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V.M. Khatsymovsky

ON THE TWO-DIMENSIONAL MODEL OF QUANTUM REGGE GRAVITY

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On the two-dimensional model
of quantum Regge gravity

V.M. Khatsymovsky

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

The Ashtekar-like variables are introduced in the Regge calculus. A simplified model of the resulting theory is quantized canonically. The consequences related to quantization of Regge areas are obtained.

1. INTRODUCTION

The new variables by Ashtekar [1] provide the polinomial form of general relativity constraints which appear in the Hamiltonian formalism. These variables were also shown [2, 3, 4] to arise naturally from a $3+1$ split of the first order tetrad formalism with the action

$$I=\frac{1}{4}\int d^4x\,\varepsilon_{abcd}\,\varepsilon^{\mu\nu\lambda\rho}e^a_\mu e^b_\nu[\partial_\lambda+\omega_\lambda,\partial_\rho+\omega_\rho]^{cd}, \qquad (1)$$

where $\omega^{ab}_\mu=-\omega^{ba}_\mu$ is an element of $so\,(3,1)$, the Lie algebra of $SO\,(3,1)$. Namely, let us (i) set the connection ω^{ab}_μ to be complex and self-dual one, i.e. an element of $SO\,(3,C)$,

$$\omega^{ab}_\mu=\omega^k_\mu\Sigma^{ab}_k/2\,,\qquad \Sigma^{ab}_k={}^*\Sigma^{ab}_k\,,\qquad k=1,2,3, \qquad (2)$$

and (ii) impose the Schwinger time gauge [5] on the tetrad e^a_μ:

$$e^0_\alpha=0\,,\qquad \alpha=1,2,3\,. \qquad (3)$$

Here ${}^*M^{ab}$ is dual to a matrix M^{ab}:

$${}^*M^{ab}\equiv\frac{1}{2}i\varepsilon^{ab}{}_{cd}M^{cd}\,,\qquad \varepsilon^{0123}=+1\,. \qquad (4)$$

The set of self-dual matrices $i^{-1}\Sigma^a_{kb}$ can be chosen to satisfy algebra of Pauli ones. Raising and lowering frame indices is performed with the help of metric $\eta_{ab}=\mathrm{diag}(-1,1,1,1)$. Under conditions (i), (ii) the action takes the form

$$I=\int d^4x\,[\,\pi_i^\alpha\dot{\omega}_\alpha^i-h^i(\partial_\alpha\pi_i^\alpha+\varepsilon_{ikl}\pi^{k\alpha}\omega_\alpha^l)-$$
$$-N_\alpha\pi_\beta^i F_i^{\beta\alpha}-N\varepsilon^{ikl}\pi_i^\alpha\pi_k^\beta F_{l\alpha\beta}\,]\,,$$

$$\pi_i^\alpha=-\frac{i}{2}\varepsilon^{\alpha\beta\gamma}\varepsilon_{ikl}e_\beta^k e_\gamma^l\,,\qquad F_{\alpha\beta}^i=\partial_\alpha\omega_\beta^i-\partial_\beta\omega_\alpha^i-\varepsilon^{ikl}\omega_{k\alpha}\omega_{l\beta}\,. \tag{5}$$

The N_α, N are linear combinations of e_0^a, $a=0, 1, 2, 3$, and $h^i=-\omega_0^i$

On the other hand, the formalism analogous to the first order tetrad one has been constructed for Regge calculus in the author's works [6, 7] including also self-dual representation and the case of the continuous time required for constructing the Hamiltonian formalism. This opens the possibility of introducing an analog of Ashtekar variables in Regge calculus or, equivalently, Regge discretization of Ashtekar gravity. Resulting formalism is still rather complicated. To get a soluble model we consider a simplified $(1+1)D$ ($1+1$-dimensional) system modelling such the Ashtekar—Regge gravity. In the continuum language the simplification amounts to limiting the range of variation of world (Greek) space indices in (5) to only one value (hereafter omitted in this case). The $SO(3, C)$ (Latin) index takes on three values as before. This model can be considered as a degenerate case of $(3+1)D$ system rather than pure $(1+1)D$ gravity. Then we are left with only three -component Gaussian constraint:

$$I=\int(\vec{\pi}\dot{\vec{\omega}}-\vec{C}\vec{h})\,dx\,dt\,,\qquad \vec{C}\vec{h}=\vec{h}(\partial\vec{\pi}+\vec{\pi}\times\vec{\omega})\,. \tag{6}$$

These constraints imposed on three canonical pairs $(\vec{\pi}, \vec{\omega})$ are first class ones. Therefore (6) presents at most finite number of the degrees of freedom which depends on the specific form of boundary conditions.

In section 2 we describe Regge counterpart of (6) which is then quantized in section 3 with the help of functional integral approach. Gauge symmetry, physical status of the model and consequences of it's quantization are disscussed in section 4.

2. THE MODEL

Let us begin with the results of [6, 7] of interest. In these works the Euclidean time version of the formalism is considered. Transition to the Minkowsky signature amounts to the formal sub-

stitution $A^{\cdots 4\cdots}\to iA^{\cdots 0\cdots}$ for the time components of tensor quantities. The set of variables consists of link vectors l_A^a and finite $SO(3, 1)$ rotations Ω_{AB}^{ab}. These are analogs of the tetrad e_μ^a and connection ω_μ^{ab}, respectively. The subscripts A, AB label links and 2-simplices (triangles) of the $3D$ leaf of the foliation ($3D$ Regge manifold), respectively. Notationally it is convenient to prescribe a definite structure of the leaf: it is topologically equivalent to the collection of cubes each divided into six tetrahedra [8]. Then multiindex A runs over seven unordered combination of unequal indices 1, 2, 3, 23, 31, 12, 123 and AB runs over twelve ordered pairs of miltiindices A, B with empty intersection. The A and AB label the links and triangles of a cube erected from one of it's vertex. Also six ordered combinations of three different indices 1, 2, 3 can be introduced to label the 3-simplices of a cube. These serve as the subscripts at the variables $h_{\alpha\beta\gamma}^{ab}\in so(3, 1)$, the analogs of $-\omega_0^{ab}$ of the continuum theory. The $h_{\alpha\beta\gamma}^{ab}$ play the role of Lagrange multipliers at the analogs of Gaussian constraints. The latter possess a simple geometrical sence: they express the closure of surfaces of the tetrahedra. Finally, let T_A be operator of translation to the neighbouring vertex along the link A. Having in view the subsequent reduction to the $2D$ model write out here only the $\dot{\Omega}$ and Gaussian constraint terms in the Lagrangian [7]:

$$2\mathscr{L}_{\dot{\Omega}}=-l_3^a l_{23}^{b\,*}(\bar{\Omega}_{32}\dot{\Omega}_{32})_{ab}-l_{23}^a l_2^{b*}(\bar{\Omega}_{23}\dot{\Omega}_{23})_{ab}-l_{123}^a l_1^{b*}(\bar{\Omega}_{1d}\dot{\Omega}_{1d})_{ab}-$$
$$-l_{23}^a l_{123}^{b\,*}(\bar{\Omega}_{d1}\dot{\Omega}_{d1})_{ab}+\text{cycle permutations of }(1, 2, 3)\,, \tag{7}$$

$$2\mathscr{L}_h=[\,l_3^a l_{23}^b+l_{23}^a l_{123}^b+l_{123}^a l_3^b-T_3(\Omega_{21}l_2)^a(\Omega_{21}l_{21})^b\,]^*h_{321,ab}+$$
$$+[\,l_{12}^a l_1^b+(\Omega_{d3}l_{123})^a(\Omega_{d3}l_{12})^b+(\Omega_{1d}l_1)^a(\Omega_{1d}l_{123})^b-$$
$$-T_1(\Omega_{23}l_{23})^a(\Omega_{23}l_2)^b\,]^*h_{123,ab}+\text{cycle perm.}(1, 2, 3)\,. \tag{8}$$

Index $1d$ means AB at $A=1$, $B=23$, etc. The $\mathscr{L}_{\dot{\Omega}}$, $\mathscr{L}_h$ should be summed over cubes (vertices) of the $3D$ leaf.

Now we can attribute to each 2-simplex the area $SO(3, C)$ vector variable,

$$\vec{S}_{AB}=\frac{i}{4}l_A^a l_{AB}^b\vec{\Sigma}_{ab}=\frac{1}{4}(l_A^0\vec{l}_{AB}-l_{AB}^0\vec{l}_A+i\vec{l}_A\times\vec{l}_{AB})\,, \tag{9}$$

and analogs of Ashtekar coordinates $\vec{\omega}$, $\vec{h}$:

$$\Omega_{AB}=\exp(\vec{\omega}_{AB}\cdot\vec{\Sigma}/2)\,,\qquad h_{\alpha\beta\gamma}^{ab}=\vec{h}_{\alpha\beta\gamma}\cdot\vec{\Sigma}^{ab}/2\,. \tag{10}$$

A definite form of Σ-matrices is used which can be read off just from (9). The analogs of Ashtekar momenta $\vec{\pi}_{AB}$ are then introduced via

$$\vec{S}=\chi(\omega)\,\vec{\pi}+\frac{1}{2}\vec{\omega}\times\vec{\pi},$$

$$\omega=(\vec{\omega}\cdot\vec{\omega})^{1/2},\quad \chi(\omega)=\frac{\omega}{2}\operatorname{ctg}\frac{\omega}{2}. \tag{11}$$

Besides that, it is convenient to denote

$$\vec{S}^{-}=\chi(\omega)\,\vec{\pi}-\frac{1}{2}\vec{\omega}\times\vec{\pi} \tag{12}$$

Then eqs (7), (8) take the form

$$\mathcal{L}_{\Omega}=\vec{\pi}_{23}\cdot\dot{\vec{\omega}}_{23}+\vec{\pi}_{32}\cdot\dot{\vec{\omega}}_{32}+\vec{\pi}_{1d}\cdot\dot{\vec{\omega}}_{1d}+\vec{\pi}_{d1}\cdot\dot{\vec{\omega}}_{d1}+$$
$$+\text{cycle perm.}(1,2,3), \tag{13}$$

$$\mathcal{L}_{h}=\vec{h}_{123}\cdot(-T_1\vec{S}^{-}_{23}-\vec{S}^{-}_{1d}+\vec{S}^{-}_{d3}+\vec{S}_{12})+\vec{h}_{321}(T_3\vec{S}^{-}_{21}+\vec{S}_{3d}-\vec{S}_{d1}-\vec{S}_{32})+$$
$$+\text{cycle perm}(1,2,3) \tag{14}$$

There are some qualitative differences from the continuum Ashtekar theory:

(i) the lack of the time gauge $l^0_A=0$ since multiindex A takes on more that three values;

(ii) the functional dependence between different $\vec{\pi}_{AB}$ since the number of 2-simplices is largen than that of 1-simplices (links).

In view of (i) we should not restrict ourselves to the purely imaginary $\vec{\pi}_{AB}$. However, according to (9) the following condition should be fullfilled:

$$\vec{S}'_{AB}\cdot\vec{S}''_{AB}=0. \tag{15}$$

One and two primes denote real and imaginary part, respectively. The point (ii) becomes irrelevant and the terms in the Lagrangian other than $\mathcal{L}_{\Omega}$, $\mathcal{L}_h$ are absent if one permits the subscript at $\vec{\pi}_{AB}$ to take on only one value (hereafter omitted). Thus the fields are defined on the $1D$ chain of vertices. In what follows it will be convenient to consider the Lagrangian of this system as $\varepsilon\to 0$ limit of the following ε-modified one:

$$\mathcal{L}_{\varepsilon}=\vec{\pi}\,\dot{\vec{\omega}}-\vec{C}_{\varepsilon}\vec{h},\quad \vec{C}_{\varepsilon}\vec{h}=\vec{h}\cdot\left[\Delta(\chi\,\vec{\pi})+\frac{(1-\varepsilon)T+1+\varepsilon}{2}\,\vec{\pi}\times\vec{\omega}\right]. \tag{16}$$

Summation should be performed over the index j at $\vec{\pi}$, $\vec{\omega}$, $\vec{h}$ (here omitted) enumerating the vertices. The operator T enlarges j by unity; $\Delta\equiv T-1$. The eqs. (16) should be supplemented by the constraint (15) of the type

$$f''=0, \tag{17}$$

where $f=\vec{S}^2/2$. Strictly speaking, h_j refers not to the vertex j but to the 1-simplex with vertices j, $j+1$. Note that in the original non-canonical variables $\vec{S}$, $\vec{\omega}$ the Gaussian constraint takes the simple form

$$R\vec{S}-T\vec{S}=0, \tag{18}$$

where $R\equiv R(-\vec{\omega})\equiv\exp(-\varepsilon^{ik}{}_l\,\omega_k)$ is $SO(3, C)$ rotation, $\bar{T}=T^{-1}$.

3. QUANTIZATION

Proceeding with functional integral quantization of complex Lagrangian (16) one should bear in mind the equivalent real one which gives the same equations of motion. Separating real and imaginary parts of these equations and using the Cauchy—Riemann relations one realizes that the same dynamics is generated by the Lagrangian which is real part of (16).

3.1. Gauge fixing

To construct the functional integral measure one should add n_1 gauge conditions χ to the system of constraints φ to get the second class system $\Phi=(\varphi, \chi)$ (that is, Det $\{\Phi, \Phi\}\neq 0$; $\{\cdot, \cdot\}$ being Poisson brackets). The n_1 is the number of first class constraints among φ. It is equal to the multiplicity of zero eigenvalue of the matrix $\{\varphi, \varphi\}$. In our case there are complex constraints $\vec{C}=\vec{C}'+i\vec{C}''$ equivalent to the pairs of real ones $\vec{C}'$, $\vec{C}''$ and the constraints (17) of the type $f''=0$ where $\vec{C}$, f are analytical functions of $\vec{\pi}$, $\vec{\omega}$. The P.B. (Poisson brackets) of real parts of two analytical functions

$$\{C_1', C_2'\} = \frac{\partial C_1'}{\partial \vec{\omega}'} \cdot \frac{\partial C_2'}{\partial \vec{\pi}'} + \frac{\partial C_1'}{\partial \vec{\pi}''} \cdot \frac{\partial C_2'}{\partial \vec{\omega}''} - \frac{\partial C_1'}{\partial \vec{\pi}'} \cdot \frac{\partial C_2'}{\partial \vec{\omega}'} -$$

$$- \frac{\partial C_1'}{\partial \vec{\omega}''} \cdot \frac{\partial C_2'}{\partial \vec{\pi}''} = \{C_1, C_2\}_C' \left(\{C_1, C_2\}_C \equiv \frac{\partial C_1}{\partial \vec{\omega}} \cdot \frac{\partial C_2}{\partial \vec{\pi}} - \frac{\partial C_1}{\partial \vec{\pi}} \cdot \frac{\partial C_2}{\partial \vec{\omega}}\right) \tag{19}$$

by Cauchy—Riemann conditions. Here $\{\cdot, \cdot\}_C$ are complex-valued P.B. Analogously.

$$\{C_1', C_2''\} = \{C_1, C_2\}_C'', \quad \{C_1'', C_2'\} = \{C_1, C_2\}_C'',$$
$$\{C_1'', C_2''\} = -\{C_1, C_2\}_C'. \tag{20}$$

Denote by $\vec{h}'$, $-\vec{h}''$, λ the (real) Lagrange multipliers at $\vec{C}'$, $\vec{C}''$ f''. Let $(\vec{h}', -\vec{h}'', \lambda)$ be eigenvector of the matrix $\{\varphi, \varphi\}$, $\varphi = (\vec{C}', \vec{C}'', f'')$ with zero eigenvalue. It satisfies the linear system

$$\left\{\begin{array}{l} \{\vec{C}, \vec{C}\}_C' \vec{h}' - \{\vec{C}, C''\}_C'' \vec{h}'' + \{\vec{C}, f\}_C'' \lambda = 0 \\ \{\vec{C}, \vec{C}\}_C'' \vec{h}' + \{\vec{C}, \vec{C}\}_C' \vec{h}'' - \{\vec{C}, f\}_C' \lambda = 0 \\ \{f, \vec{C}\}_C'' \vec{h}' + \{f, \vec{C}\}_C' \vec{h}'' - \{f, f\}_C' \lambda = 0 \end{array}\right. \tag{21}$$

or, in the complex form

$$\left\{\begin{array}{l} \{\vec{C}, \vec{C}\}_C \vec{h} - i\{\vec{C}, f\}_C \lambda = 0 \\ (\{f, \vec{C}\}_C \vec{h} - i\{f, f\}_C \lambda)'' = 0 \end{array}\right. . \tag{22}$$

The P.B. at $\varepsilon \neq 0$ can be read off from

$$\{\vec{C}_\varepsilon \vec{h} - if\lambda, \vec{C}_\varepsilon \tilde{\vec{h}} - if\tilde{\lambda}\} = -\vec{C}_\varepsilon \cdot \vec{h} \times \tilde{\vec{h}} + \vec{g}_\varepsilon \cdot \bar{\Delta}\vec{h} \times \bar{\Delta}\tilde{\vec{h}} +$$
$$+ i\chi \vec{g}_0 \cdot \vec{\pi} \times \bar{\Delta}\vec{h}\tilde{\lambda} - i\chi \vec{g}_0 \cdot \vec{\pi} \times \bar{\Delta}\tilde{\vec{h}} \cdot \lambda,$$
$$\vec{g}_\varepsilon = \vec{g}_0 - \frac{1}{4}\varepsilon^2 \vec{\pi} \times \vec{\omega} + \varepsilon\chi\vec{\pi},$$
$$\vec{g}_0 = \frac{1-\chi}{4\sin^2(\omega/2)} \vec{\pi} \times \vec{\omega},$$
$$\bar{\Delta} = \bar{T} - 1. \tag{23}$$

The first term in the expression for P.B. corresponds to usual gauge algebra of the continuum theory; others are lattice artefacts which spoil gauge algebra. Eqs. (22) take the form

$$\left\{\begin{array}{c} \Delta(-\vec{g}_\varepsilon \times \bar{\Delta}\vec{h} + i\lambda\chi\vec{g}_0 \times \vec{\pi}) = 0 \\ (\chi\vec{g}_0 \times \vec{\pi} \cdot \bar{\Delta}\vec{h})'' = 0 \end{array}\right. . \tag{24}$$

One of the solutions is $\vec{h}_j = \vec{a}$, $\lambda_j = 0$. It corresponds to the 3-component first class constraint

$$\vec{C}_\varepsilon \vec{a} = \vec{a} \cdot \sum_j \vec{\pi}_j \times \vec{\omega}_j, \tag{25}$$

the generator of global $SO(3, C)$ rotation . If requires imposing the three complex $SO(3, C)$ breaking gauge conditions. Introduce the latter as some boundary conditions. Choosing arbitrarily large number of vertices N and the fields $\Delta\vec{\pi}$, $\vec{\omega}$ sufficiently rapidly falling off at infinity we disengage ourselves from complications connected with explicit taking into account $SO(3, C)$ symmetry breaking. Other than $\vec{a}$ parameters which define solution to (24) are λ_j and complex u_j:

$$\bar{\Delta}\vec{h} = -i\lambda\chi g_\varepsilon^{-2} \vec{g}_\varepsilon \times (\vec{g}_0 \times \vec{\pi}) + u\vec{g}_\varepsilon. \tag{26}$$

Thus there are three real parameters per vertex. As corresponding gauge conditions we choose the following ones:

$$\vec{\omega}'' = 0. \tag{27}$$

Suppose reality boundary conditions are imposed on $\vec{S}$. Modulo constraints $\vec{C}_\varepsilon$ and $\vec{\omega}''$ this leads to the reality of $\vec{S}$. Possibility of such the gauge is considered in section 4. But then (17) is satisfied authomatically and should be replaced by some another condition. As the latter we choose the following one:

$$(\vec{\pi}\vec{\omega})' = 0. \tag{28}$$

The resulting equivalent set of constraints is thus $\Phi = (\vec{C}, \operatorname{Re} F)$ where $F = (-i\vec{\omega}, \vec{\pi}\vec{\omega})$, and after calculation of the P.B. $\vec{\pi}$, $\vec{\omega}$ should be set real. The determinant of P.B can be found in the following way:

$$\operatorname{Det}\{\Phi, \Phi\}^{-1} = \int \delta(\{\vec{C}, \vec{C}\}_C' \vec{h}' - \{\vec{C}, \vec{C}\}_C'' \vec{h}'' + \{\vec{C}, F\}_C' \Lambda)$$
$$\delta(\{\vec{C}, \vec{C}\})_C'' \vec{h}' + \{\vec{C}, \vec{C}\}_C' \vec{h}'' + \{\vec{C}, F\}_C'' \Lambda) \tag{29}$$
$$\delta(\{F, \vec{C}\})_C' \vec{h}' - \{F, \vec{C}\}_C'' \vec{h}'' + \{F, F\}_C' \Lambda)\, D\vec{h}'\, D\vec{h}''\, D\Lambda.$$

Here $\Lambda = (\vec{z}, \lambda)$ is real. The complex-valued P.B. at $\varepsilon \neq 0$ can be read off from

$$\{\vec{C}_\varepsilon\cdot\vec{h}-i\vec{\omega}\,\vec{z}+\vec{\pi}\,\vec{\omega}\,\lambda,\quad \vec{C}_\varepsilon\tilde{\vec{h}}-i\vec{\omega}\,\tilde{\vec{z}}+\vec{\pi}\,\vec{\omega}\,\tilde{\lambda}\}=$$

$$=-\vec{C}_\varepsilon\cdot\vec{h}\times\tilde{\vec{h}}+\vec{g}_\varepsilon\cdot\bar{\Delta}\vec{h}\times\bar{\Delta}\tilde{\vec{h}}+i\tilde{\vec{z}}\left(\chi\bar{\Delta}\vec{h}+\vec{\omega}\times\frac{\bar{T}+1}{2}\vec{h}\right)-$$

$$-i\vec{z}\left(\chi\bar{\Delta}\tilde{\vec{h}}+\vec{\omega}\times\frac{\bar{T}+1}{2}\tilde{\vec{h}}\right)-\tilde{\chi}[\,\tilde{\lambda}(\vec{\pi}\,\bar{\Delta}\,\vec{h})-\lambda(\vec{\pi}\bar{\Delta}\tilde{\vec{h}})]\;+$$

$$+i(\tilde{\vec{z}}\,\vec{\omega})\lambda-i(\vec{z}\,\vec{\omega})\tilde{\lambda},\qquad \tilde{\chi}=\chi-\omega\partial_\omega\chi \tag{30}$$

Upon setting $\vec{\pi}$, $\vec{\omega}$ real the determinant (29) becomes

$$\mathrm{Det}\{\Phi_\varepsilon,\Phi_\varepsilon\}^{-1}=\int\delta(-\Delta(\vec{g}_\varepsilon\times\bar{\Delta}\vec{h}'+\tilde{\chi}\vec{\pi}\,\lambda))$$

$$\delta\left(-\Delta(\vec{g}_\varepsilon\times\bar{\Delta}\vec{h}'')+\Delta(\chi\,\vec{z})+\frac{(1-\varepsilon)T+1+\varepsilon}{2}\,\vec{z}\times\vec{\omega}\right)$$

$$\delta\left(\chi\bar{\Delta}\vec{h}''+\vec{\omega}\times\frac{(1-\varepsilon)\bar{T}+1+\varepsilon}{2}\,\vec{h}''\right)$$

$$\delta(\tilde{\chi}\vec{\pi}\cdot\bar{\Delta}\vec{h}')\,D\vec{h}'\,D\vec{h}''\,D\vec{z}\,D\lambda=$$

$$=(\tilde{\chi}\,\vec{g}_\varepsilon\,\vec{\pi})^{-2}\,\mathrm{Det}^{-2}\left[\Delta\chi\delta^i_k+\frac{(1-\varepsilon)\,T+1+\varepsilon}{2}\,\varepsilon^i_{kl}\,\omega^l\right] \tag{31}$$

where $\vec{g}_\varepsilon\cdot\vec{\pi}=\varepsilon\chi\pi^2$. The $\mathrm{Det}\{\Phi_\varepsilon,\Phi_\varepsilon\}$ takes the form $\varepsilon^{2N}\cdot D$ where $D|_{\varepsilon=0}\neq 0$.

3.2. Functional integral

The set Φ_ε is thus second class one at $\varepsilon\neq 0$, and the functional integral measure can be defined by known expression

$$d\mu=\exp(iI)\,\mathrm{Det}\{\Phi_\varepsilon,\Phi_\varepsilon\}^{1/2}\,\delta(\Phi_\varepsilon)D\vec{\pi}D\vec{\omega} \tag{32}$$

Here $\delta(\Phi_\varepsilon)=\delta(\Phi'_\varepsilon)\,\delta(\Phi''_\varepsilon)$, $D\vec{\pi}=D\vec{\pi}'D\vec{\pi}''$, $D\vec{\omega}=D\vec{\omega}'D\vec{\omega}''$. It is well-defined at $\varepsilon\neq 0$. At $\varepsilon=0$ Det $\{\Phi_\varepsilon,\ \Phi_\varepsilon\}=0$. This nullification is in some sence accidental one connected with existence of the discrete symmetry

$$t\to -t,\quad T\to\bar{T}=T^{-1},\quad \vec{\omega}\to-\vec{\omega},\quad \vec{h}\to-\bar{T}\vec{h} \tag{33}$$

After including ε^N into overall normalization factor it becomes clear that the limit $\varepsilon\to 0$ is also well-defined and we can set $\varepsilon=0$ in the resulting expression.

Consider now integration of δ-functions in (32). The $\int D\vec{\omega}''$ is trivial due to $\delta(\vec{\omega}'')$ and $\int D\vec{\pi}''$ results in

$$\int\delta\left(\Delta(\chi\,\vec{\pi}'')+\frac{T+1}{2}\,\vec{\pi}''\times\vec{\omega}\right)D\vec{\pi}''=\mathrm{Det}^{-1}\left(\Delta\chi\,\delta^i_k+\frac{T+1}{2}\,\varepsilon^i{}_{kl}\,\omega^l\right) \tag{34}$$

which cancels the same nonlocal factor in (31). The functional measure is thus reduced to the local one in terms of real variables $\vec{\pi}=\vec{\pi}'$, $\vec{\omega}=\vec{\omega}'$.

Let us rewrite the measure in terms of original noncanonical variables $\vec{S}$, $\vec{\omega}$. Then we have in (32) δ-functions of $R(-\vec{\omega}_j)\vec{S}_j-\vec{S}_{j-1}$ and $\vec{S}_j\cdot\vec{\omega}_j$. The $\int D\vec{\omega}$ gives δ-functions of $S^2_j-S^2_{j-1}$. Integrating them we get the measure in terms of $S=|\vec{S}_j|$ and $\vec{n}_j=\vec{S}_j/S$. Let us use the same variables to rewrite the action. Upon solving the constraints $\vec{C}$, $\vec{\pi}\vec{\omega}$ the Lagrangian reads

$$L=-S\sum_j\frac{\dot{\vec{n}}_{j-1}+\dot{\vec{n}}_j}{1+\vec{n}_{j-1}\cdot\vec{n}_j}\,\vec{n}_{j-1}\times\vec{n}_j\,. \tag{35}$$

At first sight it presents one degree of freedom per vertex. This would correspond to the naive counting: originally we had six real (or three complex) canonical pairs and seven constraints per vertex three of which were first class ones. (Remind that each first class constraint or the pair of second class ones annihilates one degree of freedom). However, the sum in (35) proves to be full derivative. This remarkable property is easily seen upon substitution, $\vec{n}=z^+\vec{\sigma}z$ [9] where σ^i are Pauli matrices and z is the two-component complex ($SU(2)$) spinor normalized as $z^+z=1$:

$$L=S\dot{Q},\quad Q=2\sum_j\alpha_j\,,\quad \alpha_j=\frac{1}{2}i\ln\frac{z^+_j\,z_{j-1}}{z^+_{j-1}\,z_j}=\arg(z^+_{j-1}\,z_j)\,. \tag{36}$$

This means that we have found as a byproduct the exact discrete version of the topological θ-term in the Lagrangian density of $O(3)$ σ-model [10]:

$$-\frac{\dot{n}_{j-1}+\dot{n}_j}{1+\vec{n}_{j-1}\vec{n}_j}\,\vec{n}_{j-1}\times\vec{n}_j\Leftrightarrow\varepsilon^{\mu\nu}\,\varepsilon_{ikl}(\partial_\mu n^i)(\partial_\nu n^k)n^l,\ \varepsilon^{01}=+1\,. \tag{37}$$

Thus effectively we have only one, nonlocal degree of freedom described by the canonical pair S, Q. The resulting functional measure takes the simple form

$$d\mu = \exp\left(i\int S\dot{Q}\,dt\right) S^N dS \prod_j d^2\vec{n}_j . \qquad (38)$$

3.3. Effective action

Now we are in a position to change variables so that α_j, S form subset of the new ones and integrate out others. First, pass to the integrating over z_j. The latter are defined by $\vec{n}_j$ up to phases $\exp(i\Phi_j)$. Inserting $\int d\Phi_j/2\pi$ $(=1)$ we get

$$d^2 n_j \Rightarrow \frac{2}{\pi}\delta(|z_j|-1)\, d^4 z_j, \quad |z_j|^2 = z_j^+ z_j . \qquad (39)$$

Let us treat now z as a quaternion, i.e. make the following identification with 2×2 matrices:

$$z = (a_0 + ia_1, ia_2 + a_3) \Leftrightarrow a_0 - ia_k\sigma^k . \qquad (40)$$

Conjugation will be defined as

$$z \Rightarrow \bar{z} = a_0 + ia_k\sigma^k \qquad (41)$$

and $*$ will be usual matrix multiplication. Define also arg z as argument of complex number $a_0 + ia_1$. Then $\alpha_j = \arg(\bar{z}_{j-1} * z_j)$. Besides, the Jacobian of linear change of variables $z \to \bar{z}_0 * z$, $z_0 = \text{const}$, is $|z_0|^2 \left(= \frac{1}{2}\mathrm{tr}\,\bar{z}_0 * z_0\right)$. This prompts to make the sequence of substitutions $\bar{z}_{j-1} * z_j = u_j$. The action depends on u_j only via $\alpha_j = \arg u_j$ and integration over $d^4 u_j$ can be reduced to that over $d\alpha_j$. The result is quite simple:

$$d\mu = \exp\left(i\int S\dot{Q}\,dt\right) S^N dS \prod_j d\alpha_j, \quad Q = 2\sum_j \alpha_j . \qquad (42)$$

It describes (degenerate) system of free fields S, α_j although with a nontrivial rule of quantization. Essential is that α_j vary in the compact region from 0 to 2π. Therefore $2S$ can have only (non-negative) integer eigenvalues. Nontrivial is also the factor S^N in the measure. It can be interpreted as purely imaginary effective Hamiltonian

$$H = iN \ln(S/S_0) . \qquad (43)$$

When imposed on the Hilbert space of states the reality condition

leads to the only possibility of H being zero and $S = S_0$. For that $2S_0$ also should be (positive) integer. Corresponding physical state can be described by the wavefunction

$$\psi = \exp\left(2iS_0 \sum_j \alpha_j\right) . \qquad (44)$$

Thus the physical Hilbert space of the effective theory degenerates into the complex line.

4. DISCUSSION

4.1. On gauge fixing and physical relevance of the model

Let O_n be a set of $SO(3, C)$ matrices. Our model Lagrangian in the noncanonical variables $\vec{S}$, $\vec{\omega}$ takes the form

$$L = \sum_j \frac{1}{2}\varepsilon_{ikl} S_j^i (\bar{R}_j \dot{R}_j)^{kl} - h_{ji}(R_{j+1} S_{j+1} - S_j)^i \qquad (45)$$

where $R_j = R(-\vec{\omega}_j) = \exp(-\varepsilon^{il}{}_k\, \omega_{jl})$. It is invariant under substitution

$$R_j \to O_{j-1} R_j \bar{O}_j , \quad \vec{S}_j \to O_j \vec{S}_j , \quad h_j^i \to (O_j \vec{h}_j)^i + \frac{1}{2}\varepsilon^{ikl}(\dot{O}_j \bar{O}_j)_{kl} . \qquad (46)$$

This substitution is an analog of local gauge transformation of the continuum theory.

Under the condition (17) the imaginary parts of $\vec{S}_j$ can be nullified by an appropriate choice of matrices O_j. Indeed, in the case when O is looked for in the form of the purely imaginary rotation $\exp(i\varepsilon^{il}{}_k\,\varphi_l)$, $\varphi_l'' = 0$, we have, setting the imaginary part of $O\vec{S}$ to be zero:

$$\frac{(\vec{\varphi}\vec{S}'')}{\varphi^2}\vec{\varphi} + \left(\vec{S}'' - \frac{(\vec{\varphi}\vec{S}'')}{\varphi^2}\vec{\varphi}\right)\mathrm{ch}\,\varphi + \vec{\varphi}\times\vec{S}'\frac{\mathrm{sh}\,\varphi}{\varphi} = 0 \qquad (47)$$

This equation an $\vec{\varphi}$ has the solution if and only if $\vec{S}'\cdot\vec{S}'' = 0$. Generally O can be represented in the form $O''O'$, $O'' = \exp(i\varepsilon^{il}{}_k\,\varphi_l)$, $O'' = \exp(\varepsilon^{il}{}_k\,\psi_l)$, $\psi_l'' = 0$ (this seems to be true intuitively and can be readily proven). Now again $\vec{S}'\cdot\vec{S}'' = \tilde{\vec{S}}'\cdot\tilde{\vec{S}}'' = 0$ where $\tilde{\vec{S}} = O'\vec{S}$.

Thus the gauge with purely real $\vec{\pi}$, $\vec{\omega}$ does exist in our model.

Remind that initial point was that world space index takes on single value. Conversely, let $\vec{S}_{AB}$ be real for all AB. That is, $\vec{l}_A \times \vec{l}_{AB} = 0$ and all $\vec{l}_A$, $\vec{l}_{AB}$ are collinear. So we have effectively $(1+1)D$ theory. This shows self-consistency of the model as some degenerate case of physical $(3+1)D$ theory.

Also there is correspondence between the degrees of freedom of the model and of it's continuum counterpart. This is remarkable although well-explainable property of Regge calculus inself since it is a particular case of the continuum general relativity (for the piecewise flat manifolds). In our case due to conversion of a nimber of the first class continuum constraints into the second class discrete ones the naive counting gives some extra degrees of freedom but eventually these combine to give smaller number (one) of the degrees of freedom just corresponding to the continuum theory.

4.2. On Regge areas quantization

The results obtained indicate that Regge areas are quantized as positive integers (in units of Plank length). Eventually this is the consequence of compactness of the group $SO(3, R)$ in which dynamical part of conjugate canonical coordinates $\vec{\omega}$ vary. This rule has no analog in the continuum theory: instead of conjugate variables α_j varying in the compact region $[0, 2\pi]$ we have there the infinitesimal ones of the type $\beta_j dj$ where now j is the continuous variable and β_j is an arbitrary real function of j. Further, of integer eigenvalues of areas only one is admissible for a given world. This is due to the requirement of the effective Hamiltonian being Hermitian. In turn, possible non-Hermiticity of the Hamiltonian is due to the power-like factor in the functional measure. Appearence of such the factors is usually the case in various functional integral approaches to quantum general relativity.

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В.М. Хацимовский

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V.M. Khatsymovsky

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