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3D REGGE GRAVITY AS AN EXACTLY SOLUBLE SYSTEM

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3D Regge Gravity as an Exactly Soluble System

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ABSTRACT

The 2+1 dimensional Regge gravity is quantised canonically in the triad-connection variables in the limit of continuous time. The first class constraints are generators of gauge group of the system and form the algebra w.r.t. the commutators. The general form of the Euclidean functional integral in the full discrete theory is described which leads upon Wick rotation to the canonical quantisation anzats in the continuous time limit whatever direction is chosen as time. It is parameterized and specified by the choice of a subset of links fixed by hand. Some interesting consequences of this formalism are discussed, in particular, mechanism of arising divergences in the continuum theory, nonquantisability of timelike links and one-dimensionality of time in the pseudoRimannian manifolds.

1. INTRODUCTION

The (2+1)D general relativity is recently known as an exactly soluble system on both classical and quantum levels [1-4]. The graviton is absent in three dimensions and the only nontrivial content of the theory may be topological degrees of freedom as shown by Witten [4]. Witten has also shown in this work that (2+1)D gravity possesses renormalizable perturbative expansion. An issue point in this analysis was the triad-connection form of gravity action,

$$I = \frac{1}{2}\int \varepsilon^{\mu\nu\lambda}\varepsilon_{abc} e^{a}_{\mu} (\partial_{\nu}\omega^{bc}_{\lambda} - \partial_{\lambda}\omega^{bc}_{\nu} + [\omega_{\nu}, \omega_{\lambda}]^{bc})\, d^{3}x. \qquad (1)$$

Here $\omega^{ab}_{\mu} =: \varepsilon^{abc}\omega_{\mu c}$ is SO(2,1) connection, e^{a}_{μ} is a triad, the frame (Latin) and world (Greek) indices run over 0, 1, 2; $\varepsilon^{012}=+1$. By setting $\pi^{ia} = 2\varepsilon^{ij}e^{a}_{j}$, $\omega^{a}_{0} = -h^{a}$, $e^{a}_{0} = -N^{a}/2$ where $i, j=1, 2$, $\varepsilon^{12}=+1$ this can be brought to the canonical form

$$I = \int \vec{\pi}^{i}\vec{\omega}_{i}\, (\partial_{i}\vec{\pi}^{i} + \omega_{i}\times\vec{\pi}^{i})\, d^{3}x, \qquad (2)$$

(vector notation refers to the frame indices). Here

$$\vec{C}(\vec{h}) = \int \vec{h}\,(\partial_i \vec{\pi}^i + \vec{\omega}_i \times \vec{\pi}^i)\, d^3x, \tag{3a}$$

$$\vec{R}(\vec{N}) = \int \vec{N}(\partial_1 \vec{\omega}_2 - \partial_2 \vec{\omega}_1 + \vec{\omega}_1 \times \vec{\omega}_2)\, d^3x, \tag{3b}$$

are the constraints whose P.B. (Poisson brackets; signs are such that symbolically $\{\omega, \pi\} = +1$) obey the Lie algebra of ISO(2,1) (rotations and translations) group:

$$\begin{aligned} \{\vec{C}(\vec{h}), \vec{C}(\vec{h}')\} &= \vec{C}(\vec{h} \times \vec{h}'), \\ \{\vec{C}(\vec{h}), \vec{R}(\vec{N})\} &= \vec{R}(\vec{h} \times \vec{N}), \\ \{\vec{R}(\vec{N}), \vec{R}(\vec{N}')\} &= 0. \end{aligned} \tag{4}$$

These first class constraints require the same number of gauge conditions to be imposed; since this number locally coincides with that of canonical pairs (π, ω) the local degrees of freedom are absent.

It is interesting to ask if there is an analogous formalism in the (continuous time) Regge calculus [5] which is the general relativity for the piecewise-flat manifolds. Regge calculus in three dimensions has been considered in a number of works [6-11]. The main problem is that the constraints become generally not first class ones. (This is the problem inherent in usual lattice discretization of ordinary field theories). Then the procedure of counting the degrees of freedom changes and some superfluous ones can arise usually referred to as 'lattice artefacts'. This means that there is neither smooth transition to the continuum theory nor a regular perturbative expansion about flat

spacetime (the criterium of smoothness can be taken as $a^2R \ll 1$ where a is a typical lattice spacing and R is a typical curvature of the triangulated smooth manifold; from this it is seen that flat and smooth limits in Regge calculus are equivalent).

Here we formulate $(2+1)D$ continuous time Regge calculus so that it leads to the system of first and second class constraints similar to those suggested by Waelbroeck [12]. For that we use the triad-connection formulation of Regge calculus. The triad-connection formalism (together with $4D$ such formalism) takes it's origin in the works by Bander [13, 10] and was further developed in the works [14-16]. Besides, we perform some further extension of the phase space. The resulting first class constraints are generators of all the continuous symmetries of the system and form the algebra of commutators analogous to (4). Thereby as shown by Waelbroeck only global degrees of freedom are present identical to those in the continuum theory.

The question of local degrees of freedom is not so simple if we return to the completely discrete theory (and this is the thing which we are going to do), because there is neither canonical formalism nor the notion of the degree of freedom without notion of the continuous time. But remarkable is that the Euclidean functional integral measure exists which results in the canonical quantisation if we choose any field of directions in Regge manifold, make Wick rotation and perform the continuous time limit by bringing the neighbouring 2*D* leaves of the foliation (2*D* Regge submanifolds themselves) together along these directions. All possible such measures form the family each element of which corresponds to a definite choice of physical measurement (triangulation) process. Namely, Regge links of some field of directions should be fixed by hand while others fluctuate quantum-mechanically. Local triviality of 3*D* gravity shows up in the fact that in special case of vanishing link lengths (it is the case of the most detailed triangulation) the functional integral is up to boundary effects the product of independent measures on separate links. This factorization takes place for quite general structure of linking (supposedly for any structure), and these measures have universal form. They are positively defined and thus adopt probabilistic interpretation. The VEV (vacuum expectation value) of link length raised to any power exceeding (-2) is finite and nonzero. Moreover, if analytical continuation to pseudoEuclidean signature is made, invariant intervals for links fluctuate still in the spacelike region. This means that the only possibility for the link being timelike is to fix it by hand. In other words, time is not quantised and, due to one-dimensionality of the field of directions along which links are fixed, time is no more than one-dimensional.

The paper is organized as follows. In the next section

triad-connection formulation is introduced and continuous time limit is considered. Canonical quantisation in the form of the functional integral is obtained in sect. 3. In sect. 4 the general form of the full discrete functional integral is found, it's properties are studied and it's continuous time limit is shown to correspond to the canonical quantisation. The results are discussed in sect. 5 where also some comparison with the continuum theory is made.

2. THE TRIAD-CONNECTION 3D REGGE GRAVITY.

Let us consider 3*D* Regge manifold of the Euclidean signature (+,+,+); the problem of analytical continuation to the timelike signature will be considered in subsection 2.3.

2.1. Full discrete theory

Here we collect some results concerning the triad-connection formulation of Regge calculus [10, 13-16]. The set of variables are link vectors $l^a_{\sigma^1}$ and connection matrices $\Omega^{ab}_{\sigma^2}$ which are finite SO(3) rotations. The subscripts σ^n label *n*-simplices (vertices, links, triangles and tetrahedra for $n = 0, 1, 2, 3$ respectively) of the Regge manifold. The vector $l^a_{\sigma^1}$ is defined in a local frame corresponding to some tetrahedron containing the link σ^1. Connection $\Omega^{ab}_{\sigma^2}$ relates the frames of the two neighbouring tetrahedra σ^3_1, σ^3_2 sharing the triangle σ^2. According to whether Ω_{σ^2} transforms vectors from σ^3_1 to σ^3_2 or vice versa we shall speak of Ω_{σ^2} as being oriented from σ^3_1 to σ^3_2 or inversely, respectively. Also to each link σ^1 the curvature matrix $R^{ab}_{\sigma^1}$ can be attributed which is the product of connections ordered along the loop enclosing this (and only this) link.

Although the following analysis is valid for rather general Regge manifold, it is convenient from the notational point of view to illustrate it by the example of Regge manifold of a particular structure. Namely, let the Regge simplicial complex be topologically equivalent to the collection of cubes each divided into six tetrahedra, as in [17]. To label the simplices is sufficient to do this for the simplices of a cube erected from it's definite vertex and obtain others by translations. The links of a cube will be denoted by multiindex M running over seven unordered combinations of unequal indices α = 1, 2, 3 (the edges), $\alpha\beta$= =12, 23, 31 (diagonals of the faces spanned by the edges α,β), $\alpha\beta\gamma$ = 123 (body diagonal). The twelve triangles will then be labelled by the ordered pair of multiindices MN with empty intersection: it means the triangle spanned by the links M and MN. The objects and quantities referred to the neighbouring cubes can be obtained by action of the operator of translation T_M along the link M.

The specification of Regge action amounts to attributing to each link M the tetrahedron in which $\vec{l}_M$ and R_M are def-

ined and choosing orientations of the connections for each triangle. The action takes the form

$$I = \sum_{\text{cubes}} \sum_M l_M \sin^{-1} \frac{1}{2l_M} \vec{l}^{\,*}_M R_M , \qquad (5)$$

$$\vec{l}^{\,*} R := l^a \varepsilon_{abc} R^{bc} .$$

The sence of this is that upon excluding Ω's with the help of equations of motion it becomes indeed the Regge action in terms of purely link lengths [14]. There is some ambiguity in this equation[1)]. If R is a rotation around $\vec{l}$ by an angle φ then $\sin^{-1}$ in (5) reads

$$\sin^{-1}(\sin\varphi) \qquad (6)$$

which is φ or $\pi-\varphi$. Of these only φ leads to the known Regge calculus expression for the action. Therefore one should be careful dealing with this action and consider only that branch of $\sin^{-1}$ which is small for connections Ω close to unity.

Having chosen one such representation (5) we can pass to another one by means of the following discrete symmetry transformations:

$$\vec{l}_M \longrightarrow \bar{\Gamma}_M \vec{l}_M , \qquad (7)$$
$$R_M \longrightarrow \bar{\Gamma}_M R_M \Gamma_M ,$$

for some links M and

1) I am grateful to referee of my previous work [14] who has drawn my attention to this circumstance.

$$\Omega_{MN} \longrightarrow \bar{\Omega}_{MN} , \qquad (8)$$

for some triangles MN. The Γ_M is the product of matrices taken along *the part* of the loop enclosing link M: $R_M = \Gamma_M \dots$ Consider the choice close to the previous my work [14]:

$$R_1 = \bar{\Omega}_{12}(\bar{T}_3\bar{\Omega}_{31})(\bar{T}_{23}\Omega_{d1})(\bar{T}_2\Omega_{21})\Omega_{13}\Omega_{1d} ;$$
$$R_2 = \bar{\Omega}_{23}(\bar{T}_1\bar{\Omega}_{12})(\bar{T}_{31}\Omega_{d2})(\bar{T}_3\Omega_{32})\Omega_{21}\Omega_{2d} ;$$
$$R_3 = \bar{\Omega}_{31}(\bar{T}_2\bar{\Omega}_{23})(\bar{T}_{12}\Omega_{d3})(\bar{T}_1\Omega_{13})\Omega_{32}\Omega_{3d} ;$$
$$R_{23} = \bar{\Omega}_{32}(\bar{T}_1\Omega_{1d})\Omega_{23}\Omega_{d1} ;$$
$$R_{31} = \bar{\Omega}_{13}(\bar{T}_2\Omega_{2d})\Omega_{31}\Omega_{d2} ;$$
$$R_{12} = \bar{\Omega}_{21}(\bar{T}_3\Omega_{3d})\Omega_{12}\Omega_{d3} ;$$
$$\bar{R}_{123} = \Omega_{2d}\Omega_{d1}\Omega_{3d}\Omega_{d2}\Omega_{1d}\Omega_{d3} ; \qquad (9)$$

index $1d$ (d means 'diagonal') stands for MN at $M = 1$, $N = 23$ etc.; $\bar{T}_M = T_M^{-1}$.

2.2. Continious time limit

The aim is now to pass to the continuous time limit. Let us choose direction of links 3 as a future candidate for the timelike one and begin to bring together the neighbouring $2D$ leaves along this direction. The link vectors are

$$\vec{l}_{A3} := \vec{l}_A + \varepsilon\vec{N}_A ,$$
$$\vec{l}_3 := \varepsilon\vec{N} , \qquad (10)$$

where $A = 1, 2, d$ (:= 12) labels links in the $2D$ layer and $\varepsilon \longrightarrow 0$. As for the connections Ω_{MN} one should differ between those on the timelike triangles $MN = A3$, $3A$ and diagonal $MN = id$, di, $i=1, 2$ and spacelike ones $MN = 12, 21$. (Let us call these timelike, diagonal and spacelike connections, respectively). Of them diagonal and spacelike ones are analogs of the continuum connection for the parallel vector transport at a distance $O(\varepsilon)$. Therefore these connections are infinitesimally close to unity:

$$\Omega_{MN} = 1 + \varepsilon f_{MN}, \; MN = 12, 21, 1d, d1, 2d, d2, \qquad (11)$$
$$\bar{f}_{MN} = -f_{MN} .$$

The action should be the sum of $O(\varepsilon)$ terms and upon identification $\varepsilon = dt$ be reducable to integral over dt. First consider contribution from the diagonal link $A3$. Up to $O(\varepsilon)$ terms corresponding curvature is the product of two appropriately oriented timelike connections Ω_{A3}, Ω_{3A}:

$$R_{13} = \bar{\Omega}_{13}\Omega_{31}, \; R_{23} = \bar{\Omega}_{32}\Omega_{23}, \; \bar{R}_{123} = \Omega_{3d}\Omega_{d3} . \qquad (12)$$

With the same accuracy T_3 is equal to unit operator and

$$R_1 = \bar{R}_{13}, \; R_2 = \bar{R}_{23}, \; R_{12} = \bar{R}_{123} . \qquad (13)$$

Therefore $O(1)$ contributions from spacelike and diagonal links cancel each other. As for the $O(\varepsilon)$ terms, there may be those proportional to $\vec{N}_A$. They are given by contribution of link $A3$ and are proportional to the $O(1)$ part of R_{A3} (12). Therefore variation of these terms w. r. t. $\vec{N}_A$ gives $R_A - \bar{R}_A =$

$= O(\varepsilon)$, that is, $R_A = 1 + O(\varepsilon)$ and[2)]

$$\Omega_{31} = \Omega_{13} =: \Omega_1,\ \Omega_{32} = \Omega_{23} =: \Omega_2,\ \bar{\Omega}_{3d} = \Omega_{d3} =: \Omega_d \qquad (14)$$

with the same accuracy.

Consider other terms in the arising Lagrangian L. Time translation is converted into the time derivative of Ω_1, Ω_2, Ω_d. The infinitesimal connections f_{MN} enter L being summed over infinitesimal prisms whose bases are the triangles AB = = 12, 21 and T_3AB:

$$h_{12} = -f_{12} + f_{d2} + f_{1d},\ h_{21} = -f_{21} - f_{d1} - f_{2d}\ . \qquad (15)$$

These equations give infinitesimal connections for parallel vector transport to the neighbouring leave. The h_{AB} become multipliers at the constraints expressing the closure of the spacelike triangles. They are discrete version of the Gaussian constraints. Finally, contribution of the timelike link 3 is

$$\mathrm{N} \sin^{-1} \frac{\vec{\mathrm{N}}^* R_3}{2\mathrm{N}}\ . \qquad (16)$$

Differentiating this w.r.t. $\vec{\mathrm{N}}$ gives ${}^*R_3 = 0$ (which implies $R_3 = 1$, see footnote [2)]). Let us add this constraint to L with $\vec{\mathrm{N}}$ now as a Lagrange multiplier. Resulting L is here a sum over periodic cells (squares):

[2)] Another solution $R = -1 + O(\varepsilon), \varphi = \pi + O(\varepsilon)$(provided by ambiguity of the function $\sin^{-1}(\sin\varphi)$, now *inversed* to (6)) should be disregarded as not relevant to Regge calculus.

$$L = \sum_{\text{squares}} \mathfrak{L}\ .$$

$$2\mathfrak{L} = \vec{l}_1^{\,*}(\bar{\Omega}_1 \dot{\Omega}_1) - \vec{l}_2^{\,*}(\bar{\Omega}_2 \dot{\Omega}_2) + \vec{l}_d^{\,*}(\bar{\Omega}_d \dot{\Omega}_d) +$$
$$+ \vec{C}_{12}(h_{12}) + \vec{C}_{21}(h_{21}) + R(\vec{\mathrm{N}}),$$
$$\vec{C}_{12}(h_{12}) = (\vec{l}_1 + T_1\Omega_2\vec{l}_2 - \Omega_d\vec{l}_d)^* h_{12}\,,$$
$$\vec{C}_{21}(h_{21}) = (-T_2\Omega_1\vec{l}_1 - \vec{l}_2 + \vec{l}_d)^* h_{21}\,,$$
$$R(\vec{\mathrm{N}}) = \vec{\mathrm{N}}^*[\bar{\Omega}_1(\bar{T}_2\bar{\Omega}_2)(\bar{T}_d\Omega_d)(\bar{T}_1\Omega_1)\Omega_2\bar{\Omega}_d]. \qquad (17)$$

2.3. Wick rotation

Contrary to the Minkowsky signature (-, +, +) in the Euclidean case connections belong to the compact group SO(3). This allows us in subsection 3.2 to extend integrations over them in finite way on the whole orbit of local rotation gauge subgroup. Thereby fixing the corresponding gauge is avoided and functional integral can be written in the explicitly SO(3) invariant form. At the same time due to unboundedness of the gravity action the monotonous Euclidean exponential factor $\exp(-I)$ leads to divergences. Fortunately, the region in complex plane of field variables exists which results in both the compact rotation group SO(3) and oscillating exponent $\exp(iI)$. It can be obtained from the Euclidean formulation by passing to the purely imaginary links, $l^a \longrightarrow -il^a$. This may be conventionally referred to as a signature (-, -, -).

The suggested region in complex plane of variables is also convenient by that each coordinate of the local frame has the timelike nature and may be used as a time for the canonical quantisation. Therefore it will be used hereafter. Notationally, Euclidean region differs from this one by only absence of the factor i at the action in the exponent. It will be considered from the viewpoint of analytical continuation. In turn, it will be continued to the Minkowsky region by substitution $A^{\dots 3 \dots} \longrightarrow -iA^{\dots 0 \dots}$ for the time components of tensor (w.r.t. the local frame) field variables. In the cases of signature (+, +, +) or (-, -, -) the terms 'timelike' and 'spacelike' will be applied to the simplices (links and triangles) of measure $O(\varepsilon)$ and $O(1)$ at $\varepsilon \longrightarrow 0$, respectively.

3. CANONICAL QUANTISATION.

3.1. Extended phase space and gauge algebra

Thus obtained the continuous time *3D* Regge Lagrangian (17) is up to notations that suggested by Waelbroeck [12], and we can introduce

$$(-1)^{\eta(A)} l^{a}_{A} \varepsilon_{abc} \Omega^{bd}_{A} = P^{b}_{Ac} (\eta(1)=\eta(d)=0,\ \eta(2)=1) \qquad (18)$$

as independent variables and write

$$2L = \sum_{\text{squares}} \sum_{A} P^{ab}_{A} \Omega_{Aab} - C_{12}(h_{12}) - C_{21}(h_{21}) - R(\mathrm{N}),$$

$$C_{12}(h_{12}) = \sum_{\text{squares}} \mathrm{tr}\ h_{12}(\bar{\Omega}_1 P_1 - T_1 P_2 \bar{\Omega}_2 - P_d \bar{\Omega}_d),$$

$$C_{21}(h_{21}) = \sum_{\text{squares}} \mathrm{tr}\ h_{21}(-T_2 P_1 \bar{\Omega}_1 + \bar{\Omega}_2 P_2 + \bar{\Omega}_d P_d),$$

$$R(\mathrm{N}) = \sum_{\text{squares}} \mathrm{tr}\ \mathrm{N} \bar{\Omega}_1 (\bar{T}_2 \bar{\Omega}_2)(\bar{T}_d \Omega_d)(\bar{T}_1 \Omega_1) \Omega_2 \bar{\Omega}_d\ . \qquad (19)$$

Here N is the matrix $\mathrm{N}_{ab} = \varepsilon_{abc} \mathrm{N}^{c}$. Also the two constraints should be added ensuring that Ω is orthogonal and P has the form (18):

$$0 = \mu(M) := \sum_{\text{squares}} \sum_{A} \mathrm{tr}\ M^{A} \bar{\Omega}_A P_A\ ,$$

$$0 = \sigma(S) := \sum_{\text{squares}} \sum_{A} \mathrm{tr}\ S^{A} (\bar{\Omega}_A \Omega_A - 1)/2\ . \qquad (20)$$

The matrices M^A, S^A are symmetrical while $h_{\alpha\beta}$, N are antisymmetrical ones.

Consider the Poisson brackets defined w.r.t. the canonical pair P, Ω so that simbolically $\{\Omega, P\} = 1$. On the surface of constraints $\Phi = (C, R, \mu, \sigma)$ only $\{\mu, \mu\}$ and $\{\mu, \sigma\}$ do not vanish; $\vartheta = (\mu, \sigma)$ is the second class system and

$$\det \{\vartheta, \vartheta\}^{1/2}|_{\vartheta} = \det \{\mu, \sigma\}|_{\vartheta} = 1. \qquad (21)$$

Upon resolving and excluding μ, σ canonical structure is described by the Dirac brackets

$$\{\Phi_1, \Phi_2\}_{D(\vartheta)} := \{\Phi_1, \Phi_2\} - \{\Phi_1, \vartheta\} \{\vartheta, \vartheta\}^{-1} \{\vartheta, \Phi_2\} \qquad (22)$$

which are the P.B. projected onto the hyperplane normal to the surface $\vartheta = 0$ in the simplectic space. On this surface the

D.B. of C, R turn out to coincide with P.B. and equal to

$$\{C_\Delta(h_\Delta),\ C_{\Delta'}(h'_{\Delta'})\}_D = \delta_{\Delta\Delta'} C_\Delta([h_\Delta,\ h'_{\Delta'}]),$$

$$\{C_\Delta(h_\Delta),\ R(\mathrm{N})\}_D = \delta_{\Delta\Delta_0} R([h_\Delta,\ \mathrm{N}]),$$

$$\{R(\mathrm{N}),\ R(\mathrm{N}')\}_D = 0. \tag{23}$$

Here Δ, Δ' are triangles for which Gaussian constraints are considered, $\Delta_0 = 12$ is the triangle in the frame of which the R^{ab} is defined. This is to be compared with eqs. (4) of the continuum theory.

Thus C_Δ, R form the algebra of symmetry generators of the system. These act as the D. B.:

$$\{C(h),\ \Omega_1\}_D = \Omega_1 h_{12} - (\bar{T}_2 h_{21})\Omega_1,$$

$$\{C(h),\ \Omega_2\}_D = -(\bar{T}_1 h_{12})\Omega_2 + \Omega_2 h_{21},$$

$$\{C(h),\ \Omega_d\}_D = -h_{12}\Omega_d + \Omega_d h_{21},$$

$$\{C(h),\ \vec{l}_A\}_D = -[h_{12}\delta_{A1} + h_{21}(1-\delta_{A1})]\vec{l}_A, \tag{24}$$

$$\{R(\mathrm{N}),\ \Omega_A\}_D = 0,$$

$$\{R(\mathrm{N}),\ \vec{l}_1\}_D = \vec{\mathrm{N}} - T_1\Omega_2\bar{\Omega}_d\vec{\mathrm{N}},$$

$$\{R(\mathrm{N}),\ \vec{l}_2\}_D = \bar{\Omega}_d\vec{\mathrm{N}} - T_2\Omega_1\vec{\mathrm{N}},$$

$$\{R(\mathrm{N}),\ \vec{l}_d\}_D = \bar{\Omega}_d\vec{\mathrm{N}} - T_d(\bar{T}_1\Omega_1)\Omega_2\bar{\Omega}_d\vec{\mathrm{N}}. \tag{25}$$

Here $C(h) = C_{12}(h_{12}) + C_{21}(h_{21})$. The $R(\mathrm{N})$ generates translations of vertices whose finite form does not differ from (25). There $\vec{\mathrm{N}}$ is the displacement vector defined in the frame associated with the triangle 12 where also R is

defined. The matrices appearing in front of $\vec{\mathrm{N}}$ in (25) serve to transform $\vec{\mathrm{N}}$'s at the different vertices to the same frames. As for $C_\Delta(h_\Delta)$, these generate rotations $U_\Delta \in SO(3)$ in the triangles so that

$$\Omega_A \longrightarrow U_\Delta \Omega_A \bar{U}_{\Delta'}, \tag{26}$$

for the two triangles Δ, Δ' sharing the link A (Ω_A acts from Δ' to Δ) and

$$\vec{l}_A \longrightarrow U_\Delta \vec{l}_A, \tag{27}$$

for the vector $\vec{l}_A$ defined in the triangle Δ.

3.2. The functional integral

Our issue point is the known expression for the functional integral measure for the system subject to the set of second class constraints $\tilde{\Phi}$ ($\det\{\tilde{\Phi},\ \tilde{\Phi}\} \neq 0$):

$$d\mu = \exp(iI)\ \det\{\tilde{\Phi},\ \tilde{\Phi}\}^{1/2}\ \delta(\tilde{\Phi})\ D\eta,\quad \eta := (P,\ \Omega) \tag{28}$$

If the set $\Phi=(\vartheta,\ \psi)$ includes both first ψ (C, R in our case) and second class ϑ constraints, then $\tilde{\Phi} = (\Phi,\ \chi)$ follows by adding gauge conditions χ by the number of ψ's. Eq. (28) takes the form

$$d\mu_\chi = \exp(iI)\ \det\{\vartheta,\ \vartheta\}^{1/2}\delta(\vartheta)\times$$

$$\times\det\{\psi,\ \chi\}_{D(\vartheta)}\delta(\psi)\delta(\chi)\ D\eta\ . \tag{29}$$

If ψ's form the closed algebra w.r.t. $\{\cdot,\cdot\}_{D(\vartheta)}$, it is the algebra of generators of symmetry and eq. (29) differs from Faddeev - Popov anzats by only the factor $\det\{\vartheta,\ \vartheta\}^{1/2}\ \delta(\vartheta)$

provided by the presence of second class constraints ϑ. It can be interpreted as the naive (with non-fixed gauge) expression

$$d\mu_0 = \exp(iI)\ \det\{\vartheta,\ \vartheta\}^{1/2}\ \delta(\vartheta)\ \delta(\psi)\ D\eta\ , \tag{30}$$

divided by the (constant) volume of gauge group generated by $\{\psi, \cdot\}_D$.

In the continuum theory Witten [4] has suggested the following gauge:

$$\partial^\mu \omega^a_\mu = 0, \tag{31a}$$

$$\partial^\mu e^a_\mu = 0. \tag{31b}$$

In the full discrete theory where we are going to construct the measure all integrations over connections are compact. Therefore there will be no need in the analog of (31a). However, the subgroup of translations is noncompact and some gauge condition f on links is required to break the translational symmetry. The latter means that the system $\tilde{\vartheta} = (\mu,\ \sigma,\ R,\ f)$ is second class one while C or, more generally, some combination of C, R remains first class thus ensuring the residual local rotation symmetry. Then the completely gauge-fixed measure (29) can be written as

$$d\mu_f = \exp(iI)\ \det\{\tilde{\vartheta},\ \tilde{\vartheta}\}^{1/2}\ \delta(\tilde{\vartheta})\ \delta(C)\ D\eta, \tag{32}$$

divided by the (constant) volume of local rotations gauge subgroup. Thus (32) is the desired analog of (30) where now only local rotational symmetry is exhibited explicitly while the translational one is gauged away.

Practically, let us choose as gauge condition fixing any link vector per point, say, $\vec{l}_d$:

$$f(F) = \mathrm{tr}\ (\Omega_d P_d - A)\ F\ . \tag{33}$$

Here A, F are antisymmetric matrices and A_{ab}=const=$\varepsilon_{abc} l^c_d$. The C_{12} commute with f on the surface of constraints while C_{21}, R do not:

$$\{C_{21}(h),\ f(F)\}|_{(\Phi,f)} = \mathrm{tr}\ A\ [h,\ F],$$

$$\{R(\mathrm{N}),\ f(F)\}|_{(\Phi,f)} = \mathrm{tr}\ F\ [\bar{\Omega}_d \mathrm{N} \Omega_d - T_d(\Gamma \mathrm{N} \bar{\Gamma})], \tag{34}$$

$$\Gamma = (\bar{T}_1 \Omega_1)\Omega_2 \bar{\Omega}_d\ .$$

Therefore the first class constraints become C_{12} and

$$C'_{21}(h) = C_{21}(h) + R(\mathrm{N}_h), \tag{35}$$

where N_h is implicit function of h via

$$T_d(\Gamma \mathrm{N} \bar{\Gamma}) - \bar{\Omega}_d \mathrm{N} \Omega_d = [A,\ h]. \tag{36}$$

The sence of this is that local rotation in the triangle 21 is accompanied by the translations of vertices in order to compensate for the change of $\vec{l}_d$. The C_{12}, C'_{21} form the algebra w.r.t. the P. B. (the same as C_{12}, C_{21} in (23) do). Equally to say, C_{12}, C_{21} (unprimed) form the algebra w.r.t. the D. B. $\{\cdot,\cdot\}_{D(\tilde{\vartheta})}$ where one can put $^*R = 0$ in the strong sence i. e. before calculating these brackets. Therefore Faddeev - Popov anzats for this first class system is valid and we come to (32) by multiplying (29) by the rotational gauge subgroup volume.

It remains to calculate $\det\{\tilde\vartheta, \tilde\vartheta\}$. This turns out to be unity. Indeed, it can be represented as

$$\det\{\vartheta, \vartheta\}\Big|_{\tilde\vartheta} = \det\{\mu, \sigma\}^2 \det\{R, f\}^2, \qquad (37)$$

where the first factor is already given by (21) as unity and

$$\det\{R, f\}^{-1} = \int \prod_{t} \prod_{\text{vertices}} \delta^3(T_d \Gamma \vec{N} - \bar\Omega_d \vec{N})\, d^3\vec{N}. \qquad (38)$$

Here $\vec{N}$ can be subsequently integrated out giving 1 under appropriate boundary conditions.

Now as usual the arguments R, C of δ-functions in (32) can be raised to exponent with the help of Lagrange multipliers N, h integrations over which are provided. The remaining δ-functions of second class constraints μ, σ result in the left- (right-) invariant local measure over $\vec{l}$, Ω:

$$\int \delta^6(\bar\Omega P + \bar P \Omega)\, \delta^6(\bar\Omega\Omega - 1)\, d^9P\, d^9\Omega = \int d^3\vec{l}\, \mathcal{D}\Omega, \qquad (39)$$

where $\mathcal{D}\Omega$ is the Haar measure on SO(3). In the exponential parameterization ($\Omega = \exp(\varepsilon^{il}{}_{k}\omega_l)$)

$$\mathcal{D}\Omega = \frac{\sin^2(\omega/2)}{4\pi^2\omega^2}\, d^3\vec{\omega} \quad \Big(\int_{SO(3)} \mathcal{D}\Omega = 1 \Big). \qquad (40)$$

The resulting measure is

$$d\mu_l = \exp(iI)\Big|_{\vec{l}_d = \text{const}} \times$$

$$\times \prod_{t} \prod_{\text{vertices}} d^3\vec{l}_1\, d^3\vec{l}_2\, d^3\vec{N}\, \mathcal{D}\Omega_1\, \mathcal{D}\Omega_2\, \mathcal{D}\Omega_d\, d^3\vec{h}_{12}\, d^3\vec{h}_{21}. \qquad (32a)$$

If the values are averaged with this measure such that no divergence occurs when integrating over $\vec{l}_d$ we can consider also anzats (30) with no gauge fixing at all which reads

$$d\mu_0 = d\mu_l \prod d^3\vec{l}_d. \qquad (32b)$$

Here $\vec{l}_d$ is considered as variable. Finally, consider one else form of $d\mu$ which will be important. Let us fix $\vec{N}$ at finite times. That is, we impose the constraint R only as initial condition. This is sufficient due to conservation of the first class constraints in time. We have

$$d\mu_N = \exp(iI)\times$$

$$\times \prod d^3\vec{l}_1\, d^3\vec{l}_d\, d^3\vec{l}_2\, \mathcal{D}\Omega_1\, \mathcal{D}\Omega_d\, \mathcal{D}\Omega_2\, d^3\vec{h}_{12}\, d^3\vec{h}_{21}, \qquad (32c)$$

$$R\Big|_{t=-\infty} = 1.$$

The forms (32a) - (32c) will be used in constructing the full discrete measure.

4. THE FULL DISCRETE MEASURE.

4.1. General form

Integrations over $d^3\vec{h}$, $d^3\vec{N}$ can be considered as a particular case of those over $\mathcal{D}\Omega$, $d^3\vec{l}$ if one sets

$$\omega_{12} = \varepsilon h_{12},\ \omega_{21} = \varepsilon h_{21},\ \vec{l}_3 = \varepsilon\vec{N}, \qquad (41)$$

as $\varepsilon \longrightarrow 0$ and rescales $\mathcal{D}\Omega$, $d^3\vec{l}$ by ε^{-3}. Then the only maximally symmetrical measure which has a chance to be reducable to the canonical one $d\mu$ in the continuous time limit takes the form

$$d\mathfrak{M}_{\mathcal{F}} = \exp(i\tilde I) \left(\prod_{\substack{\text{links} \\ \notin \mathcal{F}}} d^3\vec{l} \right) \left(\prod_{\text{faces}} \mathcal{D}\Omega \right),$$

$$\tilde I = \frac{1}{2} \sum_{\text{links}} l^* R. \qquad (42)$$

Note that here not action I but $\tilde{I}$ stands in the exponent. The $\tilde{I}$ differs from the action I by the absence of $\sin^{-1}$.

Another peculiarity of this formula is that integrations over links of some family $\mathcal{F}$ are deleted. This has appeared as a consequence of the Bianchi identity. Let us consider this point in more detail. If $d\mathfrak{M}$ contained integrations over *all* the links attached to a given vertex σ_0 (0-simplex) they would result in

$$\prod_{M\supset\sigma_0} \delta^3(R_M - \bar{R}_M). \qquad (43)$$

(Here only the root $R_M = 1$ should be taken as corresponding to Regge calculus; cf. footnote[2]). Being transformed to the same local frame these R_M satisfy the Bianchi identity [5, 18]: their ordered product is unity (expression of R in terms of Ω is, in fact, solution to the Bianchi identities). If therefore all but one of these matrices are unity this one is unity too. Therefore (43) contains singularity of the type $\delta^3(0)$. Thus integration over one of these links is superfluous and this link should be fixed by hand. Such links constitute family $\mathcal{F}$ and will be reffered to as fixed links.

Let us outline the structure of $\mathcal{F}$. As we have seen each vertex should belong to some fixed link but $\mathcal{F}$ should be minimal in order to take into account all the degrees of freedom. How should fixed links be arranged to meet these requirements? The isolated fixed link with ending points not shared by any else fixed links is unacceptable. Indeed, integrating over other links meeting at it's two ends gives

twice the δ-function of curvature on this link, that is, again $\delta^3(0)$. For the same reason unacceptable is (closed or unclosed) line consisting of finite number of fixed links. On the other hand, the vertices shared by three or more fixed links should be absent as resulting in the nonminimal $\mathcal{F}$. Thus we arrive at the following picture. The family $\mathcal{F}$ consists of mutually nonintersecting infinite or semiinfinite broken lines passing through all the vertices. By it's sense each $\mathcal{F}$ corresponds to a definite physical measurement (triangulation) process which creates it's own quantum measure $d\mathfrak{M}_{\mathcal{F}}$.

It turns out that $d\mathfrak{M}_{\mathcal{F}}$ indeed possesses the true continuous time limit but we postpone proof of this till subsection 4.3 when some properties of this measure will have been considered.

4.2. Factorization

Since continuum $3D$ gravity does not describe any particle propagation neither on classical nor on quantum level it is natural to expect something similar in quantum Regge calculus. It turns out that in the latter this property develops as factorization of quantum measure under appropriate conditions. Namely, we show this for the particular case of arbitrarily small links of family $\mathcal{F}$.

The idea is to pass from the set of connection variables Ω to the set of curvature-connection variables where some Ω's are replaced by the independent (that is, not constrained by the Bianchi identities) curvatures R whose number in our particular Regge lattice is six per point. Jacobian of this change is unity in the sence that

$$\prod_{\text{cubes}}\left(\prod_{12\ \text{faces}}\mathcal{D}\Omega\right) = \prod_{\text{cubes}}\left(\prod_{6\ \text{links}}\mathcal{D}R\right)\left(\prod_{6\ \text{faces}}\mathcal{D}\Omega\right). \qquad (44)$$

Indeed, let us insert in the LHS of (44) the following representation of unity,

$$1 = \int \prod_{\text{cubes}}\ \prod_{6\ \text{links}} \delta^9(R - R(\Omega))\, d^9R, \qquad (45)$$

and integrate over some Ω's according to the formula

$$\int d^9R \int \delta^9(R - R(\Omega))\, \delta^6(\bar\Omega\Omega - 1)\, d^9\Omega =$$

$$= \int d^9R \int \delta^9(R - R(\Omega))\, \delta^6(\overline{R(\Omega)}R(\Omega) - 1)\, d^9R(\Omega) =$$

$$= \int \delta^6(\bar R R - 1)\, d^9R = \int \mathcal{D}R\ . \qquad (46)$$

Here $R(\Omega)=\Gamma_1\Omega^{\pm1}\Gamma_2$ with Γ_1, Γ_2 being some products of connection matrices Ω' other than Ω orthogonal due to the factors $\delta^6(\bar\Omega'\Omega'-1)$ entering $\mathcal{D}\Omega'$. The possibility to eliminate by (six per point) Ω-integrations all the δ-functions of (45) is equivalent to the rank of the linearized system $\vec\varphi = \vec\varphi(\vec\omega)$ ($R = \exp(-{}^*\vec\varphi)$, $\Omega=\exp(-{}^*\vec\omega)$) being just the number of independent curvatures. For instance, for our particular structure of linking this system is

$$\vec\varphi_1 = \vec\omega_{13} - \vec\omega_{12} + \vec\omega_{1d} + \vec\gamma_1\ ,$$

$$\vec\varphi_2 = \vec\omega_{21} - \vec\omega_{23} + \vec\omega_{2d} + \vec\gamma_2\ ,$$

$$\vec\varphi_3 = \vec\omega_{32} - \vec\omega_{31} + \vec\omega_{3d} + \vec\gamma_3\ ,$$

$$\vec\varphi_{12} = \vec\omega_{12} - \vec\omega_{21} + \vec\omega_{d3} + \vec\gamma_{12}\ ,$$

$$\vec\varphi_{23} = \vec\omega_{23} - \vec\omega_{32} + \vec\omega_{d1} + \vec\gamma_{23}\ ,$$

$$\vec\varphi_{31} = \vec\omega_{31} - \vec\omega_{13} + \vec\omega_{d2} + \vec\gamma_{31}\ ,$$

$$\vec\varphi_{123} = \vec\omega_{1d} + \vec\omega_{2d} + \vec\omega_{3d} + \vec\omega_{d1} + \vec\omega_{d2} + \vec\omega_{d3} + \vec\gamma_{123}\ . \qquad (47)$$

The $\vec\gamma$ are sums of $\vec\omega$'s obtained by the backward translations along the links 1, 2, 3. These $\vec\omega$ can be considered as those constituting the preceding layers. The rank of (47) is six per vertex (as a consequence of the Bianchi identity the algebraic sum of eqs. (47) does not contain $\vec\omega$'s apart from those in $\vec\gamma$'s referred to the preceding layers). Omitting one equation the system can be subsequently solved from layer to layer for some properly chosen six $\vec\omega$'s per each cube. Corresponding Ω's under suitable boundary conditions can be subsequently integrated out from (45).

Now suppose, as we were going, that fixed links are arbitrarily small. Then connections enter $\tilde I$ in the exponent only in the form of independent curvatures and, besides, $\tilde I$ is the sum of terms referred to separate links. Therefore integrations over Ω's which remain after change of variables (see RHS of (44)) are trivial and drop out (of course, it is assumed that the measure is defined as functional on the space of functions of only link vectors and independent curvatures). The measure becomes the product of

$$\exp\left(\frac{i}{2}\ \vec l^{\,*}R\right) d^3\vec l\ \mathcal{D}R, \qquad (48)$$

referred to separate links.

Note that if fixed links were not negligible or $d\mathfrak{M}_{\mathcal{F}}$ were defined on the space of functions of *all* curvatures then not only the measure would be nonfactorizable but also we could not trivially integrate over any connection. The reason is that Bianchi identity does not uniquely express the curvatures purely in terms of each other but also includes dependence on connections required to transform all the curvature matrices to the same local frame. It looks like

$$\prod_{l \supset \sigma^0} \bar{\Gamma}_l R_l^{\pm 1} \Gamma_l = 1, \qquad (49)$$

where the (ordered) product is over the links l sharing a vertex σ^0, and Γ_l are the products of connections.

4.3. Continuous time limit

Let us consider three basic choices for $\mathcal{F}$ consisting of spacelike, diagonal or timelike links, $\mathcal{F} = \{l_{12}\}$, $\{l_{123}\}$ or $\{l_3\}$, and show that limiting procedure exists which reduces $d\mathfrak{M}_{\mathcal{F}}$ to the continuous time canonical measure in the forms (32a), (32b) or (32c), respectively. Roughly speaking, this amounts to the formal substitution $\Omega = \exp(\varepsilon h)$ for the diagonal and spacelike faces, $\vec{l} = \varepsilon \vec{N}$ for the timelike links and rescaling the measure by a power of ε to get finite expression at $\varepsilon \longrightarrow 0$. More accurately, some preparative work is required. The whole procedure consists of the following three steps.

(i) Fixing the gauge. Upon integrating over six connections per point (six is the number of independent curvatures) the result does not depend on the remaining six connections. (Therefore the latter can be fixed by hand making integrations over them trivial). This assertion

becomes evident if one makes change of variables of the preceding subsection. Then integrations over part of connections are substituted by those over independent curvatures and under suitable conditions give the result independent on other connections. The term 'suitable conditions' means that measure is defined on the space of functions of link vectors and only independent curvatures and fixed links have vanishing lengths. These are not very restrictive conditions in the considered aspect. Indeed, in the continuous time limit spacelike and diagonal links become coincident. Therefore the number of curvature arguments of the functions describing quantum state of the system in this limit may be smaller. As for the fixed links, these can be chosen in a way convenient for us.

Let us fix the following connections:

$$\Omega_{1d} = \Omega_{2d} = \Omega_{12} = \Omega_{21} = 1, \qquad (50)$$

for $\mathcal{F} = \{l_{12}\}$ or $\{l_3\}$ and (50) together with

$$\Omega_{3d}\Omega_{d3} = 1, \qquad (51)$$

for $\mathcal{F} = \{l_{123}\}$. The requirement these gauges should satisfy is that remaining connections parameterize independent curvatures in a nondegenerate way. In other words, the rank of linearized system $\vec{\varphi} = \vec{\varphi}(\vec{\omega})$ upon imposing (50) or both (50) and (51) should remain six per point. It is easy to check that this is indeed so: (50), (51) is admissible gauge.

The role of the conditions (50) is that they anticipate introduction of spacelike infinitesimal connections h_{12}, h_{21}

which play the role of Lagrange multipliers in the canonical formalism. The matter is that there are three spacelike or diagonal infinitesimal connections f in each infinitesimal prism whose bases are successive-in-time spacelike triangles. The f's enter Lagrangian in the form of sums (15) only. Therefore integrations over them are divergent. The eqs. (50) just leave only one infinitesimal connection per each such prism.

The eq. (51) will ensure that there is only one independent timelike connection for the $2D$ link d in the limit $\varepsilon \longrightarrow 0$ (it will become Ω_d in the canonical formalism). For the other two choices of $\mathcal{F}$ this will follow by integration over diagonal link $\vec{l}_{123}$ in (ii).

(ii) Integrating over diagonal links and over half of connections on the adjacent timelike triangles. There are integrations over

$$d^3\vec{l}_{13}\ \mathcal{D}\Omega_{31}\ d^3 l_{23}\ \mathcal{D}\Omega_{32}\ , \qquad (52)$$

if $\mathcal{F} = \{l_{123}\}$ and over (52) and

$$d^3\vec{l}_{123}\ \mathcal{D}\Omega_{d3}\ , \qquad (53)$$

if $\mathcal{F} = \{l_{12}\}$ or $\{l_3\}$. We have, e. g.,

$$\int \exp\left(\frac{i}{2}\vec{l}_{13}{}^*R_{13}(\Omega_{31})\right) d^3\vec{l}_{13}\ \mathcal{D}\Omega_{31} =$$

$$= \int \exp\left(\frac{i}{2}\vec{l}_{13}{}^*R_{13}\right) d^3\vec{l}_{13}\ \mathcal{D}R_{13} =$$

$$=(4\pi)^3\int\delta^3(R_{13} - \bar{R}_{13})\ \mathcal{D}R_{13} = \frac{\pi}{2}\ . \qquad (54)$$

Thus instead of Ω_{A3}, Ω_{3A} there is only one independent timelike connection for $2D$ link A; let $\Omega_A := \Omega_{A3}$.

(iii) The continuous time limit itself. Let

$$\left.\begin{aligned} \Omega_{d1} &= 1 + \varepsilon h_{21} \\ \Omega_{d2} &= 1 + \varepsilon h_{12} \\ \vec{l}_3 &= \varepsilon\vec{N} \end{aligned}\right\} . \qquad (55)$$

Upon this formal substitution and rescaling by a power of ε the corresponding $\mathcal{D}\Omega$, $d^3\vec{l}$ are reduced to $d^3\vec{h}$, $d^3\vec{N}$. At the same time

$$\Omega_{3A}^{\pm 1} = \Omega_{A3} =: \Omega_A\ , \qquad (56)$$

at $\varepsilon \longrightarrow 0$ because of (51) or $R_{A3} = 1$ according to (ii). The $\tilde{I}$ in the exponent becomes the continuous time action with the Lagrangian (17).

4.4. Positivity

Naively, Euclidean (signature is (+, +, +)) measure $d\mathfrak{M}_{\mathcal{F}}$ seems to be positive as it is the product of the Haar and Lebesgue measures and of positive monotonous exponent. If, however, it is considered in the sence of analytical continuation from the region of imaginary links (thought of as that of (-, -, -) signature) where exponent is oscillating, the question becomes not so simple. We show such the positivity for the case of vanishing lengths of fixed links (of family $\mathcal{F}$). In this case $d\mathfrak{M}_{\mathcal{F}}$ factorizes and we have to deal with the measure (48) for each (nonfixed) link. It turns out that (48) defines positive measure in the Euclidean region on the space of functions of $\vec{l}$.

Consider polinomials in l^a as testing functions. On integrating over $d^3\vec{l}$ the δ-function

$$\delta^3\left(\frac{1}{2}(R-\bar{R})\right) = \delta^3\left(\frac{\sin\varphi}{\varphi}\vec{\varphi}\right), \qquad (57)$$

and it's derivatives arise. Here only the root $\vec{\varphi}=0$ should be taken (see footnote[2]). Due to isotropy it is sufficient to consider only polynomials in l^2 and find that

$$\langle l^{2k}\rangle = 2\left(-\frac{\partial^2}{\partial\vec{r}^2}\right)^k r^{-2}\left[(1-r^2)^{-1/2}-1\right]\Bigg|_{\vec{r}=0} =$$
$$= (-1)^k\,\frac{\Gamma(2k+2)^2\,4^{-k}}{\Gamma(k+2)\,\Gamma(k+1)}. \qquad (58)$$

The backward Wick rotation from (-, -, -) to (+, +, +) gives sensible positive VEV's:

$$\langle l^{2k}\rangle = \frac{\Gamma(2k+2)^2\,4^{-k}}{\Gamma(k+2)\,\Gamma(k+1)}. \qquad (59)$$

These are defined, nonzero and finite at $k>-1$. Moreover, inversed Laplace transform of (analytically continued to complex k) eq. (59) leads to positive measure:

$$\langle f(\vec{l})\rangle = \int\frac{do_l}{4\pi}\int_0^\infty f(\vec{l})\,\nu(l)\,dl,$$

$$\nu(l) = \frac{2l}{\pi}\int_0^\pi \exp\left(-\frac{l}{\sin\varphi}\right)d\varphi. \qquad (60)$$

The $\nu(l)$ is $2l$ at $l\to 0$ and falls off exponentially at $l\to\infty$. It is proportional to the integral of modified Bessel function.

5. DISCUSSION

Thus a quantum measure exists in the Euclidean 3D Regge gravity which , on the one hand, results in the canonical quantisation in the continuous time limit and, on the other hand, is positively defined at least when the links fixed by hand are arbitrarily small.

The results may be continued to the Minkowsky signature (-, +, +), and here we still have, e. g., the formula (59) for the link metric invariant, but now l^2 is Minkowsky (-, +, +) interval. The eqs. (59), (60) say that l^2 fluctuates remaining positive. Thus, whatever links are chosen as fixed these turn out to be the only candidates for having the timelike intervals $l^2 < 0$. In other words, timelike links are not subject to quantum fluctuations and are fixed by the physical measurement process. This is quite natural requirement implicit in introducing the notion 'time' in any quantum field theory. Interesting is that it arises in three dimensions as a consequence of quantum gravity. Moreover, time cannot be more than one-dimensional since the family of fixed links $\mathcal{F}$ consists of one-dimensional structures. In turn, the latter is the consequence of the Bianchi identities. Thus, one-dimensionality of time and it's nonquantisability can be traced to the Bianchi identities!

The choice of fixed links of $\mathcal{F}$ being arbitrarily small means the maximally detailed process of triangulation. At the same time the question is open whether all said here remains valid when fixed link lengths are nonvanishing. In particular, positivity of the measure in this case, if proven, would mean the well-defined concept of quantisation in discrete time.

The triad-connection formulation leads to some new content of quantum Regge calculus as compared to usual link lengths formulation. This shows up in apparent contradiction between separate quantisation of links and triangle inequalities. The matter is that the triad-connection formulation provides the closure of links into triangles only as a consequence of the equations of motion for the connections, i. e. on the classical level. Since the functional integral over Ω is nonGaussian such the closure is violated virtually.

Although the analysis made was shown as applied to Regge manifold of a particular structure, it naturally extends to quite general such manifolds. The results show that link VEV's are finite and nonzero and do not depend on the structure of linking. This means that functional integral is saturated not by smooth manifolds (which would correspond to the formal limit of zero link lengths) but by the piecewise-flat ones. How can this effect be matched on the level of continuum general relativity? It develops itself through ill-definiteness of some continuum VEV's. Let us consider the triad bilinear VEV at coinciding points, $\langle e^a_\mu(0) e^b_\nu(0)\rangle$. It is given by the only perturbative diagram of Fig. 1 with contracted legs (at least in the Witten's gauge [4]). From symmetry considerations for the background metric $g_{\mu\nu}$ the answer has the form

$$\langle e^a_\mu e^b_\nu \rangle = C(l_g \Lambda)^2 g_{\mu\nu} \eta^{ab}, \tag{61}$$

where η^{ab} is the frame metric, Λ is an UV momentum cut off, and we have restored Plank length l_g. If it were $C = 0$, we would have unphysical unbroken phase $e = 0$. It is, however, not difficult to verify that $C \neq 0$. To look for an analogy with the discrete formulation one should compare with $\langle l^2 \rangle$ not (61) but $\langle (edx)^2 \rangle$. The difference in the factor $dx \longrightarrow 0$ is just compensated for by $\Lambda \longrightarrow \infty$.

Thus the Regge calculus operates with the objects which seem to be more adequate to the description of gravity at the distances of the order of Plank length. Roughly speaking, it issues from the very beginning from the correctly chosen scale of physical quantities.

The results of this work seem to be of the same type and supplement those of the previous author's work [19]. In that work some (1+1)*D* model of Regge calculus is considered where one coordinate is timelike and another one corresponds to the 3*D* degenerate manifold. For the *timelike* Regge areas the quantum eigenvalues are shown to be positive integers in the Plank length units. In four dimensions analogs of the Euclidean link lengths of the present (2+1)*D* model will be *spacelike* areas. So we can expect that in the physical (3+1)*D* case Regge areas will either possess discrete spectrum, if timelike, with lowest eigenvalue of the order of l_g^2 or, when spacelike, simply have nonzero VEV's of such order.

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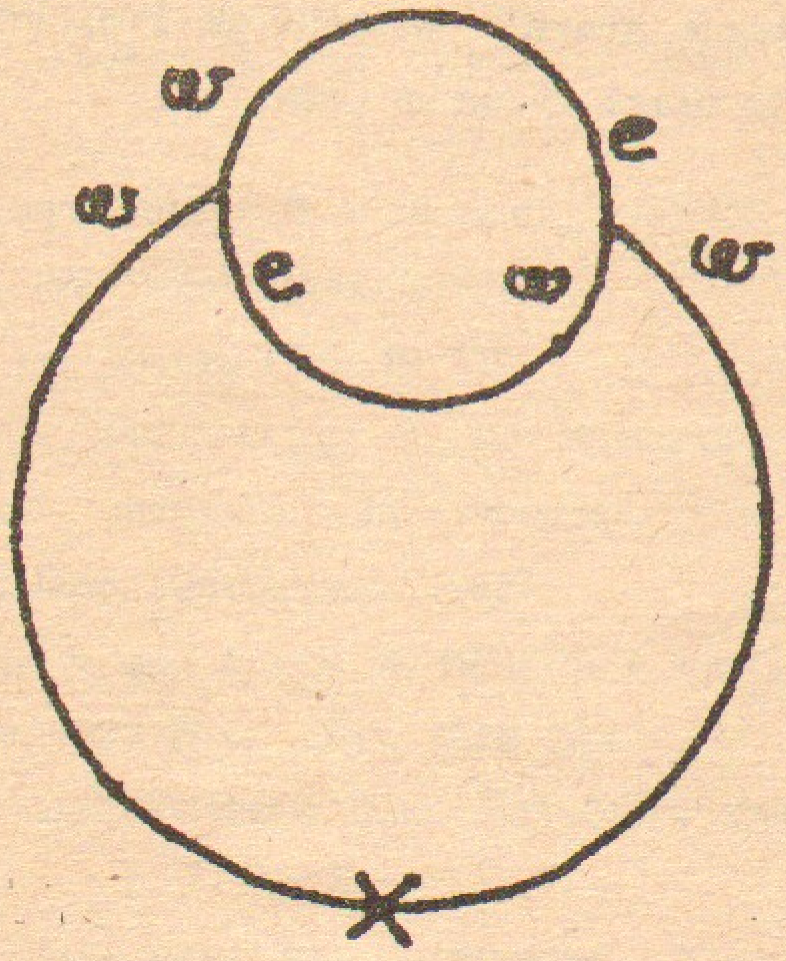

Fig. 1. The (only) diagram for the VEV $<e^a_\mu(0)e^b_\nu(0)>$ in the continuum theory with Lagrangian (1) and gauge (31).

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В.М. Хацимовский

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