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LEVEL DENSITY OF RANDOM MATRICES FOR DECAYING SYSTEMS

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НОВОСИБИРСК

Level density of random matrices for decaying systems

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ABSTRACT

We present analytical and numerical results for the level density of a certain class of random non-Hermitian matrices $\mathcal{H} = H + i\Gamma$. The conservative part H belongs to the Gaussian orthogonal ensemble while the damping piece Γ is quadratic in Gaussian random numbers and may describe the decay of resonances through various channels. In the limit of a large matrix dimension the level density assumes a surprisingly simple dependence on the relative strength of the damping and the number of channels.

Even though quantum mechanics is based on the notion of the Hamiltonian as a Hermitian operator, non-Hermitian operators are also enjoying applications. They are all related to damping or decay processes. While a microscopic treatment of dissipative motion always involves Hermitian Hamiltonians, it is often convenient to give a reduced description using generalized, non-Hermitian "Hamiltonians".

Roughly speaking, non-Hermitian generalizations of Hamiltonians $\mathcal{H}$ find three classes of applications. One arises in master equations of dissipative quantum systems[1-3], $i\hbar\dot{\rho}(t) = \mathcal{H}\rho(t)$, as generators of infinitesimal time translations for the density operator $\rho(t)$. While the Hermitian part of $\mathcal{H}$ refers to free undamped motion, the remainder describes a damping imposed on the system by some external "heat bath" which has otherwise been eliminated. Probability conservation is respected as tr $\mathcal{H}\rho = 0$. The eigenvalues $\mathcal{E} = x+iy$ of $\mathcal{H}$ are in general complex; their imaginary parts must be non-positive for damped systems and give the life times of the corresonding eigenmodes as $1/y$; the real parts x, on the other hand, become the differences of pairs of eigenenergies in the limit of zero damping. Such generators $\mathcal{H}$ can often be constructed by adiabatically eliminating the heat bath from the underlying microscopic description. The damped harmonic oscillator and spin relaxation are well-known examples of this kind[1-3].

A second type of application is concerned with the evolution of incomplete state vectors rather than density operators. To explain a typical such situation let us consider an state vector ψ, its representation in a complete basis, and its Schrödinger equation $i\hbar\dot{\psi} = H\psi$. A subset of basis vectors may have special significance and the correspondingly truncated part ψ_{trunc} of ψ merit separate consideration. In some such cases ψ_{trunc} obeys, at least approximately, an evolution equation of the form $i\hbar\dot{\psi}_{\text{trunc}} = \mathcal{H}\psi_{\text{trunc}}$ where $\mathcal{H}$ is non-Hermitian and, of course, smaller in dimension than H. Probability conservation is then not respected by $\psi_{\rm trunc}$ and $\mathcal{H}$ since there will in general be a probability flow connecting the two parts of the Hilbert space. The decomposition of ψ in ψ_{trunc} and an irrelevant remainder $\psi - \psi_{trunc}$ may be suggested by the structure of the original Hamiltonian: an unperturbed part H_0 of H may have a discrete and a continuous part in its spectrum, and the discrete part may furnish the truncated Hilbert space of interest. Excited atoms, decaying to their stable ground states through spontaneous radiation, can be described in this fashion[4]. Long-lived resonances in the continuum of nuclei, atoms, and molecules provide other examples[5].

A third, highly interesting class of applications is met with in scattering problems. Resonance structures in the energy dependence of scattering cross sections can be related to poles of the scattering matrix in the complex energy plane. In certain phenomenological representations[6, 7] of the S matrix such complex poles appear as the eigenvalues of some non-Hermitian matrix $\mathcal{H}$. The real part x of a complex eigenvalue $\mathcal{E} = x + iy$ of $\mathcal{H}$ specifies the location of the resonance along the real energy axis while the imaginary part gives the width.

Our reason for mentioning the above three types of non-Hermitian operators lies in the suspicion that they have certain statistical properties of their spectra in common. For instance, cubic level repulsion is expected in general, provided the respective dynamics are fully chaotic in the classical limit. This behavior is characteristic for Gaussian ensembles of non-Hermitian matrices[8] and has been verified for generators of strongly damped quantum systems under conditions of classical chaos[9−11]; more recently, it has been seen for the poles of the S matrix in irregular scattering[12]. Anticipating some degree of universality in the statistics of levels and eigenvectors of non-Hermitian operators we consider appropriate the study of various ensembles of random matrices.

We propose to discuss a particular class of random $N \times N$ matrices,

$$\mathcal{H} = H + i\Gamma \tag{1}$$

where H is drawn from the Gaussian orthogonal ensemble (GOE)[13, 11] and Γ a real symmetric matrix of the form

$$\Gamma_{k\ell} = -\gamma \sum_{a=1}^{M} A_k^a A_\ell^a \tag{2}$$

Since Γ consists of M separable pieces, each defined by a real N component vector A^a, one speaks of M open decay channels. This model was recently

studied[14, 15] for applications to nuclei where $\mathcal{H}$ yields the poles of some S matrix. We shall take the A_k^a as independent random numbers with identical Gaussian distributions of zero mean. The widths of the various Gaussians are chosen as

$$\langle (A_k^a)^2 \rangle = 1/N \ , \ \langle (H_{k\ell})^2 \rangle = \begin{cases} 1/N & \text{for} \quad k \neq \ell \\ 2/N & \text{for} \quad k = \ell \, . \end{cases} \tag{3}$$

First attempts at describing the spectrum of $\mathcal{H}$ as given by (1-3) have either assumed $M = 1, 2$ or employed perturbation theory[14, 15] in the limits $\gamma \to 0$ (the near conservative case) and $\gamma \to \infty$ (the limit of strongly overlapping resonances). We shall now present rigorous results for arbitrary γ. Some remarks on the method of derivation will be given further below. Denoting the eigenvalues of $\mathcal{H}$ by $\mathcal{E} = x + iy$ we discuss the level density $\rho(x, y)$ in the complex plane in the limit when both the dimension N and the number of channels M are large, keeping the ratio $m = M/N$ finite. The density is then nonzero only within a finite region defined by

$$x^2 \leq -\frac{4m}{y\gamma} - \left(\frac{1}{1/\gamma - y} + \frac{m}{y} - \frac{1}{\gamma} \right)^2 , \qquad y < 0, \tag{4}$$

the equality sign in the first of these inequalities yielding the boundary. Inside that region the density turns out independent of the real part x of the complex "energy" $\mathcal{E}$ and takes the form

$$4\pi\rho(x, y) = 1 + \frac{m}{y^2} - \left(\frac{1}{\gamma} - y \right)^{-2} . \tag{5}$$

For the weakest of dampings, $\gamma \to 0$, the eigenvalues populate a narrow strip immediately below the real axis (center at $\bar{x} = 0$, $\bar{y} = -m\gamma$; half widths $\delta x = 2$, $\delta y = 2m\gamma^2$). Note the overlapping of resonances, $\delta x/N \ll |\bar{y}|$.

For very strong damping, $\gamma \to \infty$, the region defined by (4) actually consists of two separate ones. The fraction m of eigenvalues takes off down the lower half plane, forming a cloud with center $\bar{x} = 0$, $\bar{y} = -\gamma(1 + m)$ and half widths $\delta x = \min(1/\gamma, 2\sqrt{m}/\gamma(1-m))$, $\delta y = 2\gamma\sqrt{m}$. The other $N(1-m)$ eigenvalues, however, remain in a cloud which approaches an interval of the real axis as $\bar{x} = 0, \bar{y} = -m/\gamma(1-m)$, $\delta x = 2(1-m)^{\frac{1}{2}}$, $\delta y = 2m/\gamma^2(1-m)^{\frac{5}{2}}$.

Fig. 1 illustrates the transition between the two limits just mentioned; especially, the segmentation of the single cloud (for $\gamma^{-\frac{2}{3}} + m^{\frac{1}{3}} > 1$) into two clouds (for $\gamma^{-\frac{2}{3}} + m^{\frac{1}{3}} < 1$) may be visualized from the sequence of portraits $(a-c)$. The curves in these portraits depict boundaries according to (4). The

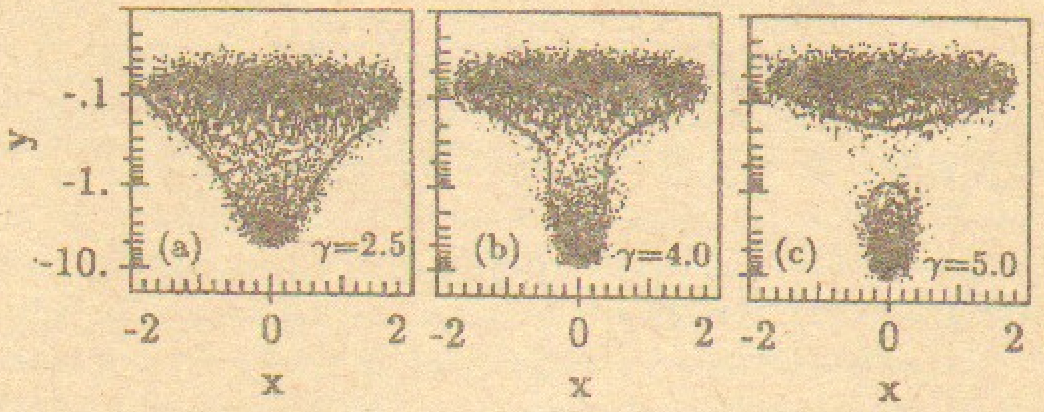

Fig. 1. Eigenvalues of 50 random matrices of the structure (1, 2, 3) with N=100, M=50 and γ=2.5 (a), γ=4.0 (b), γ=5.0 (c). The closed curves represent the contour (4). The separation of one cloud into two takes place for $\gamma \approx$4.44, i.e. in between case (b) and case(c).

clouds of points represent eigenvalues of 50 matrices of the structure (1,2) with $N = 200$, $M = 50$, and $\gamma = 2.5$ (case a), $\gamma = 4$ (case b), $\gamma = 5$ (case c); the matrix elements were drawn from the ensembles described above, and the eigenvalues determined numerically. Obviously, the asymptotic formulas (4,5) begin to work well at moderately large values of N, M.

The foregoing conclusion is further corroborated by Fig. 2 where we display the projection of the level density onto the imaginary axis, $\int dx \rho(x, y)$, for the situation of Fig. 1(b). The dip of the reduced density near $y \approx -0.5$ is a precursor of the desegregation of one cloud into two which would arise

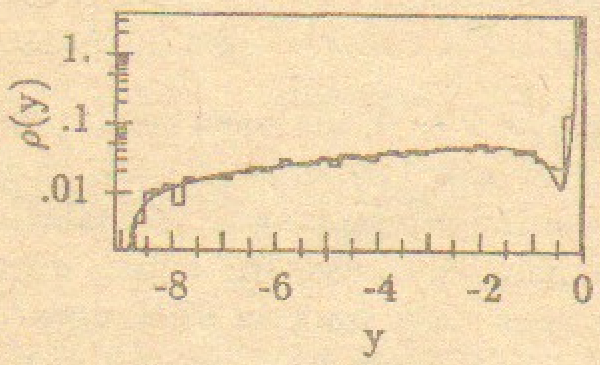

Fig. 2. Level density projected on the imaginary axis. Histogram from random matrices as in Fig. 1(b), smooth curve from (4, 5).

at a slightly larger value of γ. Incidentally, the analytic result for the desegregation point, $\gamma^{-\frac{2}{3}} + m^{\frac{1}{3}} = 1$, is well respected by all of our data.

Fig. 3 refers to $N = M$ channels open. The eigenvalues then form but

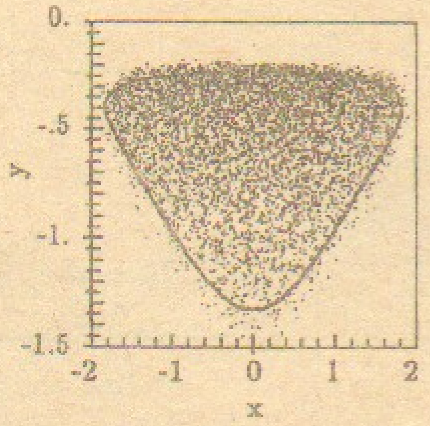

Fig. 3. As in Fig. 1, but $M = N = 200$, $\gamma = 0.5$. Only a single cloud of eigenvalues arises since all decay channels are open.

a single cloud, irrespective of the value of γ. The cloud was again obtained numerically for $N = 200$ and fits reasonably into the interior of the contour derived from (4). The damping strength was chosen as $\gamma = 0.5$.

In Fig. 4 we explore the limits of validity of the asymptotic formulas (4, 5), taking $N = 200$ as before but allowing for only two open channels, $M = 2$. For the damping strength chosen, $\gamma = 2.5$, we encounter two well separated clouds[16] (see Fig. 4(a)). The lower one contains 100 points, two for each of the 50 random matrices picked; this cloud is faithful to the asymptotic (N and M large!) result beyond resonable expectation, as is especially obvious from the reduced density $\int dx \rho(x, y)$ in Fig. 4(c). Even the upper cloud, nestled against the real axis, has its center and half width not too badly approximated by the asymptotic prediction. An interesting discrepancy is visible in Fig. 4(b), though: While the (not applicable) asymptotic result (4) implies an empty gap between the allowable region for $m = .01$ and the real axis, the reduced density $\int dx \rho(x, y)$ of the cloud seems to increase exponentially towards a finite value at the real axis. That discrepancy is expected on perturbative grounds since for the value of γ chosen H is a small perturbation to $i\Gamma$. The latter operator has $M = 2$ imaginary eigenvalues $\mathcal{E}_{1,2}^{(0)}$ of order γ and the eigenvalue 0 which is $(N - M)$ fold degenerate. Diagonalization of H in the degenerate subspace yields real approximants $\mathcal{E}_\mu^{(0)}$ of order γ^0 to the previously vanishing eigenvalues. The next corrections arise in second order, $\mathcal{E}_\mu^{(2)} = \sum_{n=1,2} H_{\mu n}^2/(\mathcal{E}_\mu^{(0)} - \mathcal{E}_n^{(0)})$, whose real parts are of the order $1/\gamma^2$ and thus hardly important. The imaginary parts $y_\mu^{(2)}$, however, are $\propto 1/\gamma$ and determine the separation of the cloud in question from the real axis. Now inasmuch as the $-y_\mu^{(2)}$ are the sums of squares of two independent random numbers, one expects the $-y_\mu$ to have a χ_M-

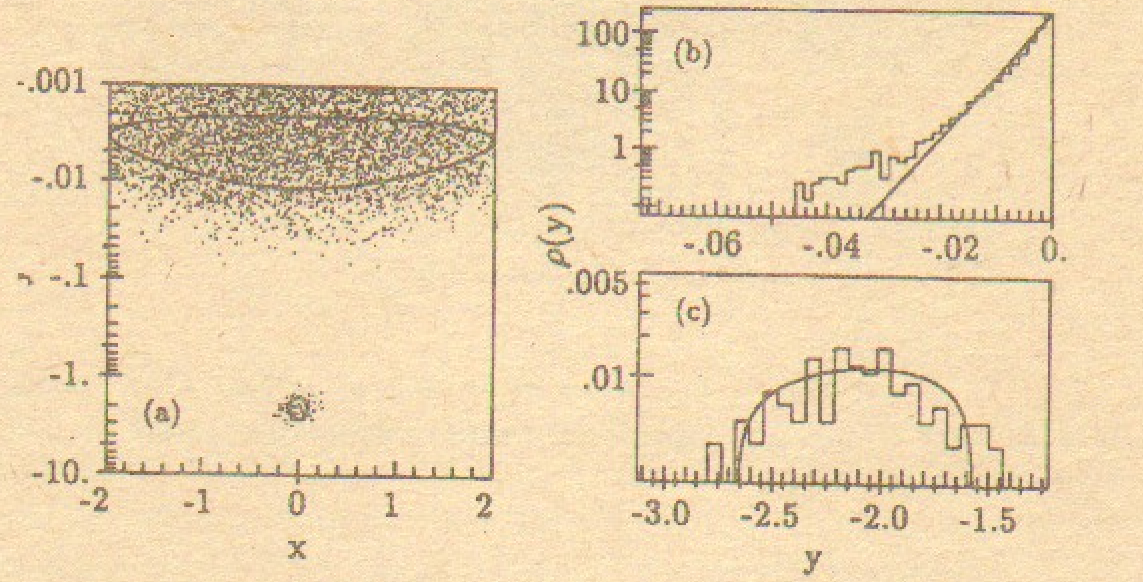

Fig. 4. (a) As Fig. 1, but $N = 200$, $M = 2$, $\gamma = 2.5$. Note that the upper cloud does not respect the gap from the real axis predicted by the asymptotic formula (4). (b) The projected eigenvalue density of the upper cloud approches an exponential (smooth curve) close to the real axis, $y \to 0$; parameters as in (a). (c) The projected eigenvalue density of the lower cloud conforms to the asymptotic prediction (smooth curve) of (4, 5).

square distribution with $M = 2$. The logarithm of that distribution displays a linear approach towards a finite value as $y \to 0$. The latter behaviour is indeed borne out by the histogram in Fig. 4(b). It may be well to add that the foregoing perturbative argument is in accord with other predictions[14, 15]. In the case of large γ and for one open channel the width distribution for $N-1$ long-lived states obeys the χ_M-square distribution with $M = 1$ (Porter-Thomas distribution). Reasoning similarly, one predicts the density going to zero for $y \to 0$ if $M \geq 3$. To summarize, the asymptotic results begin to be quantitatively reliable for $M \geq 3$, i.e. rather earlier than could be hoped.

Two special limits of (4, 5) merit mention due to their simplicity and since they constitute immediate generalizations of the perturbative results. First, for small damping and/or a small fraction m of open channels one finds semicircle laws for the reduced densities of the upper cloud

$$2\pi \int dy \rho(x, y) = \sqrt{4 - x^2}$$

$$2\pi \int dx \rho(x, y) = \frac{m}{y^2}\sqrt{4 - \left(\frac{m}{y} + \gamma + \frac{1}{\gamma}\right)^2} \quad (6)$$

$$\text{for} \quad m \ll \left(1 - \frac{1}{\gamma}\right)^2 .$$

The result for the density projected onto the real axis is identical to the GOE result for the Hermitian part of the Hamiltonian above. On the other hand, for large damping we have

$$2\pi \int_{\text{upper cloud}} dy \rho(x, y) = \sqrt{4(1-m) - x^2}$$

$$2\pi \int_{\text{lower cloud}} dx \rho(x, y) = -\frac{1}{y}\sqrt{4m - \left(\frac{y}{\gamma} + 1 + m\right)^2} \quad (7)$$

$$\text{for} \quad \frac{2}{\gamma} \ll (1-m)^{\frac{3}{2}} .$$

Again, the density for the lower cloud projected onto the imaginary axis is identical to the density of the non-zero eigenvalues of the non-Hermitian part of the Hamiltonian above. This density is an interesting new result by itself.

We would like to conclude with a remark on the derivation of (4, 5). As in Refs. 14, 15, 18 we employ a formal analogy to a two dimensional electrostatic problem, $4\pi\rho(x, y) = -\Delta\phi(x, y)$, and an identity relating the potential ϕ to the ensemble-averaged Green's function through $\phi = \langle -(1/N) \ln \det [(\mathcal{H} - \mathcal{E})(\mathcal{H} - \mathcal{E})^\dagger + 0^+] \rangle$. The potential can be evaluated with the help of the replica trick or supersymmetric methods. The method of Ref. 18 is extended here to a non-Gaussian distribution of matrix elements of the Hamiltonian $\mathcal{H}$. It may be worth mentioning that the boundary (4) results from a careful consideration of the positive infinitesimal 0^+ in the definition of ϕ. Outside the boundary ϕ becomes the real part of an analytic function of $\mathcal{E} = x + iy$.

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