ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

A.V. Burov

THE BUNCH EXPANSION AT TRANSITION CROSSING

PREPRINT 90-27

НОВОСИБИРСК

The Bunch Expansion
at Transition Crossing

A.V. Burov

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

The longitudinal phase volume of heavy particles bunch is known to increase at transition energy passing. The analytic expression for the degree of this increasing is obtained. The results are demonstrated to be in accordance with experimental data and numerical calculations have been performed for IHEP accelerator U-70.

1. THE STABILITY CRITERIUM AT FIXED ENERGY

Particles, travelling along the vacuum chamber axis, awake electromagnetic field, which acts on following ones. Quantitatively this interaction is usually discribed by means of such a concept as a vacuum chamber impedance. The last will be supposed here to be wide-band, i. e. awoked fields to dissipate faster, than for the time of revolution in the storage ring. The existance of an impedance leads to a possibility of a coherent oscillations instability, the most dangerous are oscillations with frequency greater than the synchrotron one. The stability problem for these fast oscillations for bunch at fixed energy is considered in this paragraph.

The initial point is linearized Vlasov equation:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+e\mathscr{E}\frac{\partial f_0}{\partial v}=0\,, \tag{1}$$

$$e\mathscr{E}=\frac{\Omega_s^2 Ne^2}{\varkappa}\int dx'\,G(x'-x)\,\rho(x')\,. \tag{2}$$

Here x and v are the coordinate and the velocity of a particle relatively the equilibrium one; f and f_0 are a perturbation and an unperturbed phase density correspondingly; ρ is a deviation of the linear density from a steady one (the last is normalized on the unit); Ω_s is a synchrotron frequency; $\varkappa$ is a rigidity of RF focusing: $\varkappa=q_{RF}eU_{RF}/R$ (q_{RF} is a harmonic number, U_{RF} is a voltage, R — an average radius of an orbit); N is a number of particles in a bunch;

G is a wake function, connected with an impedance by Fourier transformation.

It is necessary to pay attention that Vlasov equation (1) is not one with Hermitian operator, therefore an usual silent supposition about its spectrum existence and completeness is groundless.

Vlasov equation is considered below as a problem with an initial condition, as in the Ref. [1].

Let an initial density perturbation to be in a form of the wave packet:

$$\rho(x, 0) = \mathrm{Re}\, \rho_0 \exp\left[i\bar{k}x - \frac{x^2}{2\sigma_p^2}\right],$$

$$\rho_k(0) = \sqrt{2\pi}\, \rho_0\, \sigma_p \exp\left[-(k-\bar{k})^2 \sigma_p^2/2\right], \tag{3}$$

$\sigma_p \leqslant \sigma$, where σ is the r.m.s. bunch length. The condition of oscillation microwaveness

$$\bar{k}\sigma \gg 1 \tag{4}$$

are supposed to be satisfied. The evolution of this perturbation may be described by the following integral [1]:

$$\rho(-x, t) = \sqrt{2\pi}\, \sigma_p\, \mathrm{Re}\, \rho_0 \int_0^\infty \frac{dk}{2\pi} \exp\left[-i\omega_k t - k v_T t + ikx - (k-\bar{k})^2 \sigma_p^2/2\right] \equiv$$

$$\equiv \sqrt{2\pi}\, \sigma_p\, \mathrm{Re}\, \rho_0 \int_0^\infty \frac{dk}{2\pi} e^{\Psi(k)}, \tag{5}$$

where ω_k is the coherent oscillations frequency in the hydrodynamic limit:

$$\omega_k^2 = -i\, k \Omega_s^2\, l^2 \lambda Z_k, \tag{6}$$

$$l^2 = \frac{Ne^2}{\varkappa},$$

λ is the steady state density; an impedance

$$Z_k = \int_0^\infty G(x)\, e^{-ikx} dx \tag{7}$$

is expressed in dimensionless units

1 dimensionless impedance unit $= 30\Omega$;

v_T is the particles velocities dispersion. The integral (5) may be calculated by saddle point method:

$$\int \frac{dk}{2\pi} e^{\Psi(k)} = \frac{e^{\Psi_0}}{r_0}, \tag{8}$$

$$\Psi_0 = \Psi(k_0)\,; \quad r_0 = \sqrt{2\pi\, |\Psi''(k_0)|}\,,$$

k_0 is the saddle point of the exponent $\Psi(k)$:

$$\left.\frac{d\Psi}{dk}\right|_{k=k_0} = 0. \tag{9}$$

The stated method allows to solve the problem of a coherent oscillations stability, since an impedance is known, without any suppositions about spectrum. The results for different types of impedances are brought below.

1. The reduced impedance $|Z_k/k|$ is a monotonous decreasing function of k in the region $k \geqslant \sigma^{-1}$. In this case the criterium of stability is the same, as that for a coasting beam with the linear density equal to the maximum one of the bunch:

$$v_T \geqslant \max_{k \geqslant \sigma^{-1}} \left|\frac{\omega_k}{\kappa}\right|. \tag{10}$$

In this case

$$\max_{k \geqslant \sigma^{-1}} \left|\frac{\omega_k}{k}\right| \simeq \left|\frac{\omega_k}{k}\right|_{k=\sigma^{-1}}.$$

2. The reduced impedance has a maximum in the region $k \geqslant \sigma^{-1}$. Let us consider the situation on the example of single resonance:

$$Z_k = \frac{ik\, G_0}{\mu^2 - k^2 + 2i\nu k}. \tag{11}$$

For the most dangerous situation, when $\mu x \gg \Gamma t$ ($\Gamma = \Omega_s\, l \sqrt{\lambda G_0}$) one can obtain the solution of a saddle point equation (9) (for the appreciation $\sigma_p \ll x \simeq \sigma$ is supposed):

$$k_0 = \mu,$$

$$\mathrm{Re}\, \Psi_0 = -\nu x - \mu v_T t + \frac{3\sqrt{3}}{4} (\Gamma^2 t^2 \mu x)^{1/3}. \tag{12}$$

One can see that the instability is a convectional one: at a given x the disturbance achieves a maximum (the more, the further an observation point from an initial perturbation region), and after this achieving it exponentially decreases to zero.

The requirement of perturbation smallness $\rho<\lambda$ which is equal to

$$(\mathrm{Re}\,\Psi_0)_{max}<\ln\left(\frac{\lambda}{\rho_0}\right)$$

is satisfied, if only the velocity dispersion is high enough:

$$v_T \geqslant s\sqrt{\frac{\mu\sigma}{\nu\sigma+\ln(\lambda/\rho_0)}}, \tag{13}$$

$$s=\left.\frac{\omega_k}{k}\right|_{k\ll\mu}.$$

(The bunch length σ is connected with the velocities dispersion: $\sigma=\sigma_T=v_T/\Omega_s$.) In the case $\nu\sigma\gg\ln(\lambda/\rho_0)$ this criterium coincides with one, deduced from the coasting beam formula (10). In the opposite case the requirement (13) may be rewritten in coasting form (10) too, but the resonance width ν must be replaced by

$$\nu_{eff}=\frac{\ln(\lambda/\rho_0)}{\sigma}. \tag{14}$$

Thus, the coasting beam criterium of stability, with the pointed reserve, may (and will) be applicated to the case of bunched beam too.

2. THE TRANSITION ENERGY TRANSFER

The equations of particle motion in external RF field are suitable to be represented in the next form [2]:

$$\begin{cases}\dfrac{dx}{dt}=-\Omega_s^2 w\\[2mm]\dfrac{dw}{dt}=x\end{cases} \tag{15}$$

$$w=\frac{\varepsilon E T_0}{\varkappa};\quad \Omega_s^2=\frac{|\varkappa\eta|\,c}{E T_0};\quad \eta=\frac{1}{\gamma_t^2}-\frac{1}{\gamma^2};$$

T_0 is the period of revolution; $E=\gamma Mc^2=\gamma E_0$ — an energy of a particle; ε — a relative deviation of energy from equilibrium one, $\varepsilon=\Delta E/E$.

Supposing $\gamma=\gamma_t+\dot\gamma t$ one can find that in the vicinity of the transition energy

$$\Omega_s^2=\frac{|t|}{\tau_0^3}, \tag{16}$$

where

$$\tau_0=\left(\gamma_t^4\,\frac{E_0\,T_0}{2\varkappa\dot\gamma c}\right)^{1/3}. \tag{17}$$

By the force of (15) the bunch length σ_T and the velocities dispersion σ_T change in accordance with the next formulas.

At $|t|\gg\tau_0$ (the region of adiabatic invariant conservation):

$$\sigma_T=\left(\frac{t}{\tau_0}\right)^{1/4}\sqrt{\frac{I}{\tau_0}},\quad v_T=\left(\frac{t}{\tau_0}\right)^{3/4}\frac{1}{\tau_0}\sqrt{\frac{I}{\tau_0}}, \tag{18}$$

I is the phase volume.

At $|t|\leqslant\tau_0$:

$$\sigma_T\simeq\sqrt{\frac{I}{\tau_0}};\quad v_T\simeq\frac{t}{\tau_0^2}\sqrt{\frac{I}{\tau_0}}=\Omega_s\sigma_T\cdot\Omega_s\tau_0. \tag{19}$$

The parameter of stability $|kv_T/\omega_k|$, which may be obtained from these formulas, diminishes with an energy coming up to transition one, at $t=0$ it turns into zero. This means the existence in the vicinity of a transition energy a region of fast instability. Nevertheless, the bunch expansion (lengthening and energy widening) does not inevitably follow: the dangerous district are passed during the limited time.

Let t_k be a time moment when perturbations with wave number k become unstable,

$$\left|\frac{\omega_k}{kv_T}\right|_{t=t_k}=1, \tag{20}$$

During the whole period of an instability region passing perturbations increase in e^{S_k} times, where

$$S_k = \int_{-t_k}^{t_k} \mathrm{Im}\,\omega_k(t)\,dt \simeq |\omega_k(t_k)\; t_k| . \qquad (21)$$

The requirement of phase volume conservation may be expressed in a following form:

$$S_k \leqslant 1 \qquad (22)$$

for every $k \geqslant \sigma_T^{-1}$.

Let us suppose, for the beginning, the moment t_k to be in an adiabatic region, $t_k \gg \tau_0$. In this case

$$S_k \simeq |\omega_k t_k| = \left|\frac{kv_T}{\Omega_s}\Omega_s t_k\right| = \Omega_s t_k\, k\sigma_T \gg 1 , \qquad (23)$$

thus it situation corresponds to very far from threshold one. This is the cause, why $t_k < \tau_0$ is supposed to be. Hence

$$S_k \simeq |\omega_k t_k| = \left|\frac{kv_T}{\Omega_s}\Omega_s t_k\right| = \Omega_s t_k\, k\sigma_T\; \Omega_s \tau_0 . \qquad (24)$$

Using the formula for oscillations frequency (6), one can obtain

$$t_k = \tau_0 \frac{\sigma_k^3}{\sigma_T^3} , \qquad (25)$$

where

$$\sigma_k^3 = \frac{Ne^2}{\sqrt{2\pi}\,\varkappa} \left|\frac{Z_k}{k}\right| . \qquad (26)$$

Substituting (25) in (24) and then in (22), one can get the criterium of phase volume conservation:

$$S_k \simeq \frac{k\sigma_k^6}{\sigma_T^5} = \left(\frac{Ne^2|Z_k|}{\varkappa\sqrt{2\pi}}\right)^2 \frac{1}{kl^{5/2}} \left(\gamma_t^4 \frac{E_0 T_0}{2\varkappa\gamma c}\right)^{5/6} \equiv \left(\frac{N}{N_{th}}\right)^2 \leqslant 1 . \qquad (27)$$

for every $k \geqslant \sigma_T^{-1}$ with σ_T determined by (19).

Let us find the law of phase volume increasing in the opposite to (27) case. When $|t| < |t_k|$, oscillation's turbulization keeps up the velocities dispersion on the level

$$v_T \simeq \left|\frac{\omega_k}{k}\right| \sim \sqrt{t} ,$$

which corresponds to

$$\omega_T \sim \frac{1}{\sqrt{t}} , \quad l \sim \frac{1}{\sqrt{t}} .$$

At $|\omega_k t| \leqslant 1$ such phase volume expansion stops. Determing initial and final time moments as

$$t_i = t_k , \quad |\omega_k(t_f)\; t_f| = 1 ,$$

one can find the degree of phase volume increasing:

$$\frac{l_f}{l_i} = \sqrt{\frac{t_i}{t_f}} \qquad (28)$$

By the force of (6)

$$\frac{|t_f|}{\tau_0} = \frac{1}{(k\sigma_T)^{2/3}} \frac{\sigma_T}{\sigma_k} . \qquad (29)$$

The substitution this formula in (28) gives

$$\frac{l_f}{l_i} = S_k^{1/3} = \left(\frac{N}{N_{th}}\right)^{2/3} . \qquad (30)$$

In the case of transition energy jump, with $t_{jump} > t_f$ (in the opposite case the jump does not matter), the criterium (28) must be modified by replacement $t_f \to t_{jump}$, therefore here

$$\frac{l_f}{l_i} \sim \sqrt{N} . \qquad (31)$$

The experimental dependence for the degree of lengthening $B_f/B_i = (l_f/l_i)^{1/2}$ as a function of current, obtained at IHEP accele-

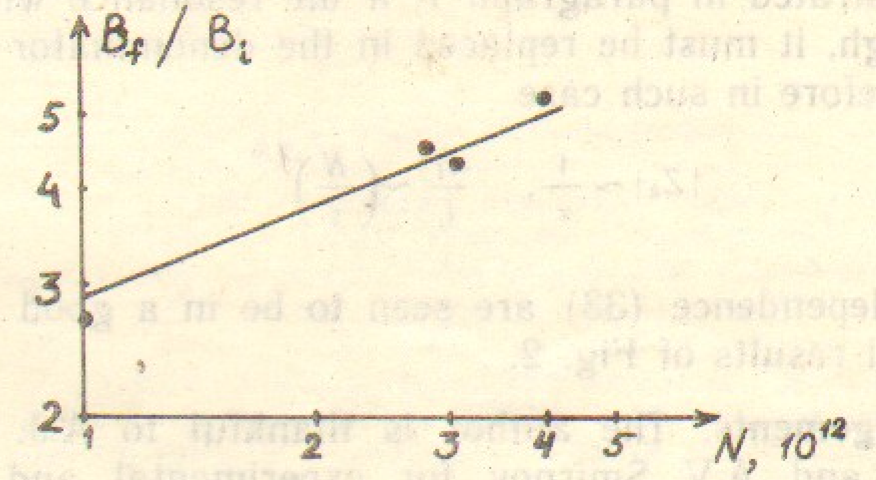

Fig. 1.

rator U-70, is demonstrated on Fig. 1. The graph was kindly suggested to author by P.T. Pashkov and A.V. Smirnov. The line slope corresponds to (30).

The proton bunch expansion because of resonant impedance of U-70 vacuum chamber goffers, represented as

$$Z = R_c \frac{2ik\tau^{-1}}{\mu^2 - k^2 + 2ik\tau^{-1} + \tau^{-2}} \qquad (32)$$

was numerically investigated in Ref. [3]. In particular, the dependence of the degree of the expansion I_f/I_i on the parameter τ was obtained. Represented in logarithmic scale, it is shown on Fig. 2. As

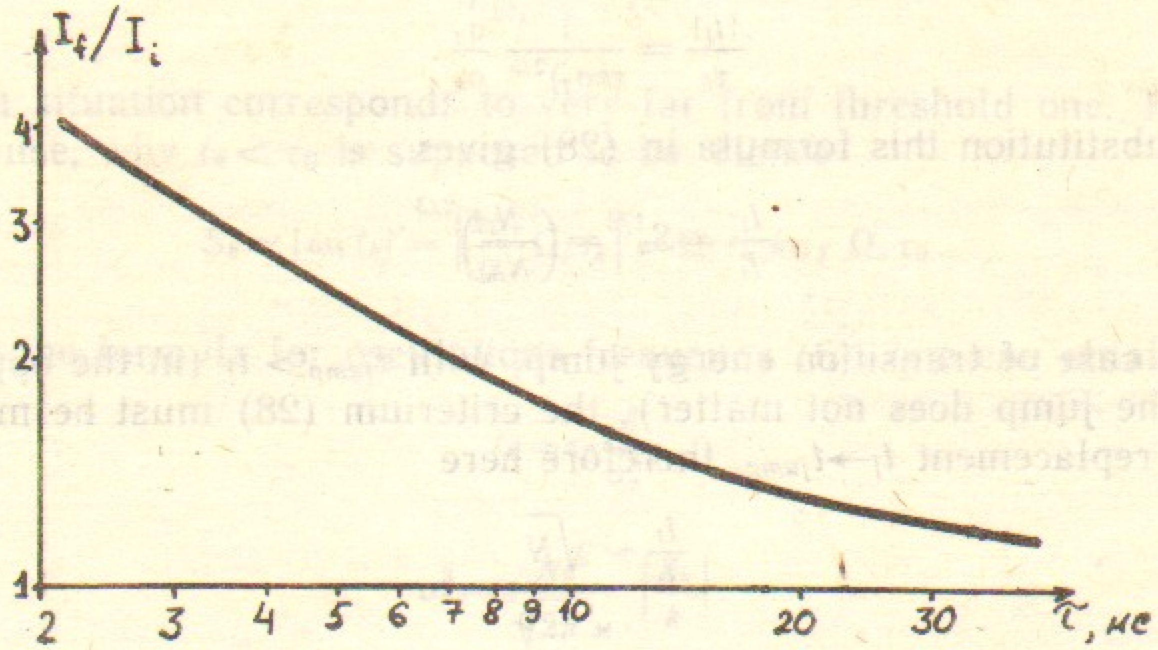

Fig. 2.

it was demonstrated in paragraph 1, if the resonance width $\nu = \tau^{-1}$ is small enough, it must be replaced in the denominator of (32) by ν_{eff} (14). Therefore in such case

$$|Z_k| \sim \frac{1}{\tau}, \quad \frac{I_f}{I_i} \sim \left(\frac{N}{\tau}\right)^{2/3}. \qquad (33)$$

An obtained dependence (33) are seen to be in a good accordance with numerical results of Fig. 2.

Acknowledgements. The author is thankful to A.J. Malovitski, P.T. Pashkov and A.V. Smirnov for experimental and numerical data, to V.I. Balbekov for discussions.

REFERENCES

1. *Burov A.V, Dikansky N.S, Pestrikov D.V.* Proc. XIII IHEA Conf., Novosibirsk, 1986, v.2, p.145.
2. *Lee M.J, Wang J.M.* IEEE Trans. Nucl. Sci., 1985, p.2323.
3. *Balbekov V.I, Malovitski A.J.* Preprint IHEP 86-74, Serpukhov, 1986.

A.V. Burov

The Bunch Expansion at Transition Crossing

А.В. Буров

Переход через критическую энергию: разбухание сгустка

Ответственный за выпуск С.Г.Попов

Работа поступила 7 февраля 1990 г.
Подписано в печать 21.02 1990 г. МН 08410
Формат бумаги 60×90 1/16 Объем 0,9 печ.л., 0,8 уч.-изд.л.
Тираж 250 экз. Бесплатно. Заказ № 27

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.