ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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THE SPIN MOTION CALCULATION USING LIE METHOD IN COLLIDER NONLINEAR MAGNETIC FIELD

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НОВОСИБИРСК

The spin motion calculation using Lie method in collider nonlinear magnetic field

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Abstract

Lie operator method of solving the spin motion equation in collider nonlinear fields is used. The matrix presentation of spin Lie transformation for particle passing through collider elements is obtained. The formulas for combined several spin turn transformations are calculated in vector, matrix and operator forms for zero, first and second powers component of dynamical variable vector. The expressions for frequency precession vector components in zero, first and second powers on orbit motion and first powers on spin motion are obtained. The computer codes algorithms for nonlinear spin motion calculation are discussed.

Introduction

Calculation of spin motion in accelerators and colliders is of interest in connection with the different schemes of experiments with polarized beams (naturally including the longitudinal polarization). Various devices are used in modern accelerators to improve beam parameters. These devices (sextupoles, octupoles etc.) strongly distort the linear motion, making the linear methods for calculation of orbital and spin motion insufficient. The Lie method, which was developed by A.Dragt, E.Forest and others [1-3], allows to take into account nonlinear corrections for orbital motion. K.Yokoya [4] proposed to use this technique in spin calculations. Unfortunately, no practical results have been obtained (the calculation formulas and algorithms, computer codes and etc.). This is due to two circumstances. First, the method of Lie operators allows to write a solution of the equation for spin motion in the form of a matrix operator, which acts on initial spin vector $\mathbf{S}(o)$. But this matrix operator depends on the synchrobetatron motion of the charged particle for each type of collider elements in rather complicated manner. The second obstacle is to find the rules for the addition of Lie operators for two successive collider elements. These formulas are necessary for calculation of one-turn map of the entire ring and, hence, the spin transformation operator for practically any number of particle turns in the collider.

This work makes an attempt to solve these problems. Its plan is as follows. The first part contains a discussion of the method of Lie operators applied to the equation of spin motion. After that a matrix form of the spin transformation operator is obtained. In the next part formulas for addition of turhs are deduced because the spin motion is just the rotation around the precession vector $\mathbf{W}$. Further the rules for addition of constant, linear and quadratic (in vector $\mathbf{Z}=(x,p_x,z,p_z,\sigma,p_\sigma)$) parts of the precession frequency vector $\mathbf{W}$ are calculated in vector and operator forms. The next part contains the formulas for $\mathbf{W}$ components which are found using the second (sextupole) order in orbital and the first order in spin component vector of dynamic variables $\mathbf{Y} = (\mathbf{Z};\mathbf{S}) = (\mathbf{Z};S_x,S_z,S_T)$. In the last part the algorithm of using the obtained results in order to create the computer code for nonlinear spin motion in collider calculation is discussed.

1. The spin motion equation

As is known (for example [5]), the classical equation of spin motion in the collider is:

$$d\mathbf{S}/ds = [\mathbf{W}\mathbf{S}], \tag{1.1}$$

where $\mathbf{S}$ is a spin vector, s is an azimuth and the precession frequency vector $\mathbf{W}$ is defined by BMT's equation [6]. The equation (1.1) is written in the frame (e_x,e_z,T), fixed relative to the collider.

The usual "classical" approach to the solution of this equation is as follows. The precession frequency vector $\mathbf{W}$ is written for each type of collider elements, after that by the method of sequential approximations a system of differential equations for $\mathbf{S}$ is integrated. However, after taking into account synchrobetatron motion (SBM) the vector $\mathbf{W}$ depends on parameters of this motion in rather complicated manner. That is why the analytical solution of the system (1.1) is possible in linear approximation (in SBM) only.

The other approach is based on using the technique of Lie operators. The vectors $\mathbf{S}$ and $\mathbf{W}$ in the equation (1.1) are considered as operators. Then for a particle with the orbital Hamiltonian H_{orb} and spin Hamiltonian $\mathbf{WS}$ one can find the solution of this equation (the semicolons (":") emphasize the operator nature of the expression):

$$\mathbf{S}(s)=\exp(-:\int_0^s ds'(H_{orb}+\mathbf{WS}):)\mathbf{S}(0). \tag{1.2}$$

Here, as usual, the exponential operator is understood as a series:

$$\exp(-:F:) = \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} :F:^n.$$

Each term of this series is the differential operator of n-th power, which action on an arbitrary function f is defined with a help of Poisson brackets:

$$:F:f=\{F,f\} = \frac{dF}{dz_i}\cdot\frac{df}{dp_{zi}} - \frac{dF}{dp_{zi}}\cdot\frac{df}{dz_i}. \quad *)$$

The operator, which is introduced in this manner is referred to as a Lie operator and exponential series - Lie transformation.

For the total Hamiltonian $H_{orb}+WS$, which does not depend on azimuth explicitly, one can find instead of (1.2):

$$S(s) = \exp(-:s(H_{orb}+WS):)\ S(0) = M\ S(0), \qquad (1.3)$$

where M is a total exponential operator. According to the Hamilton equations this operator satisfies an equation $dM/ds=M:-(H_{orb}+WS):$. Let us present this operator as a product of three exponential operators [3]: $M=M_2M_1M_0$. To this end expand the total Hamiltonian in a sum of homogeneous polynomials in powers of Z:

$$H_{orb}+WS = H_2+H_3+H_4+W_0S+W_1S+W_2S,$$

where subscripts show powers of polynomials. It is important that operators $:H_2:$ and $:W_0S:$ do not change the power of dependence on Z for any operands, but operators $:H_3:$

*) As is known, Poisson brackets for components of a vector Z are: $\{q_i,q_k\}=\{p_i,p_k\}=0$, $\{q_i,p_k\}=-\{p_i,q_k\}=\delta_{ik}$, where q,p - is a conjugate dynamical variables pair (x,p_x), (z,p_z), (σ,p_σ) and δ_{ik} - is the Kroneker symbol. For vector components one can find: $\{S_i,S_j\}=e_{ijk}S_k$, where e_{ijk} - is the tree-size completely antisymmetric tensor. Besides that for any i,j the following expression takes place: $\{Z_i,S_k\}=\{S_i,Z_k\}=0$.

increase it by one. Let us consider the operator $:W_1S:$ action on spin operand, which can be always presented as $S\cdot f(Z)$. We have:

$$:W_1S:S\cdot f(Z) = W_1\{S,S\cdot f(Z)\}+S\{W_1,S\cdot f(Z)\} =$$

$$= W_1\cdot f(Z):S:S+S^2:W_1:f(Z).$$

Since the spin vector value is proportional to Planck constant h, one can omit the second term as compared with the first one. Hence the operator $:W_1S:$ increases the power of operand dependence on Z also by one. Similarly the operators $:H_4:$ and $:W_2S:$ increase it by two. Therefore it is convenient to introduce the following operators: $H_1 = :H_3 + W_1S:$ and $H_2 = :H_4 + W_2S:$. Let us separate the total Hamiltonian $:H_{orb}+WS:$ operator into operators $H_0 = :H_2 + W_0S:$ (which does not increase the operand power) and $H_r = H_1 + H_2$, and M on M_0 and M_r. Then on one hand

$$\frac{dM}{ds} = M:-(H_{orb}+WS):=M_rM_0(-H_0-H_r)=$$

$$=M_rM_0(-H_0)+M_rM_0(-H_r)=$$

$$=M_r\frac{dM_0}{ds}+M_rM_0(-H_r),$$

and on the other hand:

$$\frac{dM}{ds} = M_r\frac{dM_0}{ds} + \frac{dM_r}{ds}M_0,$$

so that:

$$\frac{dM_r}{ds}M_0 = M_rM_0(-H_r).$$

Hence

$$\frac{dM_r}{ds} = M_rM_0(-H_r)M_0^{-1},$$

or, using the Lie transformations property

$$M_0:g(Z,S):M_0^{-1}=:g(M_0Z,M_0S)|_s:, \qquad (1.4)$$

one obtains

$$\frac{dM_r}{ds} = M_r(-H_r(M_0Z,M_0S)|_s).$$

Integrating both parts of this expression one can find (J is a unit operator) :

$$M_r = J + \int_0^s ds' M_r(s')(-H_r(M_0Z,M_0S)|_{s'}).$$

This integral equation solution is easily obtained in series form if one substitutes in the integrand the right part step by step:

$$M_r =$$

$$= J+\int_0^s ds'[J+\int_0^{s'} ds'' M_r(s'')(-H_r(M_0Z,M_0S)|_{s''})](-H_r(M_0Z,M_0S)|_{s'}) =$$

$$= J + \int_0^s ds'(-H_r(M_0Z,M_0S)|_{s'}) +$$

$$+ \int_0^s ds' \int_0^{s'} ds'' M_r(s'')(-H_r(M_0Z,M_0S)|_{s''})(-H_r(M_0Z,M_0S)|_{s'}) =$$

$$= J + \int_0^s ds'(-H_r(M_0Z,M_0S)|_{s'}) +$$

$$+ \int_0^s ds' \int_0^{s'} ds'' [J + \int_0^{s''} ds''' M_r(s''')(-H_r(M_0Z,M_0S)|_{s'''})]\cdot$$

$$\cdot(-H_r(M_0Z,M_0S)|_{s''})(-H_r(M_0Z,M_0S)|_{s'}) =$$

$$= J + \int_0^s ds'(-H_r(M_0Z,M_0S)|_{s'}) +$$

$$+ \int_0^s ds' \int_0^{s'} ds''(-H_r(M_0Z,M_0S)|_{s''})(-H_r(M_0Z,M_0S)|_{s'}) +$$

$$+ \int_0^s ds' \int_0^{s'} ds'' \int_0^{s''} ds''' M_r(s''')(-H_r(M_0Z,M_0S)|_{s'''})\cdot$$

$$\cdot(-H_r(M_0Z,M_0S)|_{s''})(-H_r(M_0Z,M_0S)|_{s'}) = \ldots .$$

A simple criterion for this series to break exists associated with the the order of its terms in powers of Z. Indeed, the operator $H_r \exp(-:w_0S:)$ increases the operand power by unit as minimum. Restricting to the terms with a power not higher than two one can omit the terms which contain H_r^3 and break the series:

$$M_r = J + \int_0^s ds'(-H_r(M_0Z,M_0S)|_{s'}) +$$
$$+ \int_0^s ds' \int_0^{s'} ds''(-H_r(M_0Z,M_0S)|_{s''})(-H_r(M_0Z,M_0S)|_{s'}). \quad (1.5)$$

Since M_r is a product of two exponential operators M_2 and M_1, let us choose their powers as operators, which increase the operand power by one $(-:f_1:)$ and two $(-:f_2:)$. Then expand the exponents into series and omit the third and higher power terms. In this way one obtains:

$$M_r = M_2M_1 = \exp(-:f_2:)\cdot\exp(-:f_1:) =$$

$$=(J - :f_2: + \frac{:f_2:^2}{2} - \ldots)\cdot(J - :f_1: + \frac{:f_1:^2}{2} - \ldots) =$$

$$= J - :f_1: - :f_2: + \frac{:f_1:^2}{2} .$$

Comparing this expression with (1.5) one can find:

$$:f_1: = - \int_0^s ds'(-H_1(M_0Z,M_0S)|_{s'}) \quad (1.6)$$

and

$$:f_2: = \frac{:f_1:^2}{2} - \int_0^s ds'(-H_2(M_0Z,M_0S)|_{s'}) -$$

$$- \int_0^s ds' \int_0^{s'} ds''(-H_1(M_0Z,M_0S)|_{s''})(-H_1(M_0Z,M_0S)|_{s'}).$$

But

$$:f_1:^2 = \int_0^s ds' \int_0^s ds''(-H_1(M_0Z,M_0S)|_{s''})(-H_1(M_0Z,M_0S)|_{s'}),$$

hence separating the second integration over s" in intervals from 0 to s' and from s' to s and changing the variable for the second interval, one can obtain

$$:f_1:^2 = \int_0^s ds' \int_0^{s'} ds''(-H_1(M_0Z,M_0S,s''))(-H_1(M_0Z,M_0S,s')) +$$

$$+ \int_0^s ds' \int_0^{s'} ds''(-H_1(M_0Z,M_0S,s'))(-H_1(M_0Z,M_0S,s'')).$$

Thus for $-:f_2:$ one finally finds:

$$:f_2: = -\int_0^s ds'(-H_2(M_0Z,M_0S,s')) - \frac{1}{2}\int_0^s ds' \int_0^{s'} ds''[-H_1(M_0Z,M_0S,s''),-H_1(M_0Z,M_0S,s')], \qquad (1.7)$$

where [,] is an operator commutator.

So, the operator M, which determines the solution (1.3) of the spin motion equation (1.1), is equal to

$$M = M_2M_1M_0 = \exp(-:f_2:)\cdot\exp(-:f_1:)\cdot\exp(-:f_0:) = \left(J - :f_1: - :f_2: + \frac{:f_1:^2}{2} \right) M_0. \qquad (1.8)$$

In some cases it is convenient to separate the expression (1.8) in successive pure spin and orbital operators. For this substitute f_1 and f_2 in (1.8) as $h_3 + w_1S$ and $h_4 + w_2S$ respectively. The operators $:h_4:$, $:h_3:^2$ and $:w_1S::h_3:$ contribution in (1.8) can be neglected: they act on an operand, which contains the first power of Z as a minimum, and increase this power by two. But operators $:w_2S:$, $:w_1S:^2$ and $:h_3::w_1S:$, can act on the operand with a

zero power in Z and they must be taken into account. Besides, let us take into account that one can omit the operator $:w_1S::h_3:$ for the same reasons and hence $:h_3::w_1S:=:(:h_3:w_1S):$. Finally one obtains:

$$M = \left(J - :h_3: - :w_1S: - :w_2S: + \frac{:w_1S:^2}{2} + \frac{:h_3::w_1S:}{2} \right) M_0 =$$

$$= \exp\left(-:\left[w_2 - \frac{(:h_3:w_1)}{2}\right]S:\right)\cdot\exp(-:w_1S:)\cdot\exp(-:h_3:)\cdot M_0.$$

Let us note that $:w_0S:$ commute with $:h_2:$ and $:h_3:$, hence one can finally write:

$$M = \exp\left(-:\left[w_2 - \frac{(:h_3:w_1)}{2}\right]S:\right)\cdot\exp(-:w_1S:)\cdot\exp(-:w_0S:)\cdot \exp(-:h_3:)\cdot\exp(-:h_2:), \qquad (1.9)$$

Sometimes it is necessary to represent the spin part of the expression (1.9) as one exponent with the precession frequency W^* which determines spin rotation for each element:

$$M = \exp(-:W^*S:)\cdot\exp(-:h_3:)\cdot\exp(-:h_2:). \qquad (1.10)$$

Let us obtain the "convolution" of three exponents operators:

$$\exp(-:sf:)=\exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:).$$

Differentiating both parts with respect to s one obtains:

$$\exp(-:sf:)\cdot:-f:=\exp(-:sf_2:)\cdot:-f_2:\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:) +$$

$$+ \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot:-f_1:\cdot\exp(-:sf_0:) +$$

$$+ \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:)\cdot:-f_0:=$$

$$= \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot(\exp(:sf_1:):-f_2:\cdot\exp(-:sf_1:))\cdot$$

$$\cdot\exp(-:sf_0:) + \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(:-sf_0:)\cdot$$

$$\cdot(\exp(:sf_0:)\cdot:-f_1:\cdot\exp(-:sf_0:)) +$$

$$+ \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:)\cdot:-f_0:.$$

But

$$:f_1:^2 = \int_0^s ds' \int_0^s ds''(-H_1(M_0Z,M_0S)|_{s''})(-H_1(M_0Z,M_0S)|_{s'}),$$

hence separating the second integration over s" in intervals from 0 to s' and from s' to s and changing the variable for the second interval, one can obtain

$$:f_1:^2 = \int_0^s ds' \int_0^{s'} ds''(-H_1(M_0Z,M_0S,s''))(-H_1(M_0Z,M_0S,s')) +$$

$$+ \int_0^s ds' \int_0^{s'} ds''(-H_1(M_0Z,M_0S,s'))(-H_1(M_0Z,M_0S,s'')).$$

Thus for $-:f_2:$ one finally finds:

$$:f_2: = -\int_0^s ds'(-H_2(M_0Z,M_0S,s')) - \qquad (1.7)$$

$$-\frac{1}{2}\int_0^s ds' \int_0^{s'} ds''[-H_1(M_0Z,M_0S,s''),-H_1(M_0Z,M_0S,s')],$$

where [,] is an operator commutator.

So, the operator M, which determines the solution (1.3) of the spin motion equation (1.1), is equal to

$$M = M_2M_1M_0 = \exp(-:f_2:)\cdot\exp(-:f_1:)\cdot\exp(-:f_0:) = \qquad (1.8)$$

$$= (J - :f_1: - :f_2: + \frac{:f_1:^2}{2}) M_0.$$

In some cases it is convenient to separate the expression (1.8) in successive pure spin and orbital operators. For this substitute f_1 and f_2 in (1.8) as $h_3 + w_1S$ and $h_4 + w_2S$ respectively. The operators $:h_4:$, $:h_3:^2$ and $:w_1S::h_3:$ contribution in (1.8) can be neglected: they act on an operand, which contains the first power of Z as a minimum, and increase this power by two. But operators $:w_2S:$, $:w_1S:^2$ and $:h_3::w_1S:$, can act on the operand with a

zero power in Z and they must be taken into account. Besides, let us take into account that one can omit the operator $:w_1S::h_3:$ for the same reasons and hence $:h_3::w_1S:=:(:h_3:w_1S):$. Finally one obtains:

$$M = (J - :h_3: - :w_1S: - :w_2S: + \frac{:w_1S:^2}{2} + \frac{:h_3::w_1S:}{2}) M_0 =$$

$$= \exp(-:[w_2 - \frac{(:h_3:w_1)}{2}]S:)\cdot\exp(-:w_1S:)\cdot\exp(-:h_3:)\cdot M_0.$$

Let us note that $:w_0S:$ commute with $:h_2:$ and $:h_3:$, hence one can finally write:

$$M = \exp(-:[w_2 - \frac{(:h_3:w_1)}{2}]S:)\cdot\exp(-:w_1S:)\cdot\exp(-:w_0S:)\cdot \qquad (1.9)$$

$$\cdot\exp(-:h_3:)\cdot\exp(-:h_2:),$$

Sometimes it is necessary to represent the spin part of the expression (1.9) as one exponent with the precession frequency W^* which determines spin rotation for each element:

$$M = \exp(-:W^*S:)\cdot\exp(-:h_3:)\cdot\exp(-:h_2:). \qquad (1.10)$$

Let us obtain the "convolution" of three exponents operators:

$$\exp(-:sf:) = \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:).$$

Differentiating both parts with respect to s one obtains:

$$\exp(-:sf:)\cdot:-f: = \exp(-:sf_2:)\cdot:-f_2:\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:) +$$

$$+ \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot:-f_1:\cdot\exp(-:sf_0:) +$$

$$+ \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:)\cdot:-f_0: =$$

$$= \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot(\exp(:sf_1:):-f_2:\cdot\exp(-:sf_1:))\cdot$$

$$\cdot\exp(-:sf_0:) + \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(:-sf_0:)\cdot$$

$$\cdot(\exp(:sf_0:)\cdot:-f_1:\cdot\exp(-:sf_0:)) +$$

$$+ \exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(-:sf_0:)\cdot:-f_0:.$$

The factor $\exp(-:sf:)$ evidently arises in second and third terms. Besides that, the expressions in figure brackets can be grouped with the help of the rule (1.4). Then:

$$\exp(-:sf:)\cdot:-f:=\exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot:-\exp(:sf_1:)f_2:\cdot$$
$$\cdot\exp(:-sf_0:)+\exp(-:sf:)\cdot:-\exp(:sf_0:)f_1:+\exp(-:sf:)\cdot:-f_0:.$$

Repeating in the first terms similar action with operator $\exp(-:sf_0:)$, one can obtain:

$$\exp(-:sf:)\cdot:-f:=\exp(-:sf_2:)\cdot\exp(-:sf_1:)\cdot\exp(:-sf_0:)\cdot$$
$$\cdot:-\exp(:sf_0:)\cdot\exp(:sf_1:)f_2:+\exp(-:sf:)\cdot:-\exp(:sf_0:)f_1: +$$
$$+\exp(-:sf:)\cdot:-f_0:=\exp(-:sf:)\cdot:-\exp(:sf_0:)\cdot\exp(:sf_1:)f_2: +$$
$$+ \exp(-:sf:)\cdot:-\exp(:sf_0:)f_1:+\exp(-:sf:)\cdot:-f_0:.$$

The required equation for f can be found after that:

$$f = f_0 + \exp(:sf_0:)f_1 + \exp(:sf_0:)\cdot\exp(:sf_1:)f_2 . \qquad (1.11)$$

The contribution of the operator $\exp(:sf_1:)$ is not lower than the third power. Then for W^* one finally obtains:

$$W^* = W_0 + \exp(:W_0S:)W_1 + \exp(:W_0S:)(W_2 - \frac{:h_3:W_1}{2}) . \qquad (1.12)$$

The operator $\exp(-:W_0S:)$ as shown below can be represented in a matrix form.

Thus using the technique of Lie operators it is necessary to substitute in (1.6) and (1.7) the expression for H_{orb} and W from BMT-formula (for each type of collider elements) and to find the transformed precession frequency w components and Hamiltonian h_{orb}. The obtained results should be substituted into (1.8) or (1.9). The operators $\exp(-:h_2:)$ and $\exp(-:W_0S:)$ are calculated by expanding to series. The equation (1.1) is indeed solved!

2. Matrix form for operator of spin transformation

The spin exponential operator, which was obtained in (1.9) and (1.12), has the form $\exp(-:SW:)$, where W is some precession frequency. If the expression for W is known, one can calculate the value of $\exp(-:SW:)$ successively finding the low-order terms of this series. For zero order, by definition, it is

$$(-:SW:)^0 S_k=S_k. \qquad (2.1)$$

The first order (taking into account the independence of spin vector components S_k on orbital variables Z):

$$(-:SW:)^1 S_k=-W_i\{S_i,S_k\}-S_i\{W_i,S_k\}=-E_{ijk}S_jW_i=[W,S]_k,$$

where [,] is the vector product. Thus:

$$(-:SW:)^1 S_k=[W,S]_k. \qquad (2.2)$$

Then for the second order one has:

$$(-:SW:)^2 S_k=(-:SW:)^1(-:SW:)^1 S_k=(-:SW:)^1[W,S]_k=:S_nW_n:E_{ijk}S_jW_i=$$
$$=E_{ijk}W_iW_n\{S_n,S_j\}+E_{ijk}S_jS_n\{W_n,W_i\}+E_{ijk}S_jW_n\{S_n,W_i\}+$$
$$+E_{ijk}W_iS_n\{W_n,S_j\}=[W,[W,S]]_k+[S_n\{W_n,W\},S].$$

The second term, which is proportional to the Planck constant h^2, can be omitted (it is rather small compared to the first one). Hence, one finally has:

$$(-:SW:)^2 S_k=[W,[W,S]]_k. \qquad (2.3)$$

Further, for the third order (omitting the terms, which are again proportional to h^2):

$$(-:SW:)^3 S_k=:-S_nW_n:[W,[W,S]]_k=:-S_nW_n:(W_kS_iW_i-S_kW_iW_i)=$$
$$=-W_nW_kW_i\{S_n,S_i\}+W_nW_iW_i\{S_n,S_k\}=-W_nW_kW_ie_{nij}S_j+W_nW_iW_ie_{nkj}S_j,$$

that is (W is an absolute value of the vector $\mathbf{W}$):

$$(-:\mathbf{SW}:)^3 S_k = -W^2[\mathbf{W},\mathbf{S}]_k . \qquad (2.4)$$

Thus we have the possibility to deduce any term of this series :

$$(:-\mathbf{SW}:)^{2n+1} S_k = (-1)^n W^{2n}[\mathbf{W},\mathbf{S}]_k \qquad n=0,1,2,\ldots;$$

$$(:-\mathbf{SW}:)^{2n+2} S_k = (-1)^n W^{2n}[\mathbf{W},[\mathbf{W},\mathbf{S}]]_k \quad n=0,1,2,\ldots .$$

Dividing the exponential series into two - even and odd degree - we can sum it completely:

$$\exp(-:\mathbf{SW}:)S_k = \sum_{n=0}^{\infty} \frac{(-:\mathbf{SW}:)^n}{n!} S_k =$$

$$=:-\mathbf{SW}:^0 S_k + \sum_{n=0}^{\infty} \left[\frac{(-:\mathbf{SW}:)^{2n+1}}{(2n+1)!} + \frac{(-:\mathbf{SW}:)^{2n+2}}{(2n+2)!}\right] S_k =$$

$$= S_k + [\mathbf{W},\mathbf{S}]_k \sin(W)/W + [\mathbf{W},[\mathbf{W},\mathbf{S}]]_k \{1-\cos(W)\}/W^2 .$$

Hence the exponential operator $\exp(-:\mathbf{SW}:)$ has a matrix form:

$$S_i = \exp(-:\mathbf{SW}:)_{ij} \cdot S_j = T_{ij} \cdot S_j ,$$

where

$$T_{ij} = E_{ij} \cdot \cos(W) + e_{ijk} \cdot W_k \cdot \frac{\sin(W)}{W} + W_i W_j \cdot \frac{1-\cos(W)}{W^2} . \qquad (2.5)$$

This representation for the solution of the spin motion equation is exact, but the accuracy is determined by that of $\mathbf{W}$ calculation. Let us confine to the second order over synchro-betatron motion in this work. Then

$$W_i = W^o{}_i + W^1{}_{im} \cdot Z_m + W^2{}_{imn} \cdot Z_m \cdot Z_n , \qquad (2.6)$$

so, for the absolute value of the spin angle turn W we have:

$$W = [(W_i)^2]^{\frac{1}{2}} = W_o \cdot [1 + \frac{W^o{}_i}{W_o{}^2} \cdot W^1{}_{im} \cdot Z_m +$$

$$+ (\frac{W^o{}_i \cdot W^2{}_{imn}}{W_o{}^2} + \frac{W^1{}_{im} \cdot W^1{}_{in}}{2 \cdot W_o{}^2} - \frac{W^o{}_i \cdot W^o{}_j \cdot W^1{}_{im} \cdot W^1{}_{jn}}{2 \cdot W_o{}^4}) \cdot Z_m \cdot Z_n] . \qquad (2.7)$$

Substituting (2.6) and (2.7) into (2.5) (the sin and cos functions, which one must decompose relative to $W_o=[(W^o{}_i)^2]^{\frac{1}{2}}$), one receives for the zero order of $\mathbf{Z}$ after some simple transformations:

$$T^o{}_{ij} = E_{ij} \cdot \cos(W_o) + e_{ijk} \cdot W^o{}_k \cdot \sin(W_o)/W_o +$$

$$+ W^o{}_k \cdot W^o{}_i \cdot \{1-\cos(W_o)\}/W_o{}^2 ; \qquad (2.8)$$

for the first order:

$$T^1{}_{ijm} = [- \sin(W_o)/W_o \cdot E_{ij} - \sin(W_o)/W_o \cdot e_{ijk} \cdot W^1{}_{km} +$$

$$+ \{1 - \cos(W_o)\}/W_o{}^2 \cdot (W^1{}_{jm} \cdot W^o{}_i + W^1{}_{im} \cdot W^o{}_j) -$$

$$- \{\cos(W_o) - \sin(W_o)/W_o\}/W_o{}^2 \cdot e_{ijk} \cdot W^1{}_{nm} \cdot W^o{}_k \cdot W^o{}_n + \qquad (2.9)$$

$$+ \{\sin(W_o) - 2 \cdot (1 - \cos(W_o)/W_o\}/W_o{}^3 \cdot W^1{}_{km} \cdot W^o{}_i \cdot W^o{}_j \cdot W^o{}_k]$$

and at last for the second order:

$$T^2{}_{ijmn} = \sin(W_o)/W_o \cdot [- E_{ij} \cdot (W^2{}_{kmn} \cdot W^o{}_k + {}^1/_2 \cdot W^1{}_{km} \cdot W^1{}_{kn}) +$$

$$+ e_{ijk} \cdot W^2{}_{kmn}] + (1 - \cos(W_o))/W_o{}^2 \cdot (W^o{}_i \cdot W^2{}_{jmn} + W^o{}_j \cdot W^2{}_{imn} +$$

$$+ W^1{}_{im} \cdot W^1{}_{jn}) + (\cos(W_o) - \sin(W_o)/W_o)/W_o{}^2 \cdot \qquad (2.10)$$

$$\cdot [e_{ijk} \cdot (W^2{}_{pmn} \cdot W^o{}_k \cdot W^o{}_p + {}^1/_2 \cdot W^1{}_{pm} \cdot W^1{}_{pn} \cdot W^o{}_k + W^1{}_{km} \cdot W^1{}_{pn} \cdot W^o{}_p) -$$

$$- E_{ij} \cdot W^o{}_k \cdot W^o{}_p \cdot W^1{}_{km} \cdot W^1{}_{pn}] + [\sin(W_o)/W_o - 2 \cdot \{1-\cos(W_o)\}/W_o{}^2]/W_o{}^2 \cdot$$

$$\cdot [W^o{}_i \cdot W^o{}_j \cdot (W^2{}_{kmn} \cdot W^o{}_k + {}^1/_2 \cdot W^1{}_{pm} \cdot W^1{}_{pn}) + (W^1{}_{jm} \cdot W^o{}_i + W^1{}_{im} \cdot W^o{}_j) \cdot$$

$$\cdot W^1{}_{kn} \cdot W^o{}_k] - [\sin(W_o)/W_o - 3 \cdot (\cos(W_o) - \sin(W_o)/W_o)/W_o{}^2]/W_o{}^2/2 \cdot$$

$$\cdot e_{ijk} \cdot W^o{}_k \cdot W^o{}_p \cdot W^o{}_q \cdot W^1{}_{pm} \cdot W^1{}_{qn} + [8 \cdot \{1-\cos(W_o)\}/W_o{}^2 + \cos(W_o) -$$

$$- 5 \cdot \sin(W_o)/W_o)]/W_o{}^4/2 \cdot W^o{}_i \cdot W^o{}_j \cdot W^o{}_p \cdot W^o{}_q \cdot W^1{}_{pm} \cdot W^1{}_{qn} .$$

3. The combination of "rotations"

As it was already mentioned above, the spin motion, which is determined by the precession frequency **W**, is simply a rotation around an axis along **W** by the angle W. The equation (2.5) describes this spin transformation for such rotation. From (2.5) one can obviously find:

$$\cos(W) = \frac{1}{2}\cdot(\mathrm{Sp}\ T^W - 1) \quad (3.1) \quad \text{and} \quad W_i = \frac{1}{2}\frac{e_{ijk}\, T^W{}_{jk}}{\sin(W)}. \quad (3.2)$$

These equations allow to find the rules for combining two and more successive spin rotations. Really, if matrixes T^V and T^u describe two successive rotations, then the total rotation is characterized by a matrix T^W, which equals: $T^W{}_{ij} = T^u{}_{ik}\cdot T^V{}_{kj}$. Hence, calculating the matrix T^W trace, we obtain

$$\cos(W) = {}^1/_2\cdot[\cos(V)\cdot\cos(U)+\cos(V)+\cos(U) + \frac{(\mathbf{VU})^2}{V^2U^2}\cdot(1-\cos(V))\cdot(1-\cos(U)) - 2\cdot\frac{(\mathbf{VU})}{V\cdot U}\cdot\sin(V)\cdot\sin(U) - 1\,]. \quad (3.3)$$

This expression can have a simpler form, if it is rewritten using "half-angles" $^V/_2$, $^U/_2$ and $^W/_2$:

$$\cos(^W/_2) = \cos(^V/_2)\cdot\cos(^U/_2) - \frac{(\mathbf{VU})}{VU}\cdot\sin(^V/_2)\cdot\sin(^U/_2). \quad (3.4)$$

To determine vector W_i it is necessary to calculate first the value of $e_{ikj}\cdot T^W{}_{jk}$. Using the total angle W value (found above) we receive the following:

$$\frac{W_i}{W} = \frac{1}{2\sin(W)} \cdot\Big[\frac{V_i}{V}\cdot(\sin(V)\cdot(1+\cos(U)) - \frac{(\mathbf{U}\cdot\mathbf{V})}{U\cdot V}\cdot\sin(U)\cdot(1-\cos(V))) + \frac{U_i}{U}\cdot(\sin(U)\cdot(1+\cos(V)) - \frac{(\mathbf{U}\cdot\mathbf{V})}{U\cdot V}\cdot\sin(V)\cdot(1-\cos(U))) + \frac{[\mathbf{V},\mathbf{U}]_i}{U\cdot V}\cdot(\sin(V)\cdot\sin(U) - \frac{(\mathbf{U}\cdot\mathbf{V})}{U\cdot V}\cdot(1-\cos(V)\cdot(1-\cos(U))))\Big]. \quad (3.5)$$

Passing to half-angles let us find a simpler variant of this formula:

$$\frac{W_i}{W} = \frac{1}{\sin(^W/_2)}\cdot\{ \frac{V_i}{V}\cdot\sin(^V/_2)\cdot\cos(^U/_2) + \frac{U_i}{U}\cdot\sin(^U/_2)\cdot\cos(^V/_2) + \frac{[\mathbf{V},\mathbf{U}]_i}{UV}\ \sin(^V/_2)\cdot\sin(^U/_2)\}. \quad (3.6)$$

Thus the pairs of expressions (3.3) and (3.4) solve the problem of combining two successive spin rotations.

4. Formulas of "addition"

To combine the successive spin transformations during the beam passing of the magnetic system let us deduce the formulas for addition of transformations in the form (1.10), i.e. let us define operators, which obey the equation:

$$\begin{aligned}&\exp(-:\mathbf{SW}:)\cdot\exp(-:h_3^{W}:)\cdot\exp(-:h_2^{W}:)=\\&=\exp(-:\mathbf{SU}:)\cdot\exp(-:h_3^{u}:)\cdot\exp(-:h_2^{u}:)\cdot\\&\cdot\exp(-:\mathbf{SV}:)\cdot\exp(-:h_3^{V}:)\cdot\exp(-:h_2^{V}:).\end{aligned}\qquad(4.1)$$

In other words, let us find Lie operators, which transform spin vector from azimuth s^0 to azimuth s" equivalent to sequential Lie operators, transforming the spin from s^0 to s' (the operators $\exp(-:\mathbf{SV}:)\cdot\exp(-:h_3^{V}:)\cdot\exp(-:h_2^{V}:)$) and then from s' to s" (the operators $\exp(-:\mathbf{SU}:)\cdot\exp(-:h_3^{u}:)\cdot\exp(-:h_2^{u}:)$). Acting similarly to the calculation of formula (1.11) for "merging" of exponential operators product one can easily obtain the following result:

$$\begin{aligned}&\exp(-:\mathbf{SW}:)=\exp(-:\mathbf{SU}:)\cdot\exp(-:\mathbf{SV}^*:),\\&\mathbf{V}^*=\exp(-:h_3^{u}:)\cdot\exp(-:h_2^{u}:)\mathbf{V},\\&h_3^{W}=h_3^{u}+\exp(-:h_2^{u}:)h_3^{V},\\&\exp(-:h_2^{W}:)=\exp(-:h_2^{u}:)\cdot\exp(-:h_2^{V}:).\end{aligned}\qquad(4.2)$$

One has now the equation $\exp(-:\mathbf{SW}:)=\exp(-:\mathbf{SU}:)\cdot\exp(-:\mathbf{SV}:)$ (a subscript "*" near **V** is omitted for simplicity). If the operators by matrixes of the form (2.5) this equation takes the form:

$$T^{W}{}_{ij} = T^{u}{}_{ik}\cdot T^{V}{}_{kj}. \qquad (4.3)$$

Since one is interested in a result in the series form in **Z** powers:

$$T^{W}{}_{ij} = T^{W0}{}_{ij} + T^{W1}{}_{ijm}\cdot Z_m + T^{W2}{}_{ijmn}\cdot Z_m\cdot Z_n, \qquad (4.4)$$

substitute to (4.3) the series for $T^{u}{}_{ik}$ and $T^{V}{}_{kj}$, which are like (4.4). After rewriting the result in powers of **Z**, one obtains:

$$\begin{aligned}T^{W0}{}_{ij} &= T^{u0}{}_{ik}\cdot T^{V0}{}_{kj}.\\T^{W1}{}_{ijm} &= T^{u0}{}_{ik}\cdot T^{V1}{}_{kjm} + T^{u1}{}_{ikm}\cdot T^{V0}{}_{kj},\\T^{W2}{}_{ijmn} &= T^{u0}{}_{ik}\cdot T^{V2}{}_{kjmn} + T^{u1}{}_{ikm}\cdot T^{V1}{}_{kjn} + T^{u2}{}_{ikmn}\cdot T^{V0}{}_{kj}.\end{aligned}\qquad(4.5)$$

In several cases it seems preferably to add directly the frequency precession vectors. To this end let us substitute the expansion (2.5) for **W** and similar to it for **V** and **U** into (3.3) and (3.4). After a simple transformation one obtains that for independent (in **Z**) parts of **W**, **V** and **U** the expression are equivalent to (3.3) and (3.4):

$$\begin{aligned}&\cos({}^{W}o/_2)=\cos({}^{V}o/_2)\cdot\cos({}^{U}o/_2) - \frac{(\mathbf{V}^o\mathbf{U}^o)}{V_oU_o}\cdot\sin({}^{V}o/_2)\cdot\sin({}^{U}o/_2),\\&\frac{W^o{}_i}{W_o} = \frac{1}{\sin({}^{W}o/_2)}\cdot\Big(\frac{V^o{}_i}{V_o}\sin({}^{V}o/_2)\cdot\cos({}^{U}o/_2) + \\&+\frac{U^o{}_i}{U_o}\sin({}^{U}o/_2)\cdot\cos({}^{V}o/_2) + \frac{[\mathbf{V}^o,\mathbf{U}^o]_i}{U_oV_o}\sin({}^{V}o/_2)\cdot\sin({}^{U}o/_2)\Big).\end{aligned}\qquad(4.6)$$

The first of these formulae determines the absolute value of the total rotation angle and the second (using the first result) - the direction of the "total" axis. For linear parts (in **Z**) corresponding expressions are the following:

$$\frac{(\mathbf{W}^o\mathbf{W}^1)}{W_o}\cdot\sin({}^{W}o/_2) = 2\cdot\frac{(\mathbf{V}^o\mathbf{U}^1)+(\mathbf{U}^o\mathbf{V}^1)}{V_o\cdot U_o}\cdot\sin({}^{V}o/_2)\cdot\sin({}^{U}o/_2) +$$

$$+ \frac{(\mathbf{V}^o\mathbf{V}^1)}{V_o}\cdot\cos({}^{U}o/_2)\cdot\sin({}^{V}o/_2) + \frac{(\mathbf{U}^o\mathbf{U}^1)}{U_o}\cdot\cos({}^{V}o/_2)\cdot\sin({}^{U}o/_2) + \tag{4.7}$$

$$+ 2\cdot\frac{(\mathbf{V}^o\mathbf{U}^o)}{V_o\cdot U_o}\cdot[\frac{(\mathbf{U}^o\mathbf{U}^1)}{U_o^2}\cdot\sin({}^{V}o/_2)\cdot(\frac{1}{2}\cos({}^{U}o/_2) - \frac{\sin({}^{U}o/_2)}{U_o})+$$

$$+ \frac{(\mathbf{V}^o\mathbf{V}^1)}{V_o^2}\cdot\sin({}^{U}o/_2)\cdot(\frac{1}{2}\cos({}^{V}o/_2) - \frac{\sin({}^{V}o/_2)}{V_o})]$$

and

$$\frac{\mathbf{W}^1}{W_o}\cdot\sin({}^{W}o/_2) = \frac{\mathbf{W}^o(\mathbf{W}^o\mathbf{W}^1)}{2\cdot W_o^2}\cdot[2\cdot\sin({}^{W}o/_2)/W_o - \cos({}^{W}o/_2)] -$$

$$- [\frac{\mathbf{U}^o(\mathbf{U}^o\mathbf{U}^1)}{U_o^3} - \frac{\mathbf{U}^1}{U_o} + \frac{[\mathbf{U}^o,\mathbf{V}^o]\cdot(\mathbf{V}^o\mathbf{V}^1)}{2\cdot U_o\cdot V_o^3}]\cdot\cos({}^{V}o/_2)\cdot\sin({}^{U}o/_2) -$$

$$- [\frac{\mathbf{V}^o(\mathbf{V}^o\mathbf{V}^1)}{V_o^3} - \frac{\mathbf{V}^1}{V_o} + \frac{[\mathbf{U}^o,\mathbf{V}^o]\cdot(\mathbf{U}^o\mathbf{U}^1)}{2\cdot V_o\cdot U_o^3}]\cdot\cos({}^{U}o/_2)\cdot\sin({}^{V}o/_2) +$$

$$+ [\frac{(\mathbf{U}^o(\mathbf{U}^o\mathbf{U}^1)}{2\cdot U_o^3} + \frac{(\mathbf{V}^o(\mathbf{V}^o\mathbf{V}^1)}{2\cdot V_o^3}]\cdot\cos({}^{V}o/_2)\cdot\cos({}^{U}o/_2) - \tag{4.8}$$

$$- \{(\frac{\mathbf{U}^o}{2} - \frac{[\mathbf{U}^o,\mathbf{V}^o]}{V_o^2})(\mathbf{V}^o\mathbf{V}^1)+(\frac{\mathbf{V}^o}{2} - \frac{[\mathbf{U}^o,\mathbf{V}^o]}{U_o^2})(\mathbf{U}^o\mathbf{U}^1)+[\mathbf{U}^o\mathbf{V}^1]+[\mathbf{U}^1\mathbf{V}^o]\}\cdot$$

$$\cdot\frac{\sin({}^{V}o/_2)\cdot\sin({}^{U}o/_2)}{V_o\cdot U_o}.$$

The last of these formulae determines the correction of the direction of the "total" axis and value $(\mathbf{W}^o\mathbf{W}^1)$, which is contained in it, is calculated firstly in agreement with expression (4.7).

5. Spin precession frequency

As is known [6], the vector $\mathbf{W}_o$ of the spin precession frequency of a charged particle (e,m are its charge, mass and β, γ are velocity and relativistic factor) in electromagnetic fields $\mathbf{H}$ and $\mathbf{E}$ is determined by the BMT expression:

$$\mathbf{W}_o = -\frac{e}{\gamma mc}\cdot\{(1+a\cdot\gamma)\cdot\mathbf{H} - \frac{a\gamma^2}{1+\gamma}\cdot(\beta\cdot\mathbf{H})\cdot\beta - (a\cdot\gamma + \frac{\gamma}{1+\gamma})[\beta,\mathbf{E}]\}, \tag{5.1}$$

where $a = 1.159...*10^{-3}$ is the dimensionless part of the electron anomalous magnetic momentum. This precession frequency determines the known equation for the particle spin $\mathbf{S}$ precession:

$$^{d\mathbf{S}}/_{dt} = [\mathbf{W}o,\mathbf{S}]. \tag{5.2}$$

Let us pass from time t to other independent variable - the azimuth s, which is calculated along the equilibrium orbit. Then in the frame $(\mathbf{e}_x,\mathbf{e}_z,\mathbf{T})$, which is connected with this orbit, the radius-vector of particle position is equal to:

$$\mathbf{r}(x,z,s)=\mathbf{r}_o+x\mathbf{e}_x+z\mathbf{e}_z$$

and

$$^{d}/_{dt} = {}^{dl}/_{dt}\ {}^{d}/_{dl} = c\beta\ {}^{d}/_{dl} = c\beta\ {}^{ds}/_{dl}\ {}^{d}/_{ds} = {}^{c\beta}/_{l'}{}^{d}/_{ds},$$

where l is the arc length along the orbit and prime means differentiation over s. Further one has:

$$^{d\mathbf{r}}/_{ds} = \mathbf{r}_o'+x'\mathbf{e}_x+z'\mathbf{e}_z+x\mathbf{e}_x'+z\mathbf{e}_z'$$

and using the Frene formulae for plane orbit (!) one can obtain

$$\mathbf{T}={}^{d\mathbf{r}}/_{ds}, \quad {}^{d\mathbf{T}}/_{ds}=-K\mathbf{n}, \quad {}^{d\mathbf{n}}/_{ds}=K\mathbf{T}, \quad {}^{d\mathbf{b}}/_{ds}=0,$$

and hence

$$^{d\mathbf{r}}/_{ds} = (1 + K_x + K_z)\mathbf{T} + x'\mathbf{e}_x + z'\mathbf{e}_z. \tag{5.3}$$

Let us assume here and further, that the equilibrium orbit is bit-planar, since either radial curvature K_x or vertical K_z is equal to zero ($K_x \cdot K_z = 0$). Let us retain in the equation for $l' = |dr/ds|$ the terms not higher than second order in deviation from the equilibrium orbit x, x', z, z'. Then one obtains

$$l' = 1 + K_x x + K_z z + \tfrac{1}{2}x'^2 + \tfrac{1}{2}z'^2. \tag{5.4}$$

In this approximation it is easy to find the expression for velocity:

$$\beta = {}^1/_c\, {}^{dr}/_{dt} = {}^{\beta}/_{l'} \cdot r' = \beta(1 + K_x x + K_z z + \tfrac{1}{2}x'^2 + \tfrac{1}{2}z'^2)^{-1} \cdot [(1 + K_x + K_z)T + x'e_x + z'e_z],$$

or

$$\beta = \beta \cdot [(1 - \tfrac{1}{2}x'^2 - \tfrac{1}{2}z'^2)T + (1 - K_x x - K_z z) \cdot (x' \cdot e_x + z' \cdot e_z)]. \tag{5.5}$$

Let us express now the relativistic factor Y and velocity β in terms of relativistic deviation p_σ of particle energy E from its equilibrium value E_o:

$$Y = \frac{E}{mc^2} = \frac{E_o + (E - E_o)}{mc^2} = \frac{E_o}{mc^2}\left(1 + \frac{(E - E_o)}{E_o}\right) = Y_o(1 + p_\sigma) \tag{5.6}$$

and

$$\beta = \left(1 - \frac{1}{Y^2}\right)^{\frac{1}{2}} = 1 - \frac{1}{2Y^2}. \tag{5.7}$$

Let us transform the equation (5.2). Substituting the expression for $\mathbf{S}$ in the introduced frame (e_x, e_z, T) in the form

$$\mathbf{S} = S_x e_x + S_z e_z + S_T T$$

one can obtain

$$\mathbf{S}' = [\mathbf{W}, \mathbf{S}], \tag{5.8}$$

where the derivative in the left part refers to components of vector $\mathbf{S}$ only (but not to vectors e_x, e_z, T !), and $\mathbf{W}$:

$$\mathbf{W} = {}^{l'}/_{c\beta}\, \mathbf{W}_o - K_z e_x + K_x e_z. \tag{5.9}$$

The expression (5.1) for W_o can be rewritten in another form (for $E=0$, that means that only the magnetic system of collider is considered whereas the cavities and other elements with electrical fields are not taken into account):

$$W_o = -\left(a + {}^1/Y\right)\frac{E_o}{mc^2}\frac{ecH}{E_o} + a\frac{Y}{1+Y}\beta\left(\beta\frac{ecH}{E_o}\frac{E_o}{mc^2}\right)$$

or

$$W_o = -Y_o c\left[\left(a + {}^1/Y\right)B - a\frac{Y}{1+Y}\beta(\beta B)\right], \tag{5.10}$$

where the "field" $B = eH/E_o$ and the ratio $Y_o = E_o/mc^2$ are introduced. Let us express the values ${}^1/_Y$ and ${}^Y/_{1+Y}$ and velocity β, which are included in (5.10), in term of p_σ:

$$Y^{-1} = Y_o^{-1}(1+p_\sigma)^{-1} = Y_o^{-1}(1 - p_\sigma + p_\sigma^2),$$

$$Y/_{1+Y} = Y_o^{-2}(1 - Y_o + p_\sigma Y_o + Y_o^2), \tag{5.11}$$

$$\beta = 1 - \tfrac{1}{2}Y_o^{-2}(1 - 2p_\sigma + 3p_\sigma^2).$$

Substituting the expressions (5.5) and (5.11) into (5.10) and the obtained result together with (5.4) into (5.9) one can find the expressions for $\mathbf{W}$ components. It is necessary to substitute there the decompositions of $\mathbf{B}$ components in the form of series in powers of x, z (naturally not more than second order):

$$B_x = B_{ox} + (q - \tfrac{1}{2}B'_{os})x + gz - \tfrac{1}{2}m_z + K_x q + K_z g + B''_{ox})x^2 + \tfrac{1}{2}m_z z^2 + m_x xz,$$

$$B_z = B_{oz} - (q + \tfrac{1}{2}B'_{os})z + gx - \tfrac{1}{2}(m_x - K_z q + K_x g + B''_{oz})z^2 + \tfrac{1}{2}m_x x^2 + m_z xz, \tag{5.12}$$

$$B_s = B_{os} + B'_{ox}x + B'_{oz}z + \tfrac{1}{2}(q' - \tfrac{1}{2}B''_{os})x^2 - \tfrac{1}{2}(q' + \tfrac{1}{2}B''_{os})z^2 + (g' - K_z B'_{ox} - K_x B'_{oz})xz.$$

The following values, which characterize the magnetic field, are introduced there:

$$K_{x,z} = \pm \frac{eH_{oz,ox}}{E_o},$$

$$g = \frac{e}{E_o}\cdot\frac{dH_z}{dx} = \frac{e}{E_o}\cdot\frac{dH_x}{dz}, \qquad q = \frac{e}{2E_o}\cdot\left(\frac{dH_x}{dx} - \frac{dH_z}{dz}\right), \tag{5.13}$$

$$m_{x,z} = \frac{e}{2E_o}\cdot\frac{d^2H_{x,z}}{dxdz} = \frac{e}{2E_o}\cdot\frac{d^2H_{z,x}}{dx^2,dz^2},$$

and the values of all quantities in the right sides are taken on the equilibrium orbit.

Thus, one obtains the final expressions for components of Ww (including the zero, first and second orders on x, z, p_σ and its derivatives $p_x = x' - \frac{1}{2}\cdot{}^e/_{Eo}\cdot H_{os}\cdot z$, $p_z = z' + \frac{1}{2}\cdot{}^e/_{Eo}\cdot H_{os}\cdot x$, p_σ):

$$\begin{aligned} W_x = &- (B_{ox} + K_z) - B_{ox}\cdot(Y_o a + \tfrac{1}{2}\cdot a/Y_o + \tfrac{1}{2}/Y_o^2) + \\ &+ (\tfrac{1}{2}\cdot B'_{os} - q)\cdot(1 + Y_o\cdot a)\cdot x + B_{os}\cdot a\cdot(Y_o - 1)\cdot p_x + \\ &+ [\tfrac{1}{2}\cdot B_{os}^2\cdot a\cdot(Y_o - 1) - (B_{ox}\cdot K_z + g)\cdot(1 + Y_o\cdot a)]\cdot z + B_{ox}\cdot p_\sigma + \\ &+ \tfrac{1}{2}\cdot(g\cdot K_z - q\cdot K_x + m_z + B''_{ox})\cdot(Y_o\cdot a + 1)\cdot x^2 + B'_{ox}\cdot Y_o\cdot a\cdot x\cdot p_x - \\ &- (g\cdot K_x + q\cdot K_z + m_x)\cdot(Y_o\cdot a + 1)\cdot x\cdot z - (\tfrac{1}{2}\cdot B'_{os} - q)\cdot x\cdot p_\sigma + \\ &+ \tfrac{1}{2}\cdot B_{ox}(Y_o\cdot a - 1)\cdot p_x^2 + B'_{oz}\cdot Y_o\cdot a\cdot p_x\cdot z + B_{oz}\cdot Y_o\cdot a\cdot p_x\cdot p_z - \\ &- (g\cdot K_z + \tfrac{1}{2}\cdot m_z)\cdot(Y_o\cdot a + 1)\cdot z^2 + (B_{ox}\cdot K_z + g)\cdot z\cdot p_\sigma - \\ &- \tfrac{1}{2}\cdot B_{ox}\cdot(Y_o\cdot a + 1)\cdot p_z^2 - B_{ox}\cdot p_\sigma^2; \end{aligned} \tag{5.14}$$

$$\begin{aligned} W_z = &- (B_{oz} - K_x) - B_{oz}\cdot(Y_o a + \tfrac{1}{2}\cdot a/Y_o + \tfrac{1}{2}/Y_o^2) - \\ &- [\tfrac{1}{2}\cdot B_{os}^2\cdot a\cdot(Y_o - 1) + (B_{oz}\cdot K_x + g)\cdot(1 + Y_o\cdot a)]\cdot x + \\ &+ (\tfrac{1}{2}\cdot B'_{os} + q)\cdot(1 + Y_o\cdot a)\cdot z + B_{os}\cdot a\cdot(Y_o - 1)\cdot p_z + B_{oz}\cdot p_\sigma - \\ &- (g\cdot K_x + \tfrac{1}{2}\cdot m_x)\cdot(Y_o\cdot a + 1)\cdot x^2 + \\ &+ (q\cdot K_x - g\cdot K_z - m_z)\cdot(Y_o\cdot a + 1)\cdot x\cdot z + B'_{ox}\cdot Y_o\cdot a\cdot x\cdot p_z + \\ &+ (B_{oz}\cdot K_x + g)\cdot x\cdot p_\sigma - \tfrac{1}{2}\cdot B_{oz}\cdot(Y_o\cdot a + 1)\cdot p_x^2 + B_{ox}\cdot Y_o\cdot a\cdot p_x\cdot p_z + \\ &+ \tfrac{1}{2}\cdot(g\cdot K_x + q\cdot K_z + m_x + B''_{oz})\cdot(Y_o\cdot a + 1)\cdot z^2 + B'_{oz}\cdot Y_o\cdot a\cdot z\cdot p_z - \\ &- (\tfrac{1}{2}\cdot B'_{os} + q)\cdot z\cdot p_\sigma + \tfrac{1}{2}\cdot B_{oz}\cdot(Y_o\cdot a - q)\cdot p_z^2 - B_{oz}\cdot p_\sigma^2; \end{aligned} \tag{5.15}$$

$$\begin{aligned} W_s = &- B_{os}\cdot(1 + a + \tfrac{1}{2}\cdot/Y_o^2) - \\ &- B'_{ox}\cdot(1 + a)\cdot x - B'_{oz}\cdot(1 + a)\cdot z + \\ &+ B_{ox}\cdot a\cdot(Y_o - 1)\cdot p_x + B_{oz}\cdot a\cdot(Y_o - 1)\cdot p_z + B_{os}\cdot(1 + a)\cdot p_\sigma - \\ &- \tfrac{1}{2}\cdot[\tfrac{1}{4}\cdot B_{os}^3\cdot(2\cdot Y_o\cdot a + 1) + (q' - \tfrac{1}{2}\cdot B''_{os})]\cdot x^2 + \\ &+ (q - \tfrac{1}{2}\cdot B'_{os})\cdot Y_o\cdot a\cdot x\cdot p_x - g'\cdot x\cdot z + \\ &+ [g\cdot Y_o\cdot a + B_{os}^2\cdot(Y_o\cdot a + \tfrac{1}{2})]\cdot x\cdot p_z + \\ &+ B'_{ox}\cdot x\cdot p_\sigma - \tfrac{1}{2}\cdot B_{os}\cdot(2\cdot Y_o\cdot a + 1)\cdot p_x^2 - \\ &- [B_{os}^2\cdot(Y_o\cdot a + \tfrac{1}{2}) - g\cdot Y_o\cdot a]\cdot p_x\cdot z - \\ &- \tfrac{1}{2}\cdot[\tfrac{1}{4}\cdot B_{os}^3\cdot(2\cdot Y_o\cdot a + 1) - (q' + \tfrac{1}{2}\cdot B''_{os})]\cdot z^2 + \\ &- (q + \tfrac{1}{2}\cdot B'_{os})\cdot Y_o\cdot a\cdot z\cdot p_z - \tfrac{1}{2}\cdot B_{os}\cdot(2\cdot Y_o\cdot a + 1)\cdot p_z^2 + \\ &+ B'_{oz}\cdot z\cdot p_\sigma - B_{os}\cdot p_\sigma^2. \end{aligned} \tag{5.16}$$

6. Algorithm of nonlinear spin motion calculation

In the practical calculation of spin motion different approaches are possible. In one of them (A) the operators - matrixes of spin transformation are calculated and then the spin vector is successively "pulled" through each element of the collider magnetic structure. In the other one (B) for calculation of one-turn transformation of the spin vector the addition of matrixes of all elements is performed after determining each of them. One can calculate the one-turn transformation only after this. In the last one (C), it is possible to calculate and combine the vectors of spin frequency precession. Let us describe the succession of actions for each of the approaches. It is necessary to note that in all cases for each collider element one must:

- calculate the vector **W** in agreement with section 5;
- calculate the integrals (1.6) and (1.7) using the orbital Hamiltonian for this element.

Then the procedures are different.

(A). The operator M of spin transformation is calculated (formulas (1.8)) for a current element and the spin vector at its entrance is "pulled" through this element. Then the operator M of the next element is calculated and so on.

(B). The "total" frequency of spin precession in the current element is calculated (formulas (1.12). Then its orbital part is transformed and orbital transformations are added (formulas (4.2)). After this the matrixes of spin transformation in this element are calculated with the help

of formulae (2.8) - (2.10). At last the "total" matrixes to current collider azimuth are determined (formulas (4.5)).

(C). The "total" frequency of spin precession in current element is calculated (formulas (1.12)). Then its orbital part is transformed and orbital transformations are added (formulas (4.2)). After this the absolute values of the "total" turn angle and direction of "total" axis are determined (formulae(5.6) - (5.8)).

* * *

Thus, the obtained results allow to create the computer code for the nonlinear spin motion calculation in a collider by the optimal method in each specific case (one- and many turns spin dynamics, the dynamical and equilibrium degree of polarization and so on).

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РАСЧЕТ ДВИЖЕНИЯ СПИНА ЭЛЕКТРОНА В НЕЛИНЕЙНОЙ МАГНИТНОЙ СИСТЕМЕ НАКОПИТЕЛЯ С ИСПОЛЬЗОВАНИЕМ ТЕХНИКИ ЛИ ОПЕРАТОРОВ

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