ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

B.G. Konopelchenko

SYMMETRIES OF GENERATING PROBLEMS AND INTEGRABLE EQUATIONS

PREPRINT 90-119

НОВОСИБИРСК

Symmetries of Generating Problems
and Integrable Equations

B.G. Konopelchenko

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

It is shown that the infinite-dimensional Abelian symmetries of the inverse spectral transform (IST) generating problems ($\bar{\partial}$-problem, Riemann—Hilbert problem etc.) give rise to the nonlinear equations for wave functions and potentials integrable by the IST method.

1. The inverse spectral transform (IST) method is a very powerful and effective method for solving nonlinear equations (see e. g. [1—3]). Linear problems which generate the corresponding inverse problem equations are the basic ingredients of the IST method. In the most known cases the Riemann—Hilbert problem (local or nonlocal) and $\bar{\partial}$-problem (local or quasi-local) play the role of such generating problems (see e. g. [1—6]). Structure of these generating problems determines the whole structure and properties of the IST integrable equations. They are also the basic equations in different versions of the dressing method [1, 5].

In the present paper we demonstrate the close interrelation between the symmetries of the linear generating problems (GP) and nonlinear integrable equations. Symmetry of GP is a consistent evolution of the wave function and the inverse problem data. The commuting set of symmetries of GP gives rise to the nonlinear equations for the wave functions and potentials. The infinite-dimensional Abelian symmetry algebra determines the infinite hierarchy of integrable equations. We will discuss several concrete examples.

2. We will consider here the nonlocal $\bar{\partial}$-problem as the representative of GP. This is rather general GP [5]. It is defined by the equation

$$\frac{\partial\hat{\chi}(\lambda,\bar{\lambda})}{\partial\bar{\lambda}}=(\hat{\chi}*R)(\lambda,\bar{\lambda})\equiv$$
$$\equiv\iint\limits_C d\lambda'\wedge d\bar{\lambda}'\hat{\chi}(\lambda',\bar{\lambda}')\,R(\lambda',\bar{\lambda}';\lambda,\bar{\lambda}) \qquad (1)$$

and the canonical normalization $\chi \xrightarrow[\lambda\to\infty]{} 1$. Here $\hat{\chi}$ and R are the matrix-valued functions of the complex variable λ (bar means the complex conjugation). The adjoint GP is of the form

$$\frac{\partial \check{\chi}(\lambda,\bar{\lambda})}{\partial \bar{\lambda}} = -(R^{\mathrm{ad}} * \check{\chi})(\lambda,\bar{\lambda}) \equiv -\iint_C d\lambda' \wedge d\bar{\lambda}'\, R(\lambda,\bar{\lambda};\lambda',\bar{\lambda}')\, \check{\chi}(\lambda',\bar{\lambda}'), \qquad (2)$$

where also $\check{\chi} \xrightarrow[\lambda\to\infty]{} 1$. We assume that the problems (1) and (2) are uniquelly solvable.

In the $\bar{\partial}$-dressing method it is assumed that R depends linearly on some additional variables and then the auxiliary linear problems are constructed which give rise to the integrable soliton equations [5].

We propose a different way. Namely, let us ask a question about symmetries of the problems (1), (2). This question is not trivial since the linear equations (1) and (2) are integral one. But the wide class of the infinitesimal symmetries

$$\chi \to \chi' = \chi + \varepsilon \frac{\partial \chi}{\partial \tau}, \qquad R \to R' = R + \varepsilon \frac{\partial R}{\partial \tau}$$

of the problems (1) and (2) can be found rather easily. Indeed, equations (1) and (2) imply that the infinitesimal symmetry transformations of $\hat{\chi}$, $\check{\chi}$ and R are related as

$$\frac{\partial}{\partial \bar{\lambda}} \frac{\partial \hat{\chi}}{\partial \tau} = \frac{\partial \hat{\chi}}{\partial \tau} * R + \hat{\chi} * \frac{\partial R}{\partial \tau} \qquad (3)$$

and

$$\frac{\partial}{\partial \bar{\lambda}} \frac{\partial \check{\chi}}{\partial \tau} = -\frac{\partial R^{\mathrm{ad}}}{\partial \tau} * \check{\chi} - R^{\mathrm{ad}} * \frac{\partial \check{\chi}}{\partial \tau} \qquad (4)$$

where τ is the symmetry group parameter.

Here we will consider the symmetry transformations of R of the form

$$\frac{\partial}{\partial \tau} R(\chi, \bar{\lambda}'; \lambda, \bar{\lambda}) = D(\lambda') R - R D^{+}(\lambda) + \delta(\lambda - \lambda') A(\lambda, \bar{\lambda}), \qquad (5)$$

where the operator D is of the form

$$D(\lambda) = \sum_{\alpha,\beta=0}^{\alpha+\beta=n} U_{\alpha\beta}(\tau,\lambda,\bar{\lambda}) \frac{\partial^{\alpha}}{\partial \lambda^{\alpha}} \frac{\partial^{\beta}}{\partial \bar{\lambda}^{\beta}}, \qquad (6)$$

where $U_{\alpha\beta}(\tau,\lambda,\bar{\lambda})$ are matrix-valued functions, D^{+} is the operator formally adjoint to D:

$$D^{+}(\lambda)\cdot = \sum_{\alpha,\beta=0}^{\alpha+\beta=n} (-1)^{\alpha}(-1)^{\beta} \frac{\partial^{\alpha}}{\partial \lambda^{\alpha}} \frac{\partial^{\beta}}{\partial \bar{\lambda}^{\beta}} (\cdot\, U_{\alpha\beta}) \qquad (7)$$

and $A(\lambda, \bar{\lambda})$ is a matrix-valued function.

Substituting (5) into (3), combining the result with (1) and (2) and assuming that $R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}) \xrightarrow[\lambda\to\infty]{} 0$, one obtains

$$\iint_C d\lambda \wedge d\bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} \left\{ \left(\frac{\partial \hat{\chi}(\lambda,\bar{\lambda})}{\partial \tau} + \hat{\chi}(\lambda,\bar{\lambda}) D^{+}(\lambda) \right) \cdot \check{\chi}(\lambda,\bar{\lambda}) \right\} =$$

$$= \iint_C d\lambda \wedge d\bar{\lambda} \left(\hat{\chi}(\lambda,\bar{\lambda}) \frac{\partial D^{+}(\lambda)}{\partial \bar{\lambda}} \cdot \check{\chi}(\lambda) + \hat{\chi}(\lambda,\bar{\lambda}) A(\lambda,\bar{\lambda}) \check{\chi}(\lambda,\bar{\lambda}) \right). \qquad (8)$$

The relation (8) gives rise to the following

$$\frac{\partial \hat{\chi}(\lambda,\bar{\lambda})}{\partial \tau} = -\hat{\chi}(\lambda,\bar{\lambda}) D^{+}(\lambda) + \mathrm{Anal}\,(\hat{\chi}(\lambda,\bar{\lambda}) D^{+}(\lambda) \cdot \check{\chi}(\lambda,\lambda)) \cdot \check{\chi}^{-1}(\lambda,\bar{\lambda}) +$$

$$+ \frac{1}{2\pi i} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda} \left(\hat{\chi}(\lambda',\bar{\lambda}') \frac{\partial D^{+}(\lambda')}{\partial \bar{\lambda}'} \cdot \check{\chi}(\lambda',\bar{\lambda}') + \right.$$

$$\left. + \hat{\chi}(\lambda',\bar{\lambda}') A(\lambda',\bar{\lambda}') \check{\chi}(\lambda',\bar{\lambda}') + \hat{B}(\lambda',\bar{\lambda}') \right) \cdot \check{\chi}^{-1}(\lambda,\bar{\lambda}), \qquad (9)$$

where $\hat{B}(\lambda', \bar{\lambda}')$ is an arbitrary matrix-valued function which obeys the constraint

$$\iint_C d\lambda \wedge d\bar{\lambda}\, \hat{B}(\lambda,\bar{\lambda}) = 0$$

and $\mathrm{Anal}\,\Phi(\lambda, \bar{\lambda})$ means the analytic part of the function Φ $\left(\frac{\partial}{\partial \bar{\lambda}} \mathrm{Anal}\,\Phi \overset{\mathrm{def}}{=} 0 \right)$.

In a similar manner one obtains

$$\frac{\partial \check{\chi}(\lambda,\bar{\lambda})}{\partial \tau} = D(\lambda) \check{\chi}(\lambda,\bar{\lambda}) + \hat{\chi}^{-1}(\lambda,\bar{\lambda})\, \mathrm{Anal}\,(\hat{\chi}(\lambda,\bar{\lambda}) D(\lambda) \check{\chi}(\lambda,\bar{\lambda})) -$$

$$- \hat{\chi}^{-1}(\lambda,\bar{\lambda}) \frac{1}{2\pi i} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda} \left(\hat{\chi}(\lambda',\bar{\lambda}') \frac{\partial D(\lambda')}{\partial \bar{\lambda}'} \cdot \check{\chi}(\lambda',\bar{\lambda}') + \right.$$

$$\left. + \hat{\chi}(\lambda',\bar{\lambda}') A(\lambda',\bar{\lambda}') \check{\chi}(\lambda',\bar{\lambda}') + \check{B}(\lambda',\bar{\lambda}') \right), \qquad (10)$$

where $\check{B}(\lambda, \bar{\lambda})$ is an arbitrary matrix-valued function such that

$$\iint_C d\lambda \wedge d\bar{\lambda} \check{B}(\lambda, \bar{\lambda}) = 0 .$$

The formulae (5), (9) and (10) give us $\frac{\partial R}{\partial \tau}$, $\frac{\partial \hat{\chi}}{\partial \tau}$ and $\frac{\partial \check{\chi}}{\partial \tau}$ which, by construction, satisfy equations (3) and (4).

Thus, the formulae (5), (9) and (10) define the class of symmetries of the linear problems (1) and (2). This class of symmetry transformations is characterized by an arbitrary differential operator $D(\lambda)$, an arbitrary function A and by almost arbitrary «constants of integration» $\hat{B}$, $\check{B}$ and $\mathrm{Anal}(\hat{\chi} D^{+} \cdot \check{\chi})$, $\mathrm{Anal}(\hat{\chi} D \check{\chi})$. These symmetry transformations are linear for the «inverse problem data» R and, in general, nonlinear and nonlocal for the «wave functions» $\hat{\chi}$ and $\check{\chi}$. Emphasize that the symmetry transformations (9) and (10) are the joint transformations of the function $\hat{\chi}$ and adjoint function $\check{\chi}$.

The nonlocal $\bar{\partial}$-problem, as it is well known [5], is reduced to the nonlocal Riemann—Hilbert (RH) problem if

$$R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}) = \delta(\Gamma(\lambda'))\, R_0(\lambda, \lambda')\, \delta(\Gamma(\lambda)) ,$$

where the equation $\Gamma(\lambda) = 0$ defines the curve on the complex plane λ. The corresponding symmetry transformation of the nonlocal RH problem in the particular case $n=0$, $(D = Y(\lambda))$, $A=0$, is of the form

$$\begin{aligned} \frac{\partial \hat{\chi}(\lambda)}{\partial \tau} &= -\hat{\chi}(\lambda)\, Y(\lambda) + U(\lambda) \cdot \check{\chi}^{-1}(\lambda) , \\ \frac{\partial \check{\chi}(\lambda)}{\partial \tau} &= Y(\lambda)\, \check{\chi}(\lambda) - \hat{\chi}^{-1}(\lambda)\, U(\lambda) , \end{aligned} \tag{11}$$

where $U(\lambda)$ is an arbitrary rational function on λ. Under the further reduction to the local RH problem $R_0(\lambda, \lambda') = \delta(\lambda - \lambda') R_0(\lambda)$ one has $\check{\chi} = \hat{\chi}^{-1}$ and the symmetry transformation converts into the well-known spectral problem [1—3]

$$\frac{\partial \Psi(\lambda, \tau)}{\partial \tau} = U(\lambda, \tau)\, \Psi , \tag{12}$$

where

$$\Psi = \hat{\chi} \exp \left(\int^{\tau} d\tau'\, Y(\lambda, \tau') \right) .$$

Note that for the local RH problem the RH transformations of the form similar to the discussed here have been considered in [7, 8] within the completely different approach.

Another particular case of (1) is the local $\bar{\partial}$-problem which arise if $R(\lambda', \bar{\lambda}'; \lambda, \bar{\lambda}) = \delta(\lambda - \lambda')\, \delta(\bar{\lambda} - \bar{\lambda}')\, R_0(\lambda, \bar{\lambda})$. In this case $\hat{B} = \check{B} \equiv 0$ and one can identify $\check{\chi} = \hat{\chi}^{-1}$. In the particular case $n=0$ (i. e. when $D(\lambda)$ is the multiplication operator) the formula (9) is reduced to that found in [9] and [10] within the different approaches. In [9] the analog of equation (9) for χ has been used for study of the forced integrable equations. In [10] the same equation has been independently treated as the nonlinear integrable equation for χ.

Equations (9) and (10) can be treated in the same manner as the nonlinear nonlocal (2+1)-dimensional equations solvable with the help of the $\bar{\partial}$-problem (1). A simple example corresponds to the choice $D = -\frac{\partial^2}{\partial \lambda \partial \bar{\lambda}}$, $\hat{B} = \check{B} = 0$ and it is of the form

$$\begin{aligned} &\frac{\partial \hat{\chi}}{\partial \tau} + \frac{\partial^2 \hat{\chi}}{\partial \lambda \partial \bar{\lambda}} - \frac{1}{2\pi i} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda}\, \hat{\chi}(\lambda')\, A(\lambda')\, \check{\chi}(\lambda') \cdot \check{\chi}^{-1}(\lambda) = 0 , \\ &\frac{\partial \check{\chi}}{\partial \tau} - \frac{\partial^2 \check{\chi}}{\partial \lambda \partial \bar{\lambda}} + \hat{\chi}^{-1}(\lambda) \frac{1}{2\pi i} \iint_C \frac{d\lambda' \wedge d\bar{\lambda}'}{\lambda' - \lambda}\, \hat{\chi}(\lambda')\, A(\lambda')\, \check{\chi}(\lambda') = 0 . \end{aligned} \tag{13}$$

The system (13) represents itself the (2+1)-dimensional integrable generalization of the nonlinear Schrödinger equation different from the Davey—Stewartson equation. In general, equations (9), (10) form a wide class of the (2+1)-dimensional nonlinear nonlocal integrable equations.

Now let us consider the infinite set of symmetries of the problem (1) of the form (5), (9) (10) which parametrized by τ_1, τ_2 $\tau_3, \ldots$ This infinite-dimensional symmetry algebra may have different structures. Here we will consider the Abelian infinite-dimensional symmetry algebras of the problem (1). The commutativity of the symmetry transformations (5) implies (at $A \equiv 0$)

$$\left[\frac{\partial}{\partial \tau_i} + D_i(\lambda) , \frac{\partial}{\partial \tau_k} + D_k(\lambda) \right] = 0 , \quad i, k = 1, 2, 3, \ldots \tag{14}$$

that is equivalent to certain nonlinear differential equations for the coefficients $U_{i\alpha\beta}(\tau_1, \ldots; \lambda, \bar{\lambda})$.

The commutativity of the flows

$$\frac{\partial\hat{\chi}}{\partial\tau_i}=-\hat{\chi}D_i^{+}(\lambda)+\hat{\Delta}_i\cdot\check{\chi}^{-1}(\lambda),$$
$$(i=1,2,\ldots) \qquad (15)$$
$$\frac{\partial\check{\chi}}{\partial\tau_i}=D_i(\lambda)\,\check{\chi}+\hat{\chi}^{-1}\,\check{\Delta}_i,$$

where $\hat{\Delta}_i$ and $\check{\Delta}_i$ are the nonlinear in $\hat{\chi}$, $\check{\chi}$ expressions defined by (9) and (10), in general case, gives rise to more complicated nonlinear equations. Equations (15) contain the wave functions $\hat{\chi}$, $\check{\chi}$ and potentials (the «constants of integrations» $\hat{B}$, $\check{B}$, $\mathrm{Anal}(\hat{\chi}D^{+}\cdot\check{\chi})$, $\mathrm{Anal}(\hat{\chi}D\check{\chi})$). The elimination of the wave functions from (15) gives rise to the nonlinear integrable equations for potentials. An alternative procedure of the elimination of potentials («constants of integrations») leads to the nonlinear integrable equations for the wave functions $\hat{\chi}$, $\check{\chi}$. Many concrete examples of the wave (eigen-) functions equations and their properties have been considered earlier in [11].

Thus, the nonlinear integrable equations associated with the $\bar{\partial}$-problem (1) express, in fact, the symmetry property of the problem (1). The following is the simple illustrative example. Let us consider the 2×2 local $\bar{\partial}$-problem and its infinite-dimensional symmetry algebra with $D_i=\lambda^i\sigma_3$ $\left(\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$, $A=0$. In this case the symmetry transformations of wave function in terms of

$$\Psi=\hat{\chi}\exp\left(\sum_{i=1}^{\infty}\sigma_3\tau_i\lambda^i\right),$$

are of the form

$$\frac{\partial\Psi}{\partial\tau_i}=\sum_{\alpha=0}^{i}\lambda^{i-\alpha}P_\alpha(\tau_1,\tau_2,\ldots)\,\Psi,\quad i=1,2,3,\ldots, \qquad (16)$$

where P_α are the 2×2 matrix-valued functions on τ_1, τ_2, ... and $P_0=\sigma_3$. The commutativity of the transformations (16) gives rise to the well-known AKNS hierarchy [12, 3]. In fact, the infinite system (16) completely coincides with that discusses in [12] within the Lie-algebraic approach. This coincidence demonstrates also that the infinite-dimensional algebra considered in (12) is, in fact, the symmetry algebra (in a sence we discussed here) of the 2×2 local $\bar{\partial}$-problem. Within such a treatment the AKNS hierarchy becomes

completely analogous to the Kadomtsev—Petviashvili (KP) hierarchy in Sato approach (see e. g. [13]). The difference is that in the last case the infinitesimal transformations

$$\frac{\partial\Psi}{\partial\tau_n}=(L^n)_+\Psi$$

are the symmetry transformations of the pseudodifferential linear problem $L\Psi=\lambda\Psi$ where [13]

$$L=\partial_x+\sum_{i=1}^{\infty}U_i\,\partial_x^{-1}.$$

The AKNS hierarchy (16) contains as the reduction the Korteweg-de Vries hierarchy which corresponds to the wave fuction Ψ of the form

$$\begin{vmatrix}\varphi(\lambda), & \varphi(-\lambda)\\ \dfrac{\partial\varphi(\lambda)}{\partial\tau_i}, & \dfrac{\partial\varphi(-\lambda)}{\partial\tau_i}\end{vmatrix}.$$

In a similar manner one can treat also the KP-hierarchy and Davey—Stewartson hierarchy.

At last, the symmetries of the generating problem are also the symmetries of the corresponding integrable equations for the potentials and wave functions. Hence, the transformations (9), (10) give us the class of hidden symmetries of these integrable equations. This problem will be discussed in a separate paper.

The author is grateful to S.V. Manakov for the useful discussion.

REFERENCES

1. *Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevski L.P.* Theory of Solitons, Nauka, Moscow (1980); Plenum (1984).
2. *Ablowitz M.Y., Segur H.* Solitons and Inverse Stattering Transform, SIAM, Philadelphia (1981).
3. *Newell A.C.* Solitons in Mathematics and Physics, SIAM, Philadelphia (1985).
4. *Fokas A.S., Ablowitz M.Y.* In: Nonlinear Phenomena, Lecture Notes in Physics, v.189, p.137, (1983), Springer, Berlin.
5. *Zakharov V.E., Manakov S.V.* Funct. Anal and Pril., **19**, N2, 11 (1985).
6. *Beals R., Coifman R.R.* Inverse Problems, **5**, 87 (1989).
7. *Ueno K, Nakamura Y.* Phys. Lett., **109B**, 273 (1982); Publ. RIMS, Kyoto Univ., **19**, 519 (1983).

8. *L.L. Chau.* In: Nonlinear Phenomena, Lecture Notes in Physics, v.**189**, p.110 (1983), Springer, Berlin.
9. *Leon J.J.* Phys. Lett., **144A**, 444 (1990).
10. *Manakov S.V.* Talk given at the Workshop «Nonlinear Evolution Equations, Solitons and the Inverse Scattering Method», Oberwolfach, Germany, Yanuary 1990.
11. *Konopelchenko B.G.* Soliton eigenfunction equations: the IST integrability and some properties, Preprint, University of Paderborn, 1990.
12. *Flaschka H., Newell A.C., Ratiu T.* Physica D, **9**, 300 (1983).
13. *Jimbo M., Miwa T.* Publ. RIMS, Kyoto Univ., **19**, 943 (1983).

B.G. Konopelchenko

Symmetries of Generating Problems and Integrable Equations

Б.Г. Конопельченко

Симметрии порождающих уравнений и интегрируемые уравнения

Ответственный за выпуск С.Г.Попов

Работа поступила 19 октября 1990 г.
Подписано в печать 19.10 1990 г.
Формат бумаги 60×90 1/16 Объем 0,9 печ.л., 0,8 уч.-изд.л.
Тираж 200 экз. Бесплатно. Заказ № 119.

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.