For such operators $\mathcal{L}_i\chi$ obey the same problem (3.1) as χ and consequently

$$\mathcal{L}_i\chi + P_i(\xi, \eta, t)\chi = 0. \tag{3.8}$$

Equations (3.8) are just the linear system we are interested in. The compatibility condition for the system (3.8) is nothing but the integrable system.

It is not difficult to show that in our case one can construct two operators which obey the condition (3.7). They are

$$\mathcal{L}_1 = \sigma_+\cdot\mathcal{D}_\xi + \sigma_-\cdot\mathcal{D}_\eta,$$

$$\mathcal{L}_2 = (\sigma_-\cdot\mathcal{D}_\xi + \sigma_+\cdot\mathcal{D}_\eta)\cdot\mathcal{D}_t + Q\cdot\mathcal{D}_t, \tag{3.9}$$

where

$$Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}, \quad \begin{aligned} q_1 &= -\chi_{11\eta}(\lambda=0)/\chi_{21}(\lambda=0) = -\chi_{12\eta}(\lambda=0)/\chi_{22}(\lambda=0), \\ q_2 &= -\chi_{21\xi}(\lambda=0)/\chi_{11}(\lambda=0) = -\chi_{22\xi}(\lambda=0)/\chi_{12}(\lambda=0). \end{aligned} \tag{3.10}$$

Consequently we have the two linear problems

$$(\sigma_+\cdot\mathcal{D}_\xi + \sigma_-\cdot\mathcal{D}_\eta + P(\xi, \eta, t))\,\chi = 0,$$

$$((\sigma_-\cdot\mathcal{D}_\xi + \sigma_+\cdot\mathcal{D}_\eta)\cdot\mathcal{D}_t + Q\cdot\mathcal{D}_t + \tilde{Q})\,\chi = 0, \tag{3.11}$$

where P and Q are 2×2 matrices. The normalization $\chi \xrightarrow[\lambda\to\infty]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ guarantees that the diagonal elements of the matrices P and Q are equal to zero. Transiting to the matrix Φ defined by

$$\Phi \overset{\mathrm{def}}{=} \chi\cdot \exp i\left(-\bar{\lambda}\cdot\sigma_-\cdot\xi + \lambda\cdot\sigma_+\cdot\eta + \left(\frac{m_1}{\lambda}\sigma_+ - \frac{m_2}{\bar{\lambda}}\sigma_-\right)t\right), \quad (3.12)$$

one gets from (3.11) the system

$$(\sigma_+\cdot\partial_\xi + \sigma_-\cdot\partial_\eta + P)\,\Phi = 0\,, \qquad (3.13)$$

$$((\sigma_-\cdot\partial_\xi + \sigma_+\cdot\partial_\eta)\,\partial_t + Q\cdot\partial_t + \tilde{Q})\,\Phi = 0\,.$$

The specialization $P = \begin{pmatrix} 0, & -i/2\,\theta_\xi \\ -i/2\,\theta_\eta, & 0 \end{pmatrix}$ with the use of the compatibility condition for (3.13) give rise to

$$Q = -\frac{i}{2}\begin{pmatrix} 0 & \theta_\eta \\ \theta_\xi & 0 \end{pmatrix}, \quad \tilde{Q} = \frac{1}{2}\begin{pmatrix} \tilde{\theta}_{t\eta} & 0 \\ 0 & \tilde{\theta}_{t\xi} \end{pmatrix}.$$

Thus, we obtain the linear system which is nothing but the system (2.5). The corresponding compatibility condition is equivalent to the system (2.3) or (2.11). Note that in the terms of the function

$$\hat{\Phi} \overset{\mathrm{def}}{=} \chi\cdot \exp i(-\bar{\lambda}\cdot\sigma_-\cdot\xi + \lambda\cdot\sigma_+\cdot\eta)\,,$$

the linear system (3.11) looks like

$$\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\eta \end{pmatrix}\cdot\hat{\Phi} - \frac{i}{2}\begin{pmatrix} 0 & \theta_\xi \\ \theta_\eta & 0 \end{pmatrix}\cdot\hat{\Phi} = 0\,, \qquad (3.14a)$$

$$\begin{pmatrix} \partial^2_{t\eta} + \frac{1}{4}\partial_\eta^{-1}(\theta_\xi\cdot\theta_\eta)_t + m_1, & -\frac{i}{2}\theta_\eta\partial_t \\ -\frac{i}{2}\theta_\xi\partial_t, & \partial^2_{t\xi} + \frac{1}{4}\partial_\xi^{-1}(\theta_\xi\cdot\theta_\eta)_t + m_2 \end{pmatrix}\cdot\hat{\Phi} + \qquad (3.14b)$$

$$+ \frac{im_1}{\lambda}\begin{pmatrix} \partial_\eta, & -i/2\,\theta_\eta \\ -i/2\,\theta_\xi, & \partial_\xi \end{pmatrix}\cdot\hat{\Phi}\cdot\sigma_+ - \frac{im_2}{\bar{\lambda}}\begin{pmatrix} \partial_\eta, & -i/2\,\theta_\eta \\ -i/2\,\theta_\xi, & \partial_\xi \end{pmatrix}\cdot\hat{\Phi}\cdot\sigma_- = 0\,.$$

The dressing method allows us also to construct the exact solutions of the system (2.11). For given matrix $R_0(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda})$ one should first solve the $\bar{\partial}$-∂-problem (3.1) or, equivalently, the system of the linear integral equations

$$\chi^{(1)}(\lambda,\bar{\lambda}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \iint_C \frac{d(\lambda',\bar{\lambda}')}{\lambda' - \lambda}\iint_C d(\lambda'',\bar{\lambda}'')R_2(\lambda'',\bar{\lambda}'';\lambda',\bar{\lambda}')\chi^{(2)}(\lambda'',\bar{\lambda}''), \qquad (3.15)$$

$$\chi^{(2)}(\lambda,\bar{\lambda}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \iint_C \frac{d(\lambda',\bar{\lambda}')}{\bar{\lambda}' - \bar{\lambda}}\iint_C d(\lambda'',\bar{\lambda}'')R_1(\lambda'',\bar{\lambda}'';\lambda',\bar{\lambda}')\chi^{(1)}(\lambda'',\bar{\lambda}''),$$

where

$$R_1 = R_{10}(\mu, \bar{\mu}; \lambda, \bar{\lambda})\cdot \exp i\left(\bar{\lambda}\xi + \mu\eta + \left(\frac{m_1}{\mu} + \frac{m_2}{\bar{\lambda}}\right)t\right), \qquad (3.16)$$

$$R_2 = R_{20}(\mu, \bar{\mu}; \lambda, \bar{\lambda})\cdot \exp i\left(-\bar{\mu}\xi - \lambda\eta - \left(\frac{m_1}{\lambda} + \frac{m_2}{\bar{\mu}}\right)t\right).$$

Then, one can calculate the potential P by the formula

$$P = i/2 \lim_{\lambda\to\infty}\begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}\cdot[\sigma_3, \chi] \qquad (3.17)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, or, equivalently, by the formulae

$$P_{12} = -i \iint_C d(\lambda,\bar\lambda) \iint_C d(\mu,\bar\mu) R_1(\mu,\bar\mu;\ \lambda,\lambda;\xi,\ \eta,\ t)\cdot\chi_{11}(\mu,\ \bar\mu)\ ,$$

$$P_{21} = i \iint_C d(\lambda,\ \bar\lambda) \iint_C d(\mu,\bar\mu) R_2(\mu,\bar\mu;\ \lambda,\bar\lambda;\ \xi,\eta,t)\cdot\chi_{22}(\mu,\bar\mu)\ , \tag{3.18}$$

where R_1 and R_2 are given by (3.16).

For small P (in a suitable sence) $\chi \sim 1$ and one has

$$P_{12} = -i \iint_C d(\lambda,\bar\lambda) \iint_C d(\mu,\bar\mu) R_{10}(\mu,\bar\mu;\lambda,\bar\lambda)\cdot \exp i\left(\bar\lambda\xi + \mu\eta + \left(\frac{m_1}{\mu} + \frac{m_2}{\bar\lambda}\right)t\right),$$

$$P_{21} = i \iint_C d(\lambda,\bar\lambda) \iint_C d(\mu,\bar\mu) R_{20}(\mu,\bar\mu;\lambda,\bar\lambda)\cdot \exp i\left(-\bar\mu\xi - \lambda\eta - \left(\frac{m_1}{\lambda} + \frac{m_2}{\bar\mu}\right)t\right). \tag{3.19}$$

The formulae (3.19) are in obvius agreement with the dispersion law (2.12)

Necessary conditions for the reduction

$$P_{12} = -i/2\ \theta_\xi\ ,\quad P_{21} = -i/2\ \theta_\eta \tag{3.20}$$

also can obtained from the formulae (3.19). In the case $\sigma^2 = 1$ this condition is of the form

$$R_{10}(\mu,\ \bar\mu;\ \lambda,\ \bar\lambda) = i\bar\lambda\cdot\tilde R(\mu,\ \bar\mu;\ \lambda,\ \bar\lambda)\ ,$$

$$R_{20}(\mu,\bar\mu;\ \lambda,\bar\lambda) = i\lambda\cdot\tilde R(-\lambda,-\bar\lambda;\ -\mu,-\bar\mu)\ , \tag{3.21}$$

where $\tilde R(\mu,\ \bar\mu;\ \lambda,\ \bar\lambda)$ is some function. The reality condition $\theta = \bar\theta$, in particular, implies Im m_1 = Im m_2 = 0 and

$$\overline{\tilde R(\mu,\bar\mu;\lambda,\bar\lambda)} = \tilde R(-\bar\mu,-\mu;\ -\bar\lambda,-\lambda)\ . \tag{3.22}$$

In the case $\sigma^2 = -1, \bar\eta = \xi = \bar z$ and the necessary condition for the reduction (3.20) is

$$R_{10}(\mu,\bar\mu;\ \lambda,\bar\lambda) = i\bar\lambda\cdot\tilde R(\mu,\bar\mu;\ \lambda,\bar\lambda)\ ,$$

$$R_{20}(\mu,\bar\mu;\ \lambda,\bar\lambda) = i\lambda\cdot\tilde R(-\lambda,-\bar\lambda;\ -\mu,-\bar\mu)\ , \tag{3.23}$$

while for the real valued θ one has $m_2 = \bar m_1 = m$ and

$$\overline{\tilde R(\mu,\bar\mu;\lambda,\bar\lambda)} = \tilde R(-\lambda,-\bar\lambda;\ -\mu,-\bar\mu)\ . \tag{3.24}$$

In all these formulae nothing is assumed about the asymptotic behaviour of θ at the infinity $\xi^2 + \eta^2 \longrightarrow \infty$. The necessary condition for the boundness of θ can be also extracted from (3.19). It is equivalent to the condition of the pure oscillating character of the exponents in (3.19). So at $\sigma^2 = 1$ one has

$$R_{10}(\mu,\ \bar\mu;\ \lambda,\ \bar\lambda) = \delta(\text{Im}\ \lambda)\cdot\delta(\text{Im}\ \mu)\cdot\tilde R_1(\mu,\ \lambda)\ ,$$

$$R_{20}(\mu,\ \bar\mu;\ \lambda,\ \bar\lambda) = \delta(\text{Im}\ \lambda)\cdot\delta(\text{Im}\ \mu)\cdot\tilde R_2(\mu,\ \lambda)\ . \tag{3.25}$$

For such R_{10} and R_{20} the $\bar{\partial}$-and ∂-problems (3.1) are reduced to the nonlocal Rieman-Hilbert problems with the jump across the real axis.

In the another case $\sigma^2 = -1$ the condition of boundness is

$$R_{10}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \delta(\lambda - \mu)\cdot\tilde{R}_1(\lambda, \bar{\lambda}) ,$$
$$\qquad (3.26)$$
$$R_{20}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \delta(\lambda - \mu)\cdot\tilde{R}_2(\lambda, \bar{\lambda}) .$$

In this case the problem (3.1) is reduced to the local mixed $\bar{\partial}$-∂-problem.

In conclusion of this section we note that in the case $\sigma^2=-1$ all the formulae presented above are obviously admit the involution (2.19). In the terms of χ it is of the form

$$\alpha\cdot\chi(\lambda, \bar{\lambda})\cdot\alpha^{-1} = \overline{\chi(\lambda,\bar{\lambda})} , \quad \alpha\cdot R\cdot\alpha^{-1} = \overline{R} . \qquad (3.27)$$

So in this case the 2x2 matrix χ is of the form

$$\chi = \begin{pmatrix} \chi_{11}, & -\overline{\chi_{21}} \\ \chi_{21}, & \overline{\chi_{11}} \end{pmatrix} . \qquad (3.28)$$

Similar representation is valid for the matrix Φ.

4. EXACT SOLUTIONS

1. **Solutions with functional parameters.** A wide class of the exact solutions corresponds, typically for the dressing method, to the general degenerated kernels R_1 and

R_2 of the $\bar{\partial}$-∂-problem. So, having in mind the conditions (3.21) and (3.23), we put

$$R_{10}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = i\bar{\lambda}\cdot\sum_{n=1}^{N} f_n(\mu, \bar{\mu})\cdot g_n(\lambda, \bar{\lambda}) ,$$
$$\qquad (4.1)$$
$$R_{20}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = i\lambda\cdot\sum_{n=1}^{N} f_n(-\lambda, -\bar{\lambda})\cdot g_n(-\mu, -\bar{\mu}) .$$

The condition of the reality of θ gives at $\sigma^2 = 1$

$$\sum_{n=1}^{N} \overline{f_n(\mu, \bar{\mu})}\cdot\overline{g_n(\lambda, \bar{\lambda})} = \sum_{n=1}^{N} f_n(-\bar{\mu}, -\mu)\cdot g_n(-\bar{\lambda}, -\lambda) . \qquad (4.2)$$

The condition (4.2) means that

$$\overline{f_n(\mu, \bar{\mu})} = f_n(-\bar{\mu}, -\mu) ,$$
$$\qquad (4.3)$$
$$\overline{g_n(\lambda, \bar{\lambda})} = g_n(-\bar{\lambda}, -\lambda), \ (n = 1,\ldots, N)$$

for arbitrary N, or N = 2M and

$$\overline{f_n(\mu, \bar{\mu})} = f_{n+M}(-\bar{\mu}, -\mu) ,$$
$$\qquad (4.4)$$
$$\overline{g_n(\lambda, \bar{\lambda})} = g_{n+M}(-\bar{\lambda}, -\lambda), \ n = 1,\ldots, M \ (\mathrm{mod}\ M) .$$

For the 2DGSG-II equation ($\sigma^2=-1$) the condition (3.23) of the reality θ gives

$$\sum_{n=1}^{N} f_n(\mu,\bar{\mu})\cdot \overline{g_n(\lambda,\bar{\lambda})} = \sum_{n=1}^{N} f_n(-\lambda,-\bar{\lambda})\cdot g_n(-\mu,-\bar{\mu}) . \qquad (4.5)$$

The condition (4.5) means that

$$g_n(\lambda, \bar{\lambda}) = \overline{f_n(-\lambda, -\bar{\lambda})} , \quad (n = 1,\dots, N) \qquad (4.6)$$

for arbitrary N, or N = 2M and

$$g_n(\lambda, \bar{\lambda}) = \overline{f_{n+M}(-\lambda, -\bar{\lambda})} , \; n = 1,\dots, M \; (\mathrm{mod}\; M). \qquad (4.7)$$

So the kernels R_{10} and R_{20} are of the form

$$\begin{aligned} R_{10}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) &= i\bar{\lambda} \sum_{n=1}^{N} f_n(\mu, \bar{\mu})\cdot \overline{f_n(-\lambda, -\bar{\lambda})} , \\ R_{20}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) &= i\lambda \sum_{n=1}^{N} f_n(-\lambda, -\bar{\lambda})\cdot \overline{f_n(\mu, \bar{\mu})} \end{aligned} \qquad (4.8)$$

for an arbitrary N, or

$$\begin{aligned} R_{10}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) &= i\bar{\lambda} \sum_{n=1}^{2M} f_n(\mu, \bar{\mu})\cdot \overline{f_{n+M}(-\lambda, -\bar{\lambda})} , \\ R_{20}(\mu, \bar{\mu}; \lambda, \bar{\lambda}) &= i\lambda \sum_{n=1}^{2M} f_n(-\lambda, -\bar{\lambda})\cdot \overline{f_{n+M}(\mu, \bar{\mu})} , \end{aligned} \qquad (4.9)$$

where $f_{n+2M} = f_n$.

To calculate θ, using the formula (3.18) or, equivalently, the formula

$$\theta_\xi = 2\iint_C d(\lambda, \bar{\lambda})\iint_C d(\mu, \bar{\mu}) \exp i\left(\bar{\lambda}\xi + \mu\eta + \left(\frac{m_1}{\mu} + \frac{m_2}{\bar{\lambda}}\right)t\right) \times$$
$$\times\, i\bar{\lambda} \sum_{n=1}^{N} f_n(\mu, \bar{\mu})\cdot g_n(\lambda, \bar{\lambda})\cdot \chi_{11}(\mu, \bar{\mu}) \qquad (4.10)$$

one should find $\chi_{11}(\mu, \bar{\mu})$. From the system (3.15) one has

$$\chi_{11}(\lambda, \bar{\lambda}) = 1 + \iint_C \frac{d(\lambda',\bar{\lambda}')}{\lambda' - \lambda} \iint_C d(\lambda'', \bar{\lambda}'')\, R_2(\lambda'', \bar{\lambda}''; \lambda', \bar{\lambda}') \times$$
$$\times \iint_C \frac{d(\mu',\bar{\mu}')}{\bar{\mu}' - \bar{\lambda}''} \iint_C d(\mu'', \bar{\mu}'')\, R_1(\mu'', \bar{\mu}''; \mu', \bar{\mu}')\, \chi_{11}(\mu'', \bar{\mu}'') . \qquad (4.11)$$

Multiplying equation (4.11) by $\exp i\left(\lambda\eta + \frac{m_1}{\lambda} t\right)\cdot f_m(\lambda, \bar{\lambda})$, integrating over λ and taking into account (4.1), we obtain the system

$$F_m = Y_m + \sum_{k=1}^{N} (A\cdot B)_{mk} F_k, \qquad m = 1,\dots, N , \qquad (4.12)$$

where

$$F_m(\xi,\eta,t) \stackrel{\mathrm{def}}{=} \iint d(\lambda,\bar{\lambda}) \exp i(\lambda\eta + \frac{m_1}{\lambda} t)\cdot f_k(\lambda,\bar{\lambda})\cdot \chi_{11}(\lambda,\bar{\lambda};\xi,\eta,t), \qquad (4.13)$$

$$Y_m(\eta\; t) \stackrel{\mathrm{def}}{=} \iint d(\lambda,\bar{\lambda}) \exp i(\lambda\eta + \frac{m_1}{\lambda} t)\cdot f_k(\lambda,\bar{\lambda}), \; (m=1,\dots,N),$$

and the matrix elements of the matrices A and B are

$$A_{mn}(\eta,t) \overset{def}{=} \iint\limits_C d(\lambda,\ \bar\lambda)\iint\limits_C d(\mu,\ \bar\mu)\,\frac{i\mu}{\mu - \lambda} \times$$

$$\times \exp i(\lambda\eta - \mu\eta + \frac{m_1}{\lambda}t - \frac{m_1}{\mu}t)\cdot f_m(\lambda,\ \bar\lambda)\cdot f_n(-\mu,\ -\bar\mu)\ , \tag{4.14}$$

$$B_{mn}(\xi,t) \overset{def}{=} \iint\limits_C d(\lambda,\ \bar\lambda)\iint\limits_C d(\mu,\ \bar\mu)\,\frac{i\bar\mu}{\bar\mu - \bar\lambda} \times$$

$$\times \exp i(-\bar\lambda\xi + \bar\mu\xi - \frac{m_2}{\bar\lambda}t + \frac{m_2}{\bar\mu}t)\cdot g_m(-\lambda,-\bar\lambda)\cdot g_n(\mu,\ \bar\mu)$$

$$(m,n = 1,\ldots N).$$

Solving the algebraic system (4.12) and substituting the results into (4.10), one gets

$$\theta_\xi = 2\cdot\sum_{n,m=1}^{N} X_{n\xi}(\xi,\ t)\cdot(1 - AB)^{-1}_{nm}\cdot Y_m(\eta,\ t)\ , \tag{4.15}$$

where

$$X_n(\xi,t) \overset{def}{=} \iint d(\lambda,\bar\lambda)\ \exp i(\bar\lambda\xi + \frac{m_2}{\bar\lambda}t)\cdot g_n(\lambda,\ \bar\lambda). \tag{4.16}$$

From (4.13), (4.14) and (4.16) it is not difficult to see that the matrices A_{mn} and B_{mn} can be expressed in the terms of the quantities $X_n(\xi,\ t)$ and $Y_n(\eta,\ t)$ by the compact formulae

$$A_{nm} = i\int^{\eta} d\eta'\, Y_n(\eta',t)\cdot Y_{m\eta'}(\eta',t)\ ,$$

$$B_{nm} = i\int^{\xi} d\xi'\, X_n(\xi',\ t)\cdot X_{m\xi'}(\xi',\ t)\ . \tag{4.17}$$

Integration in these expressions is performed over suitable contour.

Note that the functions X_n and Y_n are arbitrary solutions of the linear Klein-Gordon equations

$$X_{\xi t} + m_2 X = 0\ , \tag{4.18a}$$

and

$$Y_{\eta t} + m_1 Y = 0\ . \tag{4.18b}$$

Similar formulae can be obtained also for θ_η.

So for arbitrary functions f_n and g_n we have the solutions (4.15) which depend on the several functional parameters.

The reality conditions (4.3), (4.4) and (4.6), (4.7) imply at $\sigma^2 = 1$:

$$X_n(\xi,\ t) = \overline{X_n(\xi,\ t)},\quad Y_n(\eta,\ t) = \overline{Y_n(\eta,\ t)}\ , \tag{4.19}$$

and at $\sigma^2 = -1$:

$$Y_n(z,\ t) = \overline{X_n(\bar z,\ t)}\ ,$$

where $z = x + iy$, $\bar z = x - iy$.

The simplest solution (N=1) of this type for the 2DGSG-I equation is of the form:

$$\theta = 4 \text{ arctg } \frac{X(\xi,t)\cdot Y(\eta,t)}{2} , \qquad (4.20)$$

where $X(\xi, t) = \overline{X(\xi, t)}$ and $Y(\eta, t) = \overline{Y(\eta, t)}$. In particular, choosing

$$X = \cos\left(a\xi + \frac{m_2}{a} t - \xi_0\right), \; Y = \sin\left(b\eta + \frac{m_1}{b} t - \eta_0\right) ,$$

where a and b are arbitrary constants, one gets the periodic solution

$$\theta = 4\cdot\text{arctg}\left(\frac{1}{2}\cos\left(a\xi + \frac{m_2}{a} t - \xi_0\right)\cdot\sin\left(b\eta + \frac{m_1}{b} t - \eta_0\right)\right) .$$

The simplest solution (N=1) with functional parameters for the 2DGSG-II equation ($\sigma^2 = -1$) is of the form:

$$\theta = 4\cdot\text{arctg } \frac{|X(\bar{z},t)|^2}{2} , \qquad (4.21)$$

where $z = x + iy$.

2. **Plane solitons (plane kinks).** Plane solitons of the 2DGSG equation are the very special case of the solutions with functional parameters considered above. They correspond to the choice of the functions f_n and g_n as the Dirac delta functions

$$f_n(\lambda,\bar{\lambda}) = C_n\cdot\delta(\lambda - \lambda_n) ,$$
$$(4.22)$$
$$g_n(\mu,\bar{\mu}) = \tilde{C}_n\cdot\delta(\mu - \mu_n) ,$$

with the constraints (4.3) (σ^2=1) and (4.6) (σ^2=-1).

Let us consider first the case σ^2=1. The constraint (4.3) means

$$\bar{\lambda}_n = - \lambda_n , \quad \bar{C}_n = C_n ,$$
$$(4.23)$$
$$\bar{\mu}_n = - \mu_n , \quad \overline{\tilde{C}_n} = \tilde{C}_n , \quad (n = 1,\ldots,N) ,$$

i.e. $\lambda_n = ip_n$, $\mu_n = iq_n$, $\text{Im } p_n = \text{Im } q_n = 0$ and

$$X_n = \tilde{C}_n\cdot\exp\left(p_n\xi - \frac{m_2}{p_n} t\right), \; Y_n = C_n\cdot\exp\left(q_n\eta - \frac{m_1}{q_n} t\right) . \qquad (4.24)$$

The general N-plane soliton solution of the 2DGSG-I equation is of the form

$$\theta_\xi = 2\cdot\sum_{n,m} X_{n\xi}(\xi, t)\cdot(1 - AB)^{-1}_{nm}\cdot Y_m(\eta, t) , \qquad (4.25)$$

where

$$A_{nm} = \frac{iq_m}{q_n + q_m}\cdot Y_n(\eta, t)\cdot Y_m(\eta, t) ,$$
$$(4.26)$$
$$B_{nm} = \frac{ip_m}{p_n + p_m}\cdot X_n(\xi,t)\cdot X_m(\xi,t) .$$

The simplest plane soliton looks like

$$\theta(\xi, \eta, t) = 4\cdot\text{arctg } \frac{X\cdot Y}{2} =$$
$$(4.27)$$
$$= 4\cdot\text{arctg}\left(\frac{C\cdot\tilde{C}}{2}\cdot\exp\left(p\xi + q\eta - \frac{m_1}{q} t - \frac{m_2}{p} t\right)\right).$$

Note that $\theta \longrightarrow 2\pi$ at $p\xi + q\eta \longrightarrow \infty$ and $\theta \longrightarrow 0$ at $p\xi + q\eta \longrightarrow -\infty$. At N=2 one has the two-soliton solution

$$\theta(\xi,\eta,t) = 4\cdot\mathrm{arctg}\,\frac{X_1Y_1 + X_2Y_2}{2 - \dfrac{(q_1-q_2)(p_1-p_2)}{2(q_1+q_2)(p_1+p_2)}\cdot X_1Y_1X_2Y_2}\,. \tag{4.28}$$

The general solution (4.25) describes an elastic scattering of N plane solitons (4.27).

In the case $\sigma^2 = -1$ the reality condition (4.6) gives

$$\mu_n = -\lambda_n\,, \qquad \tilde{C}_n = \overline{C_n}\,,\quad (n = 1,\dots,N) \tag{4.29}$$

and

$$X_n(\bar z,t) = \overline{Y_n(z,t)} = \overline{C_n}\cdot\exp i\Big(-\lambda_n\bar z - \frac{\bar m}{\bar\lambda_n}t\Big)\,. \tag{4.30}$$

The simplest plane soliton (N=1) of the 2DGSG-II equation is

$$\theta(z,\bar z,t) = 4\cdot\mathrm{arctg}\,\frac{|X(\bar z,t)|^2}{2} = 4\,\mathrm{arctg}\,\frac{|C|^2}{2}\cdot\exp i\Big(\lambda z - \bar\lambda\bar z + \frac{m}{\lambda}t - \frac{\bar m}{\bar\lambda}t\Big)\,. \tag{4.31}$$

The general N-plane soliton solution has the form

$$\theta_{\bar z} = 2\cdot\sum_{n,m} X_{n\bar z}(\bar z,t)(1 + D\cdot\bar D)^{-1}_{nm}\,\overline{X_m(z,t)}\,, \tag{4.32}$$

where

$$D_{nm} = \frac{\lambda_n\bar X_n\bar X_m}{\lambda_n + \lambda_m}\,.$$

At N=2 we have the solution

$$\theta = 4\cdot\mathrm{arctg}\,\frac{(|X_1|^2 + |X_2|^2)/2}{1 - \dfrac{|\lambda_1-\lambda_2|^2}{|\lambda_1+\lambda_2|^2}\cdot\dfrac{|X_1X_2|^2}{4}}\,, \tag{4.33}$$

where X_n are given by the formula (4.30).

3. Breathers. The solutions of the 2DGSG equation which are the 2+1-dimensional analogs of the well-known breathers of the 1+1-dimensional sine-Gordon equation correspond to the delta-functional f_n and g_n which obey the conditions (4.4) ($\sigma^2 = 1$) and (4.7) ($\sigma^2 = -1$).

Namely, for $\sigma^2 = 1$

$$R_{10}(\mu,\bar\mu;\lambda,\bar\lambda) = i\bar\lambda\cdot\sum_{n=1}^{M}\big(C_n\delta(\mu-\mu_n)\delta(\lambda-\lambda_n) + \bar C_n\delta(\mu+\bar\mu_n)\delta(\lambda+\bar\lambda_n)\big)\,,$$

$$R_{20}(\mu,\bar\mu;\lambda,\bar\lambda) = i\lambda\cdot\sum_{n=1}^{M}\big(C_n\delta(\lambda+\mu_n)\delta(\mu+\lambda_n) + \overline{C}_n\delta(\lambda-\bar\mu_n)\delta(\mu-\bar\lambda_n)\big) \tag{4.34}$$

and for $\sigma^2 = -1$:

$$R_{10}(\mu,\bar{\mu};\ \lambda,\bar{\lambda}) = i\bar{\lambda}\cdot\sum_{n=1}^{M}(C_n\delta(\mu - \mu_n)\cdot\delta(\lambda + \lambda_n) +$$

$$+ \bar{C}_n\delta(\mu - \lambda_n)\cdot\delta(\lambda + \mu_n)),$$

$$\tag{4.35}$$

$$R_{20}(\mu,\bar{\mu};\ \lambda,\bar{\lambda}) = i\lambda\cdot\sum_{n=1}^{M}(C_n\delta\ (\lambda + \mu_n)\cdot\delta(\mu - \lambda_n) +$$

$$+ \bar{C}_n\delta(\lambda + \lambda_n)\cdot\delta(\mu - \mu_n)).$$

The simplest breather of the 2DGSG-I equation looks like (N=1)

$$\theta(\xi,\eta,t) = 4\cdot\mathrm{arctg}\ \frac{|C|\cdot e^{f}\cdot\cos\varphi}{1 - \dfrac{|C|^2\lambda_R\mu_R e^{2f}}{4\lambda_I\mu_I}}, \tag{4.36}$$

where

$$C = |C|e^{i\delta},$$

$$f(\xi,\eta,t) = \lambda_I\xi - \mu_I\eta - \frac{m_2\lambda_I}{|\lambda|^2}t + \frac{m_1\mu_I}{|\mu|^2}t,$$

$$\varphi(\xi,\eta,t) = \lambda_R\xi + \mu_R\eta + \frac{m_2\lambda_R}{|\lambda|^2}t + \frac{m_1\mu_R}{|\mu|^2}t + \delta,$$

$$\lambda = \lambda_R + i\lambda_I,\ \mu = \mu_R + i\mu_I.$$

The simplest breather of the 2DGSG-II equation is (N = 1):

$$\theta(z,\bar{z},t) = 4\cdot\mathrm{arctg}\ \frac{|C|\cdot e^{-f}\cdot\cos\varphi}{1 + \dfrac{|\lambda-\mu|^2|C|^2e^{-2f}}{4|\lambda+\mu|^2}}, \tag{4.37}$$

where

$$C = |C|e^{i\delta},$$

$$f = \mathrm{Im}\left(\lambda z - \bar{\mu}\bar{z} + \frac{m\bar{\lambda}}{|\lambda|^2}t - \frac{\bar{m}\mu}{|\mu|^2}t\right),$$

$$\varphi = \mathrm{Re}\left(\lambda z - \bar{\mu}\bar{z} + \frac{m\bar{\lambda}}{|\lambda|^2}t - \frac{\bar{m}\mu}{|\mu|^2}t\right) - \delta.$$

5. INITIAL VALUE PROBLEM FOR THE 2DGSG-I EQUATION

1. Now we will study the initial value problem for the 2DGSG equation (2.11) for the class of solutions which tend rapidly enough to the asymptotic value $2\pi n$.

Our task is essentially simplified due to the existence of the solution of the similar problem for the Davey-Stewartson (DS) equation [13 - 16]. Indeed the first auxiliary linear problem

$$L_1\Phi \equiv \begin{pmatrix} \partial_\xi, & -\frac{i}{2}\theta_\xi \\ -\frac{i}{2}\theta_\eta, & \partial_\eta \end{pmatrix}\Phi = 0 \tag{5.1}$$

is nothing but the reduction $q = -\frac{i}{2}\theta_\xi$, $r = -\frac{i}{2}\theta_\eta$ of the well studied spectral problem for the DS equation [13 - 16]. The only difference is that here we will consider the solutions of the system (5.1) of the form

$$\Phi = \chi(\lambda, \bar{\lambda}) \cdot \begin{pmatrix} e^{i\lambda\eta}, & 0 \\ 0, & e^{-i\bar{\lambda}\xi} \end{pmatrix} , \tag{5.2}$$

where $\chi \longrightarrow 1$ at $\lambda \longrightarrow \infty$. The presence of $\bar{\lambda}$ in (5.2) will be reflected in the substitution $\lambda \longrightarrow \bar{\lambda}$ in all formulae derived in [13-15] in the quantities which act on the second column.

We start with the 2DGSG-I equation ($\sigma^2 = 1$). In the terms of the first column $\chi^{(1)}$ and second column $\chi^{(2)}$ of χ the linear problem (5.1) looks like

$$\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\eta \end{pmatrix} \chi^{(1)} + i\lambda\cdot\sigma_-\cdot\chi^{(1)} - \frac{i}{2}\begin{pmatrix} 0 & \theta_\xi \\ \theta_\eta & 0 \end{pmatrix} \chi^{(1)} = 0 , \tag{5.3}$$

and

$$\begin{pmatrix} \partial_\xi & 0 \\ 0 & \partial_\eta \end{pmatrix} \chi^{(2)} - i\bar{\lambda}\cdot\sigma_+\cdot\chi^{(2)} - \frac{i}{2}\begin{pmatrix} 0 & \theta_\xi \\ \theta_\eta & 0 \end{pmatrix} \chi^{(2)} = 0 . \tag{5.4}$$

The Green function $G^{(1)}$ for equation (5.3) which is bounded and analytic except the real axis has been found in [13-15]. Similarly one can construct the bounded and anti-analytic except the real axis Green function $G^{(2)}$ for equation (5.4).

These Green functions allow us to construct, in a manner completely similar to that of Refs. [13 - 15], the solutions of the problems (5.3) and (5.4) which are analytic (for (5.3)) and anti-analytic (for (5.4)) and have the jumps across the real axis given by

$$\chi^{(1)+}(\lambda) - \chi^{(1)-}(\lambda) = \int_R dl\ T(\lambda,l)\cdot e^{-il\xi - i\lambda\eta}\cdot\chi^{(2)+}(l) ,$$

$$\chi^{(2)+}(\lambda) - \chi^{(2)-}(\lambda) = \int_R dl\ S(\lambda,l)\cdot e^{il\eta + i\lambda\xi}\cdot\chi^{(1)-}(l) , \tag{5.5}$$

where $\chi^{\pm}(\lambda) \overset{def}{=} \chi(\lambda \pm i0)$ and

$$S(\lambda, l) \overset{def}{=} -\frac{1}{4\pi}\iint_{R^2} d\xi\ d\eta(-\frac{i}{2}\theta_\xi)\cdot\chi^-_{22}(\lambda)\cdot e^{-i\lambda\xi - il\eta} ,$$

$$T(\lambda,l) \overset{def}{=} \frac{1}{4\pi}\iint_{R^2} d\xi\ d\eta(-\frac{i}{2}\theta_\eta)\cdot\chi^+_{11}(\lambda)\cdot e^{i\lambda\eta + il\xi} . \tag{5.6}$$

So we have the nonlocal Riemann-Hilbert problems. Their solutions are given by the well-known formulae [13-15]. For instance

$$\chi^{(1)\pm}(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\int_R dl\ T(\lambda, l)\cdot e^{-il\xi - i\lambda\eta}\cdot\chi^{(2)+}(l)\right)^{\pm} ,$$

$$\chi^{(2)\pm}(\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left(\int_R dl\ S(\lambda, l)\cdot e^{i\lambda\xi + il\eta}\cdot\chi^{(1)-}(l)\right)^{\pm} , \tag{5.7}$$

where

$$(f(\lambda))^{\pm} = \frac{1}{2\pi i}\int_R \frac{d\lambda' f(\lambda')}{\lambda' - (\lambda \pm i0)} . \qquad (5.8)$$

The corresponding reconstruction formulae are of the form [13-14]

$$\theta_\xi = -\frac{2i}{\pi}\iint_{R^2} d\lambda\ dl\ S(\lambda,l)\cdot e^{il\eta+i\lambda\xi}\cdot\chi^-_{11}(l) ,$$

$$\theta_\eta = \frac{2i}{\pi}\iint_{R^2} d\lambda\ dl\ T(\lambda,l)\cdot e^{-il\xi-i\lambda\eta}\cdot\chi^+_{22}(l) . \qquad (5.9)$$

Note that the nonlocal Riemann-Hilbert problem are in complete agreement with the corresponding problem in the case (3.25).

The reduction $q = -\frac{i}{2}\theta_\xi$, $r = -\frac{i}{2}\theta_\eta$ implies certain constraint in the inverse problem data S and T. Considering the small θ, one gets from (5.9) that

$$S(\lambda, l) = \lambda\cdot\tilde{S}(\lambda, l) ,$$

$$T(\lambda, l) = \lambda\cdot\tilde{S}(-l, -\lambda) , \qquad (5.10)$$

and

$$\theta = -\frac{2}{\pi}\iint_{R^2} d\lambda\ dl\ \tilde{S}(\lambda,l)\cdot e^{il\eta+i\lambda\xi}\cdot\chi^-_{11}(l) . \qquad (5.11)$$

The condition of the reality of θ also implies

$$\overline{\tilde{S}(\lambda,l)} = \tilde{S}(-\lambda, -l) . \qquad (5.12)$$

Equations (5.7) to ther with the reconstruction formula (5.11) are the invers. problem equations for the linear problem (5.1).

2. To solve the initial value problem for the 2DGSG-I equation one have to find the time evolution of the inverse problem data. To do this, typical for the IST method step, one should use the second auxiliary equation $L_2\Phi = 0$. In our case this step is not trivial since, first, we need to consider the nonzero boundaries (2.10) and, second, the operator representation of the compatibility condition is the quartet operator equation (2.6).

In the presence of the nontrivial boundaries one should modify the second auxiliary equation similar to the DS and Ishimori equations cases [17 - 19]. Indeed, the solution Φ (5.2) of the problem (5.1) with the asymptotic $\Phi \longrightarrow \begin{pmatrix} e^{i\lambda\eta}, & 0 \\ 0, & e^{-i\lambda\xi} \end{pmatrix}$ at $\xi^2+\eta^2 \longrightarrow \infty$ obviously is not a solution of the linear equation $L_2\Phi = 0$ with L_2 given by (2.5.b).

Let the modified second auxiliary linear problem is of the form

$$L_{2M}\Phi \equiv (L_2 + \Delta)\Phi = 0 , \qquad (5.13)$$

where the operator Δ should be found. From operator equation (2.6) it follows that

$$(L_1 - A_2)\cdot\Delta\cdot\Phi = 0\ , \tag{5.14}$$

i.e.

$$\begin{pmatrix} \partial_\xi\ , & -\frac{i}{2}\theta_\eta \\ -\frac{i}{2}\theta_\xi\ , & \partial_\eta \end{pmatrix}\cdot\Delta\cdot\Phi = 0\ . \tag{5.15}$$

So $\Delta\cdot\Phi$ obeys the linear system of the same type as Φ but with the different potential. Using the idea of the Darboux or gauge transformation (see e.g. [20, 21]), one can show that the solution of the system

$$\begin{pmatrix} \partial_\xi\ , & -\frac{i}{2}\theta_\eta \\ -\frac{i}{2}\theta_\xi\ , & \partial_\eta \end{pmatrix} V = 0\ , \tag{5.16}$$

can be expressed via the solution Φ of the system (5.1) as

$$V = \left(\partial_\xi + \partial_\eta - \frac{i}{2}(\theta_\xi + \theta_\eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\Phi\ . \tag{5.17}$$

Then using (5.13) and taking into account the asymptotic behaviour of Φ at $\xi^2 + \eta^2 \longrightarrow \infty$, one finds the asymptotic behaviour of $\Delta\cdot\Phi$:

$$\Delta\cdot\Phi \xrightarrow[\xi^2+\eta^2\longrightarrow\infty]{} \begin{pmatrix} -m_1(\eta,t)\cdot e^{i\lambda\eta}\ , & 0 \\ 0\ , & -m_2(\xi,t)\cdot e^{-i\bar{\lambda}\xi} \end{pmatrix}\ . \tag{5.18}$$

Comparing now the integral equations for Φ and $\Delta\cdot\Phi$ which correspond to the systems (5.1) and (5.15), we obtain

$$\Delta\cdot\Phi(\lambda) = \left(\partial_\xi + \partial_\eta - \frac{i}{2}(\theta_\xi + \theta_\eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \times$$
$$\times\left(\int dl\,\frac{\hat{m}_1(\lambda-l,t)}{l}(\Phi\cdot\sigma_+)(l) + \int d\bar{l}\,\frac{\hat{m}_2(\bar{\lambda}-\bar{l},t)}{\bar{l}}(\Phi\cdot\sigma_-)(\bar{l})\right)\ , \tag{5.19}$$

where

$$\hat{m}_1(\lambda - l,\ t) \overset{def}{=} \frac{i}{2\pi}\int_R d\eta\ m_1(\eta,\ t)\cdot e^{i\eta(\lambda-l)}\ ,$$

$$\hat{m}_2(\bar{\lambda} - \bar{l},\ t) \overset{def}{=} \frac{-i}{2\pi}\int_R d\xi\ m_2(\xi,\ t)\cdot e^{-i\xi(\bar{\lambda}-\bar{l})}\ . \tag{5.20}$$

Thus the 2DGSG equation with the nontrivial boundaries is equivalent to the compatibility condition for the system of equation (5.1) and equation

$$L_{2M}\Phi = \begin{pmatrix} \partial^2_{t\eta} + m_1(\eta,t) + \frac{1}{4}\partial_\eta^{-1}(\theta_\xi\theta_\eta)_t\ , & -\frac{i}{2}\theta_\eta\cdot\partial_t \\ -\frac{i}{2}\theta_\xi\cdot\partial_t\ , & \partial^2_{t\xi} + m_2(\xi,t) + \frac{1}{4}\partial_\xi^{-1}(\theta_\xi\theta_\eta)_t \end{pmatrix}\Phi +$$
$$+ \left(\partial_\xi + \partial_\eta - \frac{i}{2}(\theta_\xi + \theta_\eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \times \tag{5.21}$$
$$\times\left(\int dl\,\frac{\hat{m}_1(\lambda-l,t)}{l}\cdot(\Phi\cdot\sigma_+)(l) + \int d\bar{l}\,\frac{\hat{m}_2(\bar{\lambda}-\bar{l},t)}{\bar{l}}\cdot(\Phi\cdot\sigma_-)(\bar{l})\right).$$

Substituting now the formulae (5.5) into (5.21), we obtain from the equation for the second column $\Psi^{(2)}=\chi^{(2)}\cdot e^{-i\lambda\xi}$ at $\xi \longrightarrow -\infty$ the following

$$(\partial^2_{t\eta} + m_1(\eta, t))(\lim_{\xi \longrightarrow -\infty} \Psi_{12}(\lambda, t)) +$$

$$+ \int \frac{d\bar{l}}{\bar{l}} \hat{m}_2(\bar{\lambda} - \bar{l}, t) \cdot \partial_\eta (\lim_{\xi \longrightarrow -\infty} \Psi_{12}(l, t)) = 0 \ . \qquad (5.22)$$

Since [18]

$$\lim_{\xi \longrightarrow -\infty} \Psi_{12}(\xi, \eta; \lambda, t) = \int_R dl \ e^{il\eta} \cdot i\lambda \cdot \tilde{S}(\lambda, l, t) \ , \qquad (5.23)$$

one obtains from (5.22) the following equation for the inverse problem data

$$\frac{\partial}{\partial t} \tilde{S}(\lambda, l, t) = \frac{i}{\sqrt{2\pi}\, l} \int_R d\mu \ \hat{m}_1(\mu, t) \cdot \tilde{S}(\lambda, l-\mu, t) +$$

$$+ \frac{i}{\sqrt{2\pi}\, \bar{\lambda}} \int_R d\mu \ \hat{m}_2(\mu, t) \cdot \tilde{S}(\lambda-\mu, l, t) \ . \qquad (5.24)$$

In the terms of the Fourier transform $\hat{S}(\xi, \eta, t)$:

$$\hat{S}(\xi, \eta, t) \stackrel{def}{=} \iint_{R^2} d\lambda \ dl \ \tilde{S}(\lambda, l, t) \cdot e^{i\lambda\xi + il\eta}$$

of the data $S(\lambda, l, t)$ equation (5.24) looks like

$$\hat{S}_{t\xi\eta} + m_1(\eta, t) \cdot \hat{S}_\eta + m_2(\xi, t) \cdot \hat{S}_\xi = 0 \ . \qquad (5.25)$$

Note that equation (5.25) is nothing but the linear part of the 2DGSG equation with nontrivial boundaries. For small potential this coincidence is the trivial consequence of the formula (5.11).

Equation (5.25) for given boundaries $m_1(\eta, t)$ and $m_2(\xi, t)$ can be solved by separation of variables. Using the corresponding solutions one can construct the localized solutions of the 2DGSG-I equation with nontrivial boundaries. This problem will be considered in the part II of the paper.

For the constant boundaries m_1 and m_2 one has

$$\hat{m}_1(\lambda - l) = i \cdot m_1 \delta(\lambda - l) \ ,$$
$$\hat{m}_2(\bar{\lambda} - \bar{l}) = -i \cdot m_2 \delta(\lambda - l) \ . \qquad (5.26)$$

So for the 2DGSG-I equation (2.11) the second modified auxiliary linear problem is

$$L_{2M}\Phi = \begin{pmatrix} \partial^2_{t\eta} + m_1 + \frac{1}{4}\partial_\eta^{-1}(\theta_\xi \theta_\eta)_t , & -\frac{i}{2}\theta_\eta \cdot \partial_t \\ -\frac{i}{2}\theta_\xi \partial_t , & \partial^2_{t\xi} + m_2 + \frac{1}{4}\partial_\xi^{-1}(\theta_\xi \theta_\eta)_t \end{pmatrix} \Phi +$$

$$+ \left(\partial_\xi + \partial_\eta - \frac{i}{2}(\theta_\xi + \theta_\eta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \cdot \Phi \cdot \begin{pmatrix} im_1/\lambda , & 0 \\ 0 , & -im_2/\bar{\lambda} \end{pmatrix} = 0, \qquad (5.27)$$

and the time evolution of $\tilde{S}(\lambda, l, t)$ is defined by the equation

$$\frac{\partial}{\partial t}\tilde{S}(\lambda, l, t) + i\left(\frac{m_1}{l} + \frac{m_2}{\lambda}\right)\cdot\tilde{S}(\lambda, l, t) = 0 . \qquad (5.28)$$

Thus for the constant boundaries m_1 and m_2 one has

$$\tilde{S}(\lambda, l, t) = \tilde{S}(\lambda, l, 0)\cdot e^{-i(m_1/l + m_2/\lambda)t} . \qquad (5.29)$$

The time evolution law (5.29) of the inverse problem data the formula (5.6) and the inverse problem equations (5.7) and (5.11) allow us to solve the initial value problem for the 2DGSG-I equation by the standard IST scheme

$$\theta(\xi, \eta, 0) \longrightarrow \tilde{S}(\lambda, l, 0) \longrightarrow \tilde{S}(\lambda, l, t) \longrightarrow \theta(\xi, \eta, t). \qquad (5.30)$$

These formulae allow us also to construct the exact solutions of the 2DGSG-I equation with functional parameters which correspond to the degenerated data $\tilde{S}(\lambda, l, t)$. These solutions are the particular cases of those with functional parameters constructed in section 4 .

Note one interesting fact. Comparing the linear auxiliary problems (3.14b) and (5.27) it is not difficult to show that they can be transformed to each other with the use of the first problem (5.1). So the auxiliary linear system (3.14) at $\sigma^2 = 1$ constructed by the $\bar{\partial}$-∂ - dressing method is equivalent to the modified linear system (5.1), (5.27) for the 2DGSG-I equation with the constant boundaries.

Note also that considering instead of (5.2) the solutions of (5.1) defined in the usual manner [13, 14]

$$\Phi = \chi\cdot\begin{pmatrix} e^{i\lambda\eta} & 0 \\ 0 & e^{-i\lambda\xi} \end{pmatrix} ,$$

one will obtain the same results as above.

6. INITIAL VALUE PROBLEM FOR THE 2DGSG-II EQUATION

The 2DGSG-II equation with the constant boundaries has the form

$$\begin{gathered} \theta_{tz\bar{z}} + m\cdot\theta_{\bar{z}} + \bar{m}\cdot\theta_z + \frac{1}{4}\theta_{\bar{z}}\cdot\partial_{\bar{z}}^{-1}(\theta_z\theta_{\bar{z}})_t + \\ + \frac{1}{4}\theta_z\cdot\partial_z^{-1}(\theta_z\theta_{\bar{z}})_t = 0 , \end{gathered} \qquad (6.1)$$

where $z = x + iy$, $\bar{z} = x - iy$ and

$$\begin{gathered} (\partial_z^{-1}f)(z,\bar{z}) \overset{def}{=} \iint_C \frac{d(z',\bar{z}')\ f(z',\bar{z}')}{\bar{z}' - \bar{z}} , \\ (\partial_{\bar{z}}^{-1}f)(z,\bar{z}) \overset{def}{=} \iint_C \frac{d(z',\bar{z}')\ f(z',\bar{z}')}{z' - z} . \end{gathered} \qquad (6.2)$$

Note that equation (6.1) does not possesses the scale invariance of the type (2.13) and (2.14).

The auxiliary linear system for equation (6.1) is

$$L_1\Phi = \begin{pmatrix} \partial_{\bar z} , & -\frac{i}{2}\theta_{\bar z} \\ -\frac{i}{2}\theta_z , & \partial_z \end{pmatrix} \Phi = 0 , \qquad (6.2a)$$

$$L_{2M}\Phi = \begin{pmatrix} \partial^2_{tz} + m + \frac{1}{4}\partial_{\bar z}^{-1}(\theta_z\theta_{\bar z})_t , & 0 \\ 0 , & \partial^2_{t\bar z} + \bar m + \frac{1}{4}\partial_z^{-1}(\theta_z\theta_{\bar z})_t \end{pmatrix} \Phi +$$

(6.2b)

$$+\left(\partial_z + \partial_{\bar z} - \frac{i}{2}(\theta_z + \theta_{\bar z})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)\cdot\Phi\cdot\begin{pmatrix} im/\lambda , & 0 \\ 0 , & -i\bar m/\bar\lambda \end{pmatrix} = 0 .$$

The linear system (6.2) for real θ possesses the involution

$$\alpha\cdot L_1\cdot\alpha^{-1} = \bar L_1 ,$$

(6.3)

$$\alpha\cdot\Phi\cdot\alpha^{-1} = \bar\Phi , \quad (i= 1,2) .$$

The linear system (6.2a) is the special reduction of the wellstudied spectral problem for the DS-II equation [13-16]. In contrast to [13-16] we will consider the solutions of (6.2) of the type

$$\Phi = \chi\cdot\begin{pmatrix} e^{i\lambda z} & , 0 \\ 0 , & e^{-i\bar\lambda\bar z} \end{pmatrix} , \qquad (6.4)$$

where $\chi\longrightarrow 1$ at $\lambda\longrightarrow\infty$. The function χ obeys the equation

$$\begin{pmatrix} \partial_{\bar z} , & 0 \\ 0 , & \partial_z \end{pmatrix}\chi - \frac{i}{2}\begin{pmatrix} \bar\lambda & 0 \\ 0 & \lambda \end{pmatrix}[\sigma_3,\chi] - \frac{i}{2}\begin{pmatrix} 0 & \theta_{\bar z} \\ \theta_z & 0 \end{pmatrix}\chi = 0 , \quad (6.5)$$

and possesses the involution

$$\alpha\cdot\chi\cdot\alpha^{-1} = \overline{\chi(\lambda)} \text{ i.e. } \chi = \begin{pmatrix} \chi_{11}, & -\bar\chi_{21} \\ \chi_{21}, & \bar\chi_{11} \end{pmatrix} . \qquad (6.6)$$

The derivation of the part of the inverse problem equations for (6.5) is similar to those for the DS-II equation. By this reason we will omit the most part of intermediate calculations. The bounded Green function for equation (6.5) is

$$G = E_\lambda^{-1}\cdot\mathcal{D}^{-1}\cdot E_\lambda , \qquad (6.7)$$

where

$$\mathcal{D}^{-1} \overset{def}{=} \begin{pmatrix} \partial_{\bar z}^{-1} & 0 \\ 0 & \partial_z^{-1} \end{pmatrix} ,$$

and operator E_λ acts as follows

$$(E_\lambda\cdot B)(z) \overset{def}{=} \begin{pmatrix} B_{11}(z) , & e^{-i\lambda z - i\bar\lambda\bar z}\cdot B_{12}(z) \\ e^{i\lambda z + i\bar\lambda\bar z}\cdot B_{21}(z) , & B_{22}(z) \end{pmatrix} . \qquad (6.8)$$

The Green function (6.7) is neither analytic nor anti-analytic in the whole complex plane. So the solutions χ of the integral equation

$$\chi(z,\bar z;\lambda,\bar\lambda) = 1 - (G(\cdot, \lambda, \bar\lambda)P(\cdot)\chi(\cdot, \lambda, \bar\lambda))(z, \bar z) , \qquad (6.9)$$

which corresponds to equation (6.5) are nonanalytic and non-anti-analytic in the whole complex plane. In addition the homogeneous equation (6.9) may have nontrivial solutions. So a solution χ of equation (6.5) is of the form

$$\chi = \tilde{\chi} + \sum_i \frac{c_i \chi_i}{\lambda - \lambda_i}, \qquad (6.10)$$

where it is assumed that the solution χ of (6.5) have only simple poles and c_i are some constants which are connected with normalization of χ_i.

Using the symmetry property of the Green function

$$\begin{aligned}(G(\cdot;\lambda,\bar{\lambda})f(\cdot)\Sigma_\lambda(\cdot))(z,\bar{z}) &= \\ &= (G(\cdot;\bar{\lambda},\lambda)f(\cdot))(z,\bar{z})\cdot\Sigma_\lambda(z,\bar{z}),\end{aligned} \qquad (6.11)$$

where

$$\Sigma_\lambda \stackrel{def}{=} \begin{pmatrix} 0, & e^{i\lambda z + i\bar{\lambda}\bar{z}} \\ e^{-i\lambda z - i\bar{\lambda}\bar{z}}, & 0 \end{pmatrix}, \qquad (6.12)$$

one can show that the nonsingular part $\tilde{\chi}$ obeys the following $\bar{\partial}$-∂ - equation

$$\begin{aligned}\mathcal{D}_{\lambda,\bar{\lambda}}\cdot\tilde{\chi}(z,\bar{z};\lambda,\bar{\lambda}) &= \tilde{\chi}\begin{pmatrix} \overleftarrow{\partial/\partial\bar{\lambda}}, & 0 \\ 0, & \overleftarrow{\partial/\partial\lambda} \end{pmatrix} = \\ &= \tilde{\chi}\begin{pmatrix} 0, & \overline{\lambda\cdot F(\lambda,\bar{\lambda})}\cdot e^{i\lambda z + i\bar{\lambda}\bar{z}} \\ \lambda\cdot F(\lambda,\bar{\lambda})\cdot e^{-i\lambda z - i\bar{\lambda}\bar{z}}, & 0 \end{pmatrix},\end{aligned} \qquad (6.13)$$

where

$$F(\lambda,\bar{\lambda},t) \stackrel{def}{=} -\frac{1}{2\lambda}\iint_C d(z,\bar{z}) e^{-i\lambda z - i\bar{\lambda}\bar{z}}\cdot\theta_{\bar{z}}(z,\bar{z},t)\cdot\chi_{22}(z,\bar{z};\lambda,t). \qquad (6.14)$$

Note that the $\bar{\partial}$-∂-problem obviously possesses the involution (6.6). To find the structure of the singular part $\chi^{(S)}$ of χ one should take into account the fact each column $\chi^{(S)}$ may have its own set of poles, involution (6.6) and the symmetry (6.11). The involution (6.6) implies that

$$\chi_i^{(S)} = \begin{pmatrix} \chi_{11i}^{(S)}, & -\overline{\chi_{21i}^{(S)}} \\ \chi_{21i}^{(S)}, & \overline{\chi_{11i}^{(S)}} \end{pmatrix}, \qquad (6.15)$$

while the symmetry (6.11) implies that together with $\chi_i^{(S)}$ the matrix $a\cdot\chi_i^{(S)}(z,\bar{z})\cdot\Sigma_\lambda$ where a is an arbitrary constant, is the solution of the homogeneous integral equation (6.9) too .

It is not difficult to show that the nontrivial $\chi^{(S)}$ which obeys the involution (6.15) correspond to the choice a = 0. So we have

$$\chi^{(S)} = \sum_{i=1}^{N} \frac{c_i}{\lambda - \lambda_i}\cdot\begin{pmatrix} \chi_{11i}, & 0 \\ \chi_{21i}, & 0 \end{pmatrix} + \sum_{i=1}^{N} \frac{\bar{c}_i}{\bar{\lambda} - \bar{\lambda}_i}\cdot\begin{pmatrix} 0, & -\overline{\chi_{21i}} \\ 0, & \overline{\chi_{11i}} \end{pmatrix}. \qquad (6.16)$$

Note that for the DS-II equation such a structure of the pole terms has been proposed in [16].

So the full $\bar{\partial}$-∂ - problem for the problem (6.5) is of the form

$$\mathcal{D}_{\lambda,\bar{\lambda}}\cdot\chi = \chi\begin{pmatrix} 0, & \overline{\lambda\cdot F(\lambda,\bar{\lambda})}\cdot e^{i\lambda z + i\bar{\lambda}\bar{z}} \\ \lambda\cdot F(\lambda,\bar{\lambda})\cdot e^{-i\lambda z - i\bar{\lambda}\bar{z}}, & 0 \end{pmatrix} + \qquad (6.17)$$

$$+\sum_{i=1}^{N} c_i \pi\delta(\lambda-\lambda_i)\cdot\begin{pmatrix}\chi_{11i}, & 0\\ \chi_{21i}, & 0\end{pmatrix} + \sum_{i=1}^{N} \bar{c}_i \pi\delta(\lambda-\lambda_i)\cdot\begin{pmatrix}0, & -\overline{\chi_{21i}}\\ 0, & \overline{\chi_{11i}}\end{pmatrix} .$$

Emphasize that the matrix problem (6.17) consists from the usual $\bar{\partial}$-problem for the first column $\begin{pmatrix}\chi_{11}\\ \chi_{21}\end{pmatrix}$ and from the ∂-problem for the second column $\begin{pmatrix}\chi_{12}\\ \chi_{22}\end{pmatrix}$ which is complex conjugation of the first $\bar{\partial}$-problem. So one can restricted by the consideration only of the equation for the first column $\chi^{(1)}=\begin{pmatrix}\chi_{11}\\ \chi_{21}\end{pmatrix}$. It is

$$\frac{\partial\chi^{(1)}(\lambda,\bar{\lambda})}{\partial\bar{\lambda}} = \pi\cdot\sum_{i=1}^{N}\delta(\lambda-\lambda_i)\cdot\chi_i^{(1)} + \qquad (6.18)$$
$$+\ \lambda\cdot F(\lambda,\bar{\lambda})\cdot e^{-i\lambda z\ -i\bar{\lambda}\bar{z}}\cdot\alpha\cdot\overline{\chi^{(1)}(\lambda,\bar{\lambda})}\ ,$$

where we normalize $\chi_i^{(1)}$ as $\chi_i^{(1)}\xrightarrow[|z|\to\infty]{}\frac{i\lambda_i}{z}\begin{pmatrix}1\\0\end{pmatrix}$, i.e.$c_i = i\lambda_i$.

The generalized Cauchy formula gives

$$\chi^{(1)}(\lambda,\bar{\lambda}) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{i=1}^{N}\frac{i\lambda_i\cdot\chi_i^{(1)}(z,\bar{z})}{\lambda-\lambda_i} + \qquad (6.19)$$
$$+\iint_C \frac{d(\lambda',\bar{\lambda}')}{\lambda'-\lambda}\lambda'\cdot F(\lambda',\bar{\lambda}')\cdot e^{-i\lambda' z\ -\ i\bar{\lambda}'\bar{z}}\alpha\cdot\overline{\chi^{(1)}(\lambda',\bar{\lambda}')}\ .$$

Equation (6.19) is the part of the inverse problem equations. To derive the rest of such equations one should

proceed in equation (6.19) to the limit $\lambda\longrightarrow\lambda_k$ to obtain the system of equations for $\chi_i^{(1)}$. To do this we must calculate the limit

$$\lim_{\lambda\to\lambda_i}\left(\chi^{(1)} - \frac{i\lambda_i\chi_i^{(1)}}{\lambda-\lambda_i}\right) . \qquad (6.20)$$

Completely similar to the DS equation case [14] one can show that the quantity

$$\lim_{\lambda\to\lambda_i}\left(\chi^{(1)} - \frac{i\lambda_i\chi_i^{(1)}}{\lambda-\lambda_i}\right) - \lambda_i z\cdot\chi_i^{(1)}\ , \qquad (6.21)$$

also is the solution of the homogeneous equation (6.9) provided the certain constraints on θ and $\chi_i^{(1)}$. Therefore the expression (5.21) is the linear superposition of the two independent solutions of the homogeneous equation (5.9), i.e.

$$\lim_{\lambda\to\lambda}\left(\chi^{(1)} - \frac{i\lambda_i\chi_i^{(1)}}{\lambda-\lambda_i}\right) - \lambda_i z\cdot\chi_i^{(1)} = \qquad (6.22)$$
$$= \gamma_i\chi_i^{(1)} + \mu_i\cdot\alpha\cdot\overline{\chi_i^{(1)}}\ .$$

where γ_i and μ_i are some constants. Note that the observation that the l.h.s. of the relation of the type (6.22) should be the superposition of the two independent solutions has been made for the first time in [16] for the DS-II equation.

Now, proceeding to the limit $\lambda \longrightarrow \lambda_k$ in equation (6.19) and taking into account the identity (6.22), we obtain

$$(\lambda_k z + \gamma_k)\cdot\chi_k^{(1)} + \mu_k\cdot\alpha\cdot\bar{\chi}_k^{(1)}\cdot e^{-i\lambda_k z - i\bar{\lambda}_k\bar{z}} = \tag{6.23}$$

$$= \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j\neq k}^{N} \frac{i\lambda_j\chi_j^{(1)}}{\lambda_j - \lambda_k} + \iint_C \frac{d(\mu,\bar{\mu})}{\mu - \lambda_k}\,\mu\cdot F(\mu,\bar{\mu})\cdot e^{-i\mu z - i\bar{\mu}\bar{z}}\cdot\alpha\cdot\overline{\chi^{(1)}(\mu,\bar{\mu})}, \quad (k = 1,\ldots,N) .$$

At last, the reconstruction formula for the potential θ is given by

$$\theta_z = 2\lim_{\lambda\longrightarrow\infty}(\lambda\cdot\chi_{21}) = \tag{6.24}$$

$$= 2i\cdot\sum_{k=1}^{N}\lambda_k\chi_{21k} - 2\iint_C d(\mu,\bar{\mu})\mu\cdot F(\mu,\bar{\mu})\cdot e^{-i\mu z - i\bar{\mu}\bar{z}}\cdot\overline{\chi_{11}(\mu,\bar{\mu})} .$$

The formulae (6.19), (6.23) and (6.24) form the complete set of the inverse problem equations for the linear problem (6.5). The quantities $\{F(\lambda,\bar{\lambda}),\ \lambda_i,\ \gamma_i,\ \mu_i,\ (i=1,\ldots,N)\}$ are the inverse problem data.

Now one should find the time evolution of the inverse problem data. Substituting (6.18) and (6.22) into (6.2b) and considering the limit $|z|\longrightarrow\infty$, one obtains

$$\frac{dF}{dt} = -\,i(m/\lambda + \bar{m}/\bar{\lambda}) ,$$

$$\frac{d\lambda_i}{dt} = 0 , \quad \frac{d\gamma_i}{dt} = 0 , \tag{6.25}$$

$$\frac{d\mu_i}{dt} = -i(\,m/\lambda_i + \bar{m}/\bar{\lambda}_i) , \quad (i = 1,\ldots, N) .$$

So

$$F(\lambda,\bar{\lambda},t) = F(\lambda,\bar{\lambda},0)\cdot e^{-i(m/\lambda + \bar{m}/\bar{\lambda})t} ,$$

$$\gamma_i(t) = \gamma_i(0) , \tag{6.26}$$

$$\mu_i(t) = \mu_i(0)\cdot e^{-i(m/\lambda_i + \bar{m}/\bar{\lambda}_i)t} , \quad (i = 1,\ldots,N) .$$

The inverse problem equations (6.19), (6.23), (6.24) and time evolution (6.26) reduce the solution of the initial value problem for the 2DGSG-II equation to the standard set (5.30) of the linear problems.

As usual, for the pure discrete data $(F(\lambda,\ \bar{\lambda}) = 0)$ the inverse problem equations are reduced to the system of the linear algebraic equations and can be solved explicitly. The corresponding solutions with the real valued θ are not found.

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