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P.G. Silvestrov and A.S. Yelkhovsky

TWO-DIMENSIONAL GRAVITY AS ANALYTICAL CONTINUATION OF THE RANDOM MATRIX MODEL

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TWO-DIMENSIONAL GRAVITY AS ANALYTICAL CONTINUATION OF THE RANDOM MATRIX MODEL

P. G. Silvestrov and A. S. Yelkhovsky

Institute of Nuclear Physics
630090, Novosibirsk, USSR

Abstract

Quantum gravity in two dimensions is formulated by the analytical continuation of the random matrix model. The smooth solution without poles is found for the exact equation, which determines the partition function dependence on the cosmological constant.

1. Recently the great progress has been made in the formulation and solution of the two-dimensional quantum gravity [1,2,3]. For the first time the universal (i.e., regularization - independent) description of the system was found outside the perturbative framework. The problem of the rigorous definition and exact evaluation of a sum over random surfaces has been reduced to the problem of solving the ordinary differential equation.

In the present paper we would like to discuss several important issues which, in our opinion, were not completely clarified in the Refs [1-3]. First of all, it has to do with the formulation of the model itself. Random surfaces appear in the matrix models as the dual Feynman graphs in the perturbative expansion of the integral over a Hermitian $N \times N$ matrix M:

$$\int dM \exp(-N \operatorname{tr} V(M)) \qquad (1)$$

where V(M) is polynomial in M. Here we shall discuss in detail only the simplest theory with the potential

$$V(M) = \frac{1}{2} M^2 - \frac{g}{48} M^4 \qquad (2)$$

In this theory random surfaces are glued from the unit-area squares. If we discuss the pure gravity, each surface must contribute to the partition function with the positive weight (this weight is merely the number of surfaces of the area A times $\exp(-\Lambda A)$, Λ being the cosmological constant). That's why the coupling constant $g = 48 \exp(-\Lambda)$ in the potential (2) must be positive. Moreover, the interesting critical behavior was observed also at the positive value of the constant $g = g_c = 1$.

On the other hand, the integral (1) exists outside the perturbation theory prescription only at $\mathrm{Re}\, g < 0$. For those values of the constant g, which are reliable to the quantum gravity in two dimensions, the integral (1) diverges. If we want to consider this integral as the exact definition of the theory, not relying on the perturbative expansion only, we have to do something with its divergency.

In order to avoid the contradiction between the positivity of the perturbative expansion and the convergency of the integral we may consider as a non-perturbative definition of the quantum gravity not the integral (1) itself, but its analytical continuation off the half-line $g < 0$ to the half-line $g > 0$ (a possibility of such a continuation was mentioned in [3]). It is easy to

perform this analytical continuation if one replaces the integration over the Hermitian matrix M by that over its eigenvalues. The crucial observation is that while varying the coupling constant in its complex plane one should simultaneously modify the contour of integration. For the specific model with the potential (2) we start with the real negative value of g and then rotate it in the complex plane. Simultaneously we rotate the contours of integration over each of the matrix M eigenvalues in the opposite direction in order not to spoil the convergency of the integral. The result of this analytical continuation can be easily represented in terms of Hermitian matrices. The potential (2) should just be replaced by

$$V(M) = \frac{i}{2} M^2 + \frac{g}{48} M^4 \qquad (3)$$

Here $g > 0$, the sign in front of i is generally not fixed.

The above procedure is easily generalized for an arbitrary polynomial potential. Even for potentials with an odd maximal power, the complexification of the integral allows one to associate the two-dimensional quantum gravity with the convergent matrix integral. For example, the theory with the

potential

$$V(M) = \frac{\exp(i\pi/3)}{2} M^2 + igM^3 \qquad (4)$$

generates the sum over triangulated random surfaces. Obviously, any model defined by a polynomial potential with real coefficients and an odd maximal power can be treated only perturbatively for any values of coupling constants.

Thus, we have defined the model in which real and positive terms of perturbation theory series are generated, while non-perturbative effects lead to the complexity of the final result. An imaginary part of the partition function cannot be expanded in powers of the coupling constant and hence cannot be interpreted in terms of random surfaces. It is the consideration of this complexity that allows us to take a fresh look at the results of [1-3]. The most important conclusion is the following. The real solutions of the Painleve equation, which describes the critical behavior in the quantum gravity, inevitably turns into infinity at some real value of the coupling constant. On the other hand complex solutions which we get in the model with the potential (3), as will be shown below, have no poles on the real axis and are therefore finite.

2. Let us consider now the model (1) with the potential (3) in more detail. As usual, it is convenient [5] to introduce the polynomials $P_n(\lambda)$, $n = 0,1,\ldots$, orthogonal with respect to the measure

$$d\mu(\lambda) = d\lambda \exp(-NV(\lambda)) \qquad (5)$$

Here $V(\lambda)$ is the complex potential (3). These polynomials satisfy the recursion relation:

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda) \qquad (6)$$

All the observables in the theory can be expressed in terms of the coefficients R_n. For example, in order to calculate the partition function, one should find the R_n's with $n = 0,1,\ldots,N$. These coefficients can be found from the recursion relation [7]:

$$\frac{n}{N} = z_n \left(1 - \frac{g}{12}\left(z_{n-1} + z_n + z_{n+1}\right)\right) \qquad (7)$$

Here $z_n \equiv iR_n$. When the parameter N is very large, z_n's are well approximated by the slowly-changing function of number [5]:

$$z_n^{(0)} = 2\,\frac{1-\sqrt{1 - gn/N}}{g} \qquad (8)$$

At $n < N/g$ this solution can be expanded in a power series in the coupling constant g and corresponds to the contribution of random surfaces with the spherical topology. At $n > N/g$, the solution (8) becomes complex. Our formulation of the model provides the realness of the perturbative series only. So, since at $n > N/g$ the function $z_n^{(0)}$ cannot be expanded in powers of g, its complexity causes no confusion.

The corrections to $z_n^{(0)}$ arising due to surfaces with more complicated topologies and due to non-perturbative effects are most important for numbers close to the critical value N/g. There z_n is conveniently represented in the form [1]:

$$z_n = \frac{2}{g} - N^{-2/5} f(N^{4/5}\Delta) \qquad (9)$$

where $\Delta \equiv (N/g - n)/N$. For the function $f(x)$ the Painleve equation is readily obtained:

$$x + \frac{1}{6} f''(x) - \frac{g}{4} f^2(x) = 0 \qquad (10)$$

At $|x| \gg 1$ a solution of this equation is naturally expanded in the asymptotic series [3]:

$$f(x) \sim \frac{2}{\sqrt{g}}\sqrt{x} - \frac{1}{12g} x^{-2} - \frac{49}{576} g^{-3/2} x^{-9/2} - \ldots \qquad (11)$$

where the coefficients are determined by the Eq.(10). The series (11) corresponds to the expansion of z_n in powers of N^{-2}. For large positive x a solution of (10) is determined by the series (11) up to the non-perturbative contribution

$$\delta f(x) = A\, x^{-1/8} \exp(-\frac{4\sqrt{6}}{5} g^{1/4} x^{5/4}) + \qquad (12)$$
$$+ B\, x^{-1/8} \exp(\frac{4\sqrt{6}}{5} g^{1/4} x^{5/4})$$

Similar terms exist at large negative x as well. If we want to have the solution, approaching the classical one (8) far from the critical point, the coefficients in front of the growing with $|x|$ exponents should be taken equal to zero both at large positive (B=0) and large negative x. Two these conditions are enough to determine the solution of the Eq.(10) unambiguously. Strictly speaking, there can be a discrete number of such solutions. Indeed, the real and imaginary parts of the Eq.(10) can be considered as the Newton's equations for a classical point particle, moving on the plane under the

influence of the time-dependent force. In fact, we would like to find the trajectory for this particle, going from the given initial point to the given final one. There can exist a number of distinct trajectories, sufficiently different from each other.

Having fixed the boundary conditions, we got numerical solutions of the Eq.(10). The most smooth solution is shown on the Figure together with the classical one. The difference between these two solutions is due to the higher topologies contributions as well as to non-perturbative effects. For the discrete approximation of the Painleve equation we used the recursion relation (7). The values of the parameters were chosen to be the following: $g = 1$, $N = (10)^5$, $(20)^5$, and $(30)^5$. With the high accuracy (of about 10^{-3}) we observed the expected scaling behavior (9). Our numerical scheme was close to that used in Ref.[8].

The simple analysis shows that any real solution of the Painleve equation (10) inevitably has a pole at some point x_0 on the real axis. The turning of the function f into infinity was interpreted in [1] as some new phase transition (the condensation of handles). In our approach, after the analytical continuation the complex solutions naturally appear. As a result, all the poles of the Painleve

transcendent get away from the real axis and we obtain here the smooth and finite solution.

We would like to note also that the physical argument exists against the interpretation of the function f poles in terms of surfaces with the infinite number of handles. In fact, the contributions of higher genus surfaces decrease the free energy - the appearance of a new handle increases the entropy. Owing to this fact, all the terms of the asymptotic series for z_n have the same sign. Thus the condensation of handles would turn the function f into the *minus* infinity while a pole of the Painleve transcendent corresponds to the infinite value of f with the *plus* sign.

3. As it was pointed out above, there exists a discrete series of solutions of the Painleve equation satisfying the boundary conditions we use. At the same time the recursion relation (7) determines z_n uniquely through the first two values,

$$z_0 = 0 \; ; \quad z_1 = i \frac{\int d\mu(\lambda)\, \lambda^2}{\int d\mu(\lambda)} \tag{13}$$

The Painleve equation describes the behavior of z_n only in the vicinity of the critical point. Now we would like to convince ourselves that we have found the right solution (i.e., the solution of the

recursion relation (7) with the initial data (13)). After reducing to zero the coefficients in front of the exponents, growing with $|x|$ at large $|x|$, the solution of the Painleve equation is uniquely determined by a value of the coefficient A in (12). We mean here only the imaginary part of A, since only the latter can be reliably extracted in the background of the real asymptotic series (11). Below we shall calculate Im A directly from the recursion relation (7) with the initial data (13) in order to check whether it coincides with the value obtained from the numerical solution shown at the Figure.

In the following we fix the value of the coupling constant g to be the critical one $g_c = 1$. Generalization to any other value of g is evident. Let $y_n \equiv \mathrm{Im}\, z_n$. To get into the range of validity of the Painleve equation, it is necessary to express y_n's with the numbers satisfying the inequality $1 - n/N \ll 1$, in terms of y_0 and y_1. At the same time large values of x, relevant for discussion of the asymptotic behavior, correspond to n's, satisfying $N - n \gg N^{1/5}$. That enables us to replace the unknown real part of z_n in the relation (7) by its classical value (8). After such replacement the equation for y_n takes the form:

$$y_{n+1} - 2 F_n y_n + y_{n-1} = 0 , \qquad (14)$$

where $F_n \equiv \dfrac{1 + 2\sqrt{1 - n/N}}{1 - \sqrt{1 - n/N}}$.

It is convenient to rewrite the equation (14) in terms of two-component vectors and the transfer matrix, connecting them:

$$\begin{pmatrix} y_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 2F_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_n \\ v_n \end{pmatrix} , \quad \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} y_1 \qquad (15)$$

The solution of this system is the product of n transfer matrices times the initial column. It is suitable to diagonalize each of transfer matrices:

$$\begin{pmatrix} 2F_n & -1 \\ 1 & 0 \end{pmatrix} = T_n \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix} T_n^{-1} , \quad T_n^{(-1)} = \frac{1}{\sqrt{\lambda_n^2 - 1}} \begin{pmatrix} \lambda_n & (-)1 \\ (-)1 & \lambda_n \end{pmatrix} ,$$

$$\lambda_n = F_n + \sqrt{F_n^2 - 1} \qquad (16)$$

The product of T_k^{-1} and T_{k-1} is very close (up to corrections $\sim N^{-1}$) to the unit matrix. It is easy to show that for λ_k not too close to unity the matrix

products $T_k^{-1}T_k$ don't affect the final result. Thus, up to corrections $\sim N^{-1}$ one obtains:

$$y_n = \left(\lambda_{n-1}^2 - 1 \right)^{-1/2} \prod_{k=1}^{n-1} \lambda_k \, y_1 \qquad (17)$$

Using the Euler-Maclaurin formula we come to the result:

$$\log \prod_{k=1}^{n-1} \lambda_k = \sum_{k=1}^{n-1} \cosh^{-1} F_k \simeq$$

$$\simeq 3N - \frac{4\sqrt{6}}{5} N \left(1 - \frac{n}{N}\right)^{5/4} - \frac{1}{2} \log 24\pi N \qquad (18)$$

Finally, y_n is connected with y_1 through

$$y_n = \frac{\exp(3N)}{6^{3/4}\sqrt{2\pi}} N^{-2/5} x^{-1/8} \exp\left(-\frac{4\sqrt{6}}{5} x^{5/4}\right) y_1 \qquad (19)$$

where $x \equiv N^{4/5}(1-n/N)$. The appearance of the large factor $\exp(3N)$ seems to be an important feature of this result. If the smallness of y_1 doesn't compensate the large exponential factor, one cannot even use the scaling *ansatz* (9) for the recursion relation (7) with the initial data (13). Fortunately, such the compensation really occurs. Simple calculation of the imaginary part of (13) yields (up to corrections $\sim N^{-1}$):

$$y_1 = 3\sqrt{2} \exp(-3N) \qquad (20)$$

Thus, the imaginary part of the non-perturbative contribution to the function f (12) has the coefficient

$$\mathrm{Im}\, A = \frac{6^{1/4}}{2\sqrt{\pi}} \simeq 0.4415 \qquad (21)$$

The numerical value of this parameter varied from 0.420 to 0.435 for different values of N . Comparing these numerical values with (21) it should be taken into account that accuracy of the discrete approximation varied correspondingly from 0.1 to 0.01. So, we can state rather firmly that the solution, shown on the Figure actually takes place in the matrix model under consideration.

4. Let us discuss now the issue of the partition function complexity. The standard formulation of the quantum gravity in two dimensions as the sum over random surfaces (with the weight $\exp(-\Lambda A)$) leads naively, i.e. without an explicit definition of the summation procedure, to the real results. On the other hand, in the matrix model considered here, its complexification seems to be the only way to get the real and fixed-sign perturbative expansion of the

convergent integral. The analysis of non-perturbative effects, arising due to this complexity, was the main theme of the present paper. At the same time the complexity of the partition function deserves a physical interpretation.

The partition function complexity may arise, to our mind, as the result of some instability in the theory, or, more formally, due to the unitarity violation. If a theory allows the Hamiltonian formulation, the unitarity is violated when one tries to consider the evolution inside some subspace of the total Hilbert space. The classical example is the pair creation from the vacuum during collisions of some particles. If the collision energy lies below the threshold of the pair creation, we can stay in the range of the two-particle Hilbert subspace, while at the energies higher than this threshold, it is necessary to take into account the transitions at least between two- and four-particle subspaces in order to provide the unitarity.

The standard way to conserve the unitarity consists in an extension of the Hilbert space. In the functional approach this procedure is equivalent to an extension of the range of functional integration. As for random matrix models, such an extension may consist, for example, in the transition from the one-

to two-matrix theory with the "Hermitian" potential, $V(A,B) = V^{*}(B,A)$. The simplest case of such a potential is the following:

$$V(A,B) = \frac{i}{2}(A^2 - B^2) + \frac{g}{48}(A^4 + B^4) \qquad (22)$$

According to the usual interpretation, in this model the random surfaces of two types are generated, without any interaction between them. The free energy in this model is merely twice the real part of the free energy for the model (1) with the potential (3). Introducing the simplest interaction $\delta V(A,B) = cAB$ we get the Ising model on random surfaces [9], where the matrix integrals are regularized by the analytical continuation.

As a matter of fact, two-matrix models may naturally appear on the physical grounds. In the quantum gravity we have to integrate over *all* metrics (of course, if there is no dynamical principle, restricting the range of integration). Two-dimensional theory, in particular, should include the summation over two possible signatures - Euclidean and Minkowskian. To obtain surfaces with general metrics on them (i.e., metrics of varying signature) one can introduce the term $\delta V(A,B) = k(AB)^2$ in the potential (22). In the perturbation

theory this term will generate the squares with the Minkowskian metric, while the quartic terms in (22) generate that of the Euclidean one.

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Note added

After this work has been prepared for publication we were informed [10] that F.David has also discussed the complexification of a matrix integral as a possible way to define non-perturbatively the sum over topologies [11].

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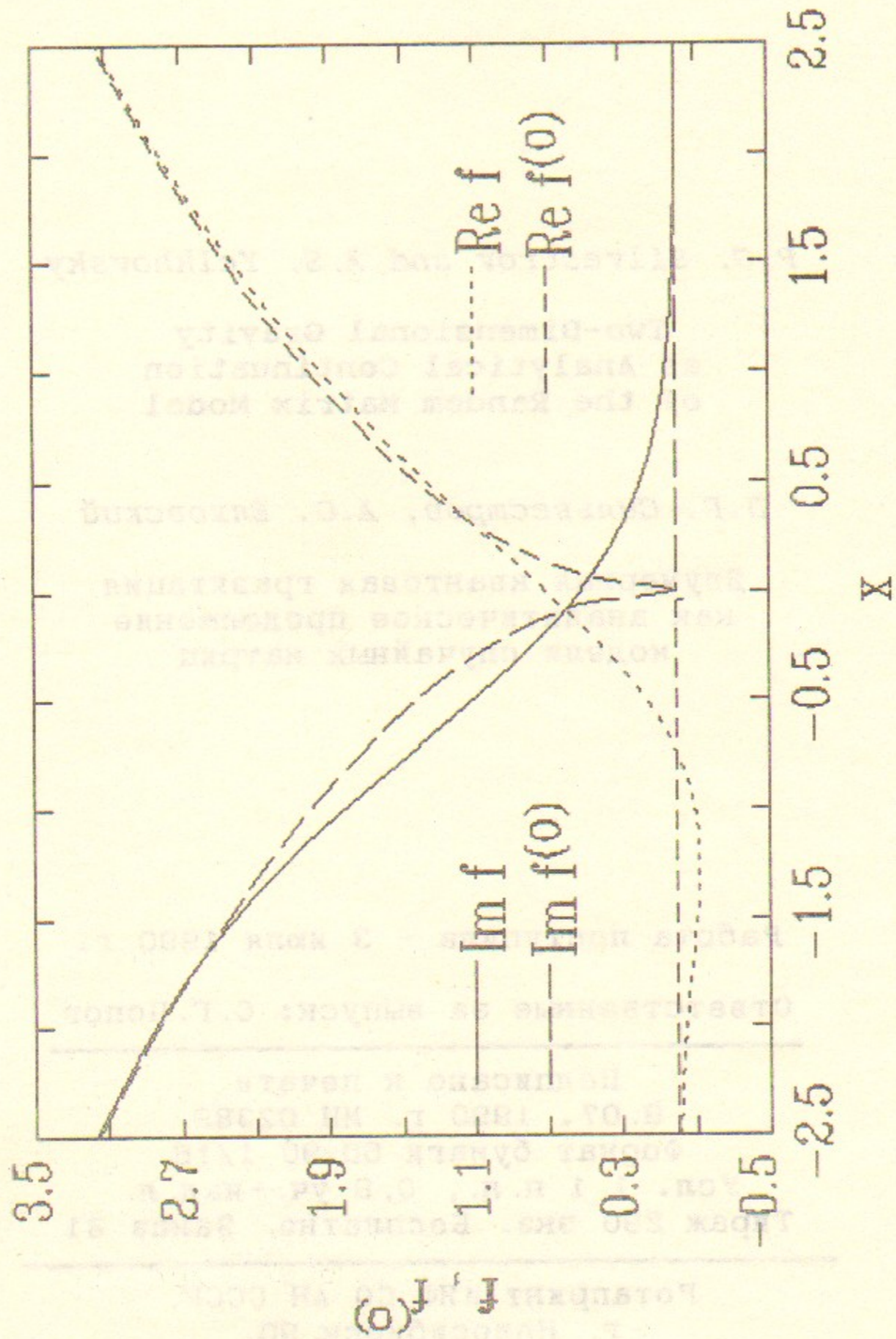
Figure caption

The smooth solution of the Painleve equation together with the classical solution.



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