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INTEGRABILITY OF EQUATIONS FOR SOLITON'S EIGENFUNCTIONS

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Integrability of Equations
for Soliton's Eigenfunctions

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ABSTRACT

Eigenfunctions of the auxiliary linear problems for the soliton equations obey the nonlinear evolution equations. It is shown that these eigenfunction's equations are integrable by the inverse spectral transform (IST) method. Eigenfunction's equations are also the generating equations. Several $(1+1)$- and $(2+1)$-dimensional eigenfunction's equations and their properties are considered.

The inverse spectral transform (IST) method is the very effective and powerful method of investigation of the partial differential equations (see e. g. [1—4]). A starting point of this method is the representation of given nonlinear differential equation as the compatibility condition of the auxiliary linear system

$$L_1(U;\lambda)\,\Psi=0\,, \tag{1}$$

$$L_2(U;\lambda)\,\Psi=0\,, \tag{2}$$

where L_1 and L_2 are some linear operators the coefficients of which depend on U, U_x, ... and on a spectral parameter λ. The nonlinear integrable equation arises after the elimination of the eigenfunction Ψ from the system (1) — (2), for instance, from the commutativity condition $[L_1, L_2]=0$. The nonlinear integrable equations for potential U possess a number of remarkable properties [1—4].

In the present paper we will demonstrate that one can extract the nonlinear differential equations for the eigenfunctions Ψ from the linear systems (1) — (2). It is achieved by the elimination of the potential U from the system (1) — (2). These eigenfunction equations are integrable by the IST method, i. e. they are representable in the form of the compatibility condition for the certain (distinct from (1) — (2)) linear systems with the quartet operator representations. The nonlinear integrable equation for the potential U and the nonlinear integrable equation for the eigenfunction Ψ can be treated as the two irreducible forms of the mixed (reducible) linear system (1) — (2).

We will show also that the eigenfunction's equations are the generating equations since their solutions generate the solutions of the other nonlinear integrable equations. The eigenfunction's equations possess a peculiar linear superposition principle and they are directly linearisable via the certain linear integral equations.

We will consider here the eigenfunction equations for the several well-known soliton equations in $1+1$ and $2+1$ dimensions.

1. The famous Korteweg-de Vries (KdV) equation

$$U_t+U_{xxx}+6UU_x=0 \tag{3}$$

is our first example. The KdV equation is equivalent to the compatibility condition of the linear system [1—4]

$$(\partial_x^2+U-\lambda^2)\,\Psi=0, \tag{4a}$$

$$(\partial_t+4\partial_x^3+6U\partial_x+3U_x)\,\Psi=0. \tag{4b}$$

The elimination of Ψ from (4) gives rise, as usual, to the KdV equation. Now, let us eliminate the potential U from the system (4). Equation (4a) gives $U=\lambda^2-\Psi^{-1}\Psi_{xx}$. Substituting this expression for U into (4b), we obtain the nonlinear equation for the KdV eigenfunction Ψ

$$\Psi_t+6\lambda^2\Psi_x+\Psi_{xxx}-3\Psi^{-1}\Psi_x\Psi_{xx}=0. \tag{5}$$

Equation (5) is integrable by the IST method. Indeed it is equivalent to the compatibility condition for the linear system

$$L_1^{\Psi}\varphi \overset{def}{=} (\Psi\partial_x^2+2\Psi_x\partial_x-\lambda^2\Psi)\,\varphi=0,$$

$$L_2^{\Psi}\varphi \overset{def}{=} (\Psi\partial_t+4\Psi\partial_x^3+12\Psi_x\partial_x^2+6(\Psi_{xx}+\lambda^2\Psi)\,\partial_x)\,\varphi=0 \tag{6}$$

with the quartet operator representation

$$[L_1^{\Psi}, L_2^{\Psi}]=\gamma_1 L_1^{\Psi}+\gamma_2 L_2^{\Psi}, \tag{7}$$

where

$$\gamma_1=-12\Psi_x\partial_x^2-12\Psi_{xx}\partial_x-3\Psi_{xxx}+3\Psi^{-1}\Psi_x\Psi_{xx},$$

$$\gamma_2=2\Psi_x\partial_x+\Psi_{xx}.$$

Equation (5) is representable also in the equivalent left-current's form

$$\Psi_t\Psi^{-1}+6\lambda^2\Psi_x\Psi^{-1}+(\Psi_x\Psi^{-1})_{xx}+2(\Psi_x\Psi^{-1})^3=0, \tag{8}$$

in the bilinear form

$$\Psi\Psi_t+6\lambda^2\Psi\Psi_x+\Psi\Psi_{xxx}-3\Psi_x\Psi_{xx}=0$$

and in the conservation law's form

$$(\Psi^2)_t+(3\lambda^2\Psi^2+(\Psi^2)_{xx}-6(\Psi_x)^2)_x=0.$$

Equation (5) has a plane wave solution $\Psi_0=e^{\lambda x-4\lambda^3 t}$. Introducing the function χ as $\Psi=\chi\exp(\lambda x-4\lambda^3 t)$, one arrives at the following equation

$$\chi_t+\chi_{xxx}-3\chi^{-1}\chi_x\chi_{xx}-6\lambda\chi^{-1}(\chi_x)^2=0, \tag{9}$$

which has quartet operator representation too.

Equation (9) can be easily rewritten in the left-current's, bilinear and conservation law's forms. This equation is of interest since it directly leads to the two nonlinear equations without λ-dependence. Indeed, substituting the asymptotic expansion $\chi=1+\lambda^{-1}\chi_1+$ $+\lambda^{-2}\chi_2+\ldots$ into (9), we obtain $\chi_{1t}+\chi_{1xxx}-6(\chi_{1x})^2=0$ that is nothing but the KdV equation in the potential form ($U=-2\chi_{1x}$) while for $\chi_0=\chi(x, t, \lambda=0)$ one has $\chi_{0t}+\chi_{0xxx}-3\chi_0^{-1}\chi_{0x}\chi_{0xx}=0$ that, of course, coincides with equation (5) for $\Psi(x, t, \lambda=0)$.

Equations (5) and (9) can be considered as the generating equations. Indeed, if for instance, χ obeys equation (9) then the variable $U=-\chi^{-1}(\chi_{xx}+2\lambda\chi_x)$ obviously obeys the KdV equation. At the same time the variable $V=\chi^{-1}\chi_x$ obeys the Gardner equation $V_t+V_{xxx}+6V^2V_x-12\lambda VV_x=0$ or mKdV equation at $\lambda=0$. This equivalence immediately gives as a wide class of exact solutions of the Gardner (or mKdV) equation.

2. The second example is given by the nonlinear Schrödinger (NLS) equation

$$iP_t-\sigma_3P_{xx}-2\sigma_3P^3=0, \tag{10}$$

where $P=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$ and $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ which is the compatibility condition for the linear system [1—4]

$$(-\sigma_3\partial_x+\lambda+P)\,\Psi=0,$$

$$(i\partial_t-2\sigma_3\partial_x^2+2P\partial_x+P_x-\sigma_3P^2)\,\Psi=0, \tag{11}$$

where Ψ is the 2×2 matrix-valued function.

Eliminating the potential P from (11), we arrive at the following nonlinear equation for the NLS eigenfunction Ψ:

$$i\Psi_t-\lambda^2\sigma_3\Psi-2\lambda\Psi_x-\sigma_3\Psi_{xx}+2\sigma_3\Psi_x\Psi^{-1}\Psi_x=0. \tag{12}$$

with the constraint $(\sigma_3\Psi_x\Psi^{-1})_{diag}=\lambda$. Considering this equation for $\Psi(x, t, \lambda=0)$, we obtain the equation [5]

$$i\Psi_t-\sigma_3\Psi_{xx}+2\sigma_3\Psi_x\Psi^{-1}\Psi_x=0. \qquad (13)$$

Equation (12) is equivalent to the compatibility condition for the linear system

$$L_1^{\Psi}\varphi \overset{def}{=} (-\sigma_3\Psi\partial_x+\lambda\Psi)\,\varphi=0,$$

$$L_2^{\Psi}\varphi \overset{def}{=} (i\Psi\partial_t-4\lambda\Psi\partial_x-2\lambda\Psi_x)\,\varphi=0 \qquad (14)$$

with the quartet operator representation $[L_1^{\Psi}, L_2^{\Psi}]=\gamma_1 L_1^{\Psi}+\gamma_2 L_2^{\Psi}$ where

$$\gamma_1=2\lambda\Psi_x-\lambda^2\Psi\sigma_3+\Psi\sigma_3(\Psi_x\Psi^{-1})_x+\Psi\sigma_3(\Psi_x\Psi^{-1})^2,$$

$$\gamma_2=[\Psi, \sigma_3]\,\partial_x-\Psi\sigma_3\Psi_x\Psi^{-1}.$$

So, equation (13) for the NLS eigenfunction is integrable by the IST method too.

Equation (13) is also a generating equation. Indeed, if Ψ obeys equation (13) then the variable $P=\sigma_3\Psi_x\Psi^{-1}-\lambda$ obeys obviously the NLS equation. The fact that at the same time the variable $S=-\Psi^{-1}\sigma_3\Psi$ obeys the Heisenberg ferromagnet equation

$$iS_t+\frac{1}{2}[S, S_{xx}]=0$$

(see e. g. [1—4]) is the more nontrivial and interesting result [5].

In the left-current's form equation (13) looks like

$$i\Psi_t\Psi^{-1}-\sigma_3(\Psi_x\Psi^{-1})_x+\sigma_3(\Psi_x\Psi^{-1})^2=0. \qquad (15)$$

3. Now let us consider the (2+1)-dimensional soliton equations. The Kadomtsev—Petviashvili (KP) equation (see [1—4])

$$(U_t+U_{xxx}+6UU_x)_x+3\alpha^2U_{yy}=0, \qquad (16)$$

the most well-known from them, is equivalent to the compatibility condition for the system [1—4]

$$(\alpha\partial_y+\partial_x^2+U)\,\Psi=0, \qquad (17a)$$

$$(\partial_t+4\partial_x^3+6u\partial_x+3u_x-3\alpha W_y)\,\Psi=0, \qquad (17b)$$

where $W_x=U$ and $\alpha^2=\pm1$. Equation (17a) gives $U=-\Psi^{-1}(\alpha\Psi_y+\Psi_{xx})$. Substituting this expression into (17b), one

obtains [5] the following nonlinear equation

$$\Psi_t+\Psi_{xxx}-3\Psi^{-1}\Psi_x\Psi_{xx}-3\alpha\Psi^{-1}\Psi_x\Psi_y-3\alpha\Psi_{xy}-3\alpha\Psi W_y=0, \qquad (18)$$

$$W_x+\alpha\Psi_y\Psi^{-1}+\Psi^{-1}\Psi_{xx}=0. \qquad (19)$$

Equation (18) for the KP eigenfunction Ψ is equivalent to the compatibility condition for the linear system [5]

$$L_1^{\Psi}\varphi \overset{def}{=} (\alpha\Psi\partial_y+\Psi\partial_x^2+2\Psi_x\partial_x)\,\varphi=0,$$

$$L_2^{\Psi}\varphi \overset{def}{=} (\Psi\partial_t+4\Psi\partial_x^3+12\Psi_x\partial_x^2+6(\Psi_{xx}-\alpha\Psi_y)\,\partial_x)\,\varphi=0 \qquad (20)$$

with the quartet operator representation $[L_1^{\Psi}, L_2^{\Psi}]=\gamma_1 L_1^{\Psi}+\gamma_2 L_2^{\Psi}$ with

$$\gamma_1=-12\Psi_x\partial_x^2-12\Psi_{xx}\partial_x-3\Psi_{xxx}+3\Psi^{-1}\Psi_x\Psi_{xx}+$$

$$+3\alpha\Psi^{-1}\Psi_x\Psi_y-3\alpha\Psi_{xy}-3\alpha\Psi W_y,$$

$$\gamma_2=2\Psi_x\partial_x+\alpha\Psi_y+\Psi_{xx}.$$

Equation (18)—(19) is the (2+1)-dimensional analog of equation (6). Similar to equation (6) it can be considered as the generating equation: namely, introducing the variable $U=-\Psi^{-1}(\alpha\Psi_y+\Psi_{xx})$, we obtain the KP equation, while the variable $V=\Psi^{-1}\Psi_x$ obeys the equation

$$V_t+V_{xxx}+6V^2V_x-6\alpha V_x\partial_x^{-1}V_y+3\alpha^2\partial_x^{-1}V_{yy}=0, \qquad (21)$$

which is the modified KP equation, introduced in [6, 7]. At the same time the relation (19) is transformed into the two-dimensional Miura transformation $U=-V_x-V^2-\alpha\partial_x^{-1}V_y$ [6, 7]. The equivalence of equation (18)—(19) to the mKP equation allows us to construct a wide class of exact solutions of the mKP equation, using the known KP eigenfunctions.

4. The next (2+1)-dimensional example is given by the Devay—Stewartson (DS) equation (see [1—4])

$$iP_t-\sigma_3(P_{xx}+\alpha^2P_{yy})+\sigma_3\,\mathrm{tr}(\sigma_3Q_D)\,P=0,$$

$$(\alpha\partial_y-\sigma_3\partial_x)\,Q_D=\sigma_3(\alpha\partial_y+\sigma_3\partial_x)\,P^2, \qquad (22)$$

where $P=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$, Q_D is a diagonal matrix and $\alpha^2=\pm1$. Equation (22) is the compatibility condition for the system

$$(\alpha\partial_y-\sigma_3\partial_x+P)\,\Psi=0, \qquad (23a)$$

$$(i\partial_t - 2\sigma_3\partial_x^2 + 2P\partial_x + Q_D + \sigma_3(\alpha P_y + \sigma_3 P_x))\,\Psi = 0\,, \qquad (23b)$$

where Ψ is the 2×2-nondegenerate matrix. Let us eliminate the potential P from the system (23). Equation (23a) gives $P = (\sigma_3\Psi_x - \alpha\Psi_x)\Psi^{-1}$. Substituting this expression into (23b), after some transformations we obtain [5]

$$\begin{aligned} & i\Psi_t - \sigma_3(\Psi_{xx} + \alpha^2\Psi_{yy}) + \sigma_3\Psi_x\Psi^{-1}\Psi_x + \alpha^2\sigma_3\Psi_y\Psi^{-1}\Psi_y - \\ & - \alpha\Psi_y\Psi^{-1}\Psi_x - \alpha\Psi_x\Psi^{-1}\Psi_y + (\alpha C_y + \sigma_3 C_x)\,\Psi = 0\,, \qquad (24) \\ & \alpha C_y - \sigma_3 C_x - \sigma_3(\alpha\Psi_y\Psi^{-1} - \sigma_3\Psi_x\Psi^{-1})^2 = 0\,. \end{aligned}$$

Equation (24) for the DS eigenfunction Ψ is integrable by the IST method. Indeed, it is equivalent to the compatibility condition for the following linear system [5]

$$\begin{aligned} & L_1^{\Psi}\varphi \stackrel{def}{=} (\alpha\Psi\partial_y - \sigma_3\Psi\partial_x)\,\varphi = 0\,, \\ & L_2^{\Psi}\varphi \stackrel{def}{=} (i\Psi\partial_t - 2\alpha\Psi\partial_x\partial_y - 2\alpha\Psi_x\partial_y - 2\alpha\Psi_y\partial_x)\,\varphi = 0\,. \qquad (25) \end{aligned}$$

In the operator form equation (24) looks like

$$[L_1^{\Psi}, L_2^{\Psi}] = \gamma_1 L_1^{\Psi} + \gamma_2 L_2^{\Psi}\,, \qquad (26)$$

where L_1^{Ψ}, L_2^{Ψ} are given by (25) and

$$\begin{aligned} & \gamma_1 = 2\alpha\Psi_y\partial_x + 2\alpha\Psi_x\partial_y + a\,, \\ & \gamma_2 = [\Psi, \sigma_3]\,\partial_x + \Psi(\alpha\Psi_y - \sigma_3\Psi_x)\,\Psi^{-1}, \\ & a = \Psi\sigma_3(\Psi_x\Psi^{-1})_x - \alpha^2\Psi\sigma_3(\Psi_y\Psi^{-1})_y + \\ & + \alpha\Psi[\Psi_y\Psi^{-1}, \Psi_x\Psi^{-1}] - \Psi(\alpha C_y + \sigma_3 C_x)\,. \end{aligned}$$

The IST method is applicable to equation (24) is a rather standard manner [8]. The use of the $\bar\partial$-problem and nonlocal Riemann—Hilbert problem method allows one to solve the initial value problem for equation (24) [8]. The $\bar\partial$-dressing method gives also the possibility to construct a wide class of the exact solutions of equation (24) with the functional parameters and rational-exponential solutions [8].

Equation (24) has also a direct algebra-geometric sence since it can be rewritten in the left-current's form

$$\begin{aligned} & i\Psi_t\Psi^{-1} - \sigma_3(\Psi_x\Psi^{-1})_x - \alpha^2\sigma_3(\Psi_y\Psi^{-1})_y - \\ & - \alpha\Psi_y\Psi^{-1}\Psi_x\Psi^{-1} - \alpha\Psi_x\Psi^{-1}\Psi_y\Psi^{-1} + \alpha C_y + \sigma_3 C_x = 0\,, \qquad (27) \\ & \alpha C_y - \sigma_3 C_x - \sigma_3(\alpha\Psi_y\Psi^{-1} - \sigma_3\Psi_x\Psi^{-1})^2 = 0\,. \end{aligned}$$

So, equation (24) is of the principal chiral type equation (see e. g. [1, 4]).

Similar to equation (13) equation (24) is the generating equation. Indeed, if Ψ obeys equation (24) then the combination

$$P = (\sigma_3\Psi_x - \alpha\Psi_y)\,\Psi^{-1}$$

obviously obeys the DS equation (22) and at the same time the variables [5, 8]

$$\begin{aligned} & S = \vec{S}\,\vec{\sigma} = -\Psi^{-1}\sigma_3\Psi\,, \\ & \Phi = 2i\alpha \ln\det\Psi\,, \qquad (29) \end{aligned}$$

where σ_1, σ_2, σ_3 are the Pauli matrices, obey the Ishimori equation [9]

$$\begin{aligned} & \vec{S}_t + \vec{S}\times(\vec{S}_{xx} + \alpha^2\vec{S}_{yy}) + \Phi_x\vec{S}_y + \Phi_y\vec{S}_x = 0\,, \\ & \Phi_{xx} - \alpha^2\Phi_{yy} + 2\alpha^2\vec{S}\cdot(\vec{S}_x\times\vec{S}_y) = 0\,, \qquad (30) \end{aligned}$$

where $\vec{S}\vec{S} = 1$. This circumstance allows one to construct the new exact solutions of equation (30), in particular, the exponentially localized solutions. The interrelation between equation (24) and DS and Ishimori equations has been discussed in [8] in more details.

5. At last, the final example. This is the equation

$$P_t - A\,\mathrm{ad}_A^{-1}[B, P_x] - \mathrm{ad}_A^{-1}[B, P_y] + [\mathrm{ad}_A^{-1}[B, P], P] = 0\,, \qquad (31)$$

which describes the resonantly interacting waves in $2+1$ dimensions x, y, t [1—4]. Here P is the $N\times N$ off-diagonal matrix, A and B are diagonal matrices with the distinct elements and $\mathrm{ad}_A\Phi \stackrel{def}{=} [A, \Phi]$. Equation (31) is integrable with the help of the linear system [1—4]

$$\begin{aligned} & \Psi_y + A\Psi_x + P\Psi = 0\,, \\ & \Psi_t + B\Psi_x + (\mathrm{ad}_A^{-1}[B, P])\,\Psi = 0\,, \qquad (32) \end{aligned}$$

where Ψ is the $N\times N$ matrix.

Eliminating the potential P from (32), we arrive at the following nonlinear equation for the matrix-valued eigenfunction Ψ [5]

$$\Psi_t + B\Psi_x - \mathrm{ad}_A^{-1}[B, (\Psi_y + A\Psi_x)\,\Psi^{-1}]\,\Psi = 0\,. \qquad (33)$$

Equation (33) is integrable by the IST method too: it is equivalent to the compatibility condition for the linear system [5]

$$L_1^\Psi \varphi \overset{def}{=} (\Psi\partial_y + A\Psi\partial_x)\varphi = 0,$$
$$L_2^\Psi \varphi \overset{def}{=} (\Psi\partial_t + B\Psi\partial_x + \Psi)\varphi = 0 \qquad (34)$$

with the quartet operator representation $[L_1^\Psi, L_2^\Psi] = \gamma_1 L_1^\Psi + \gamma_2 L_2^\Psi$, where

$$\gamma_1 = [\Psi, B]\,\partial_x - \Psi\,\mathrm{ad}_A^{-1}[B, \Psi_y\Psi^{-1} + A\Psi_x\Psi^{-1}],$$
$$\gamma_2 = [A, \Psi]\,\partial_x + \Psi(\Psi_y + A\Psi_x)\Psi^{-1}.$$

As in the previous cases equation (33) is the generating equation. Namely, if Ψ is the solution of equation (33) then the variable $P = -(\Psi_y + A\Psi_x)\Psi^{-1}$ obviously obeys equation (31). The equation for the quantity $S = \Psi^{-1}A\Psi$ should be the second generated equation. In the case $B = \sum_{n=0}^{N-1} C_n A^n$ where C_n are some constants, this equation looks like [5]

$$S_t + \sum_{n=0}^{N-1} C_n (S^n S_x - S(S^n)_x - (S^n)_y) = 0. \qquad (35)$$

Equation (35) is equivalent to the commutativity condition $[L_1^S, L_2^S] = 0$ with

$$L_1^S = \partial_y + S\partial_x, \quad L_2^S = \partial_t + \sum_{n=0}^{N-1} C_n S^n \partial_x. \qquad (36)$$

Note that equation (35) is the local (2+1)-dimensional integrable equation.

The simplest equation (35) corresponds to the choice $C_0 = C_1 = C_3 = \ldots = 0$, $C_2 = 1$ and is of the form

$$S_t - SS_xS - (S^2)_y = 0. \qquad (37)$$

Under Z_N reduction or in the scalar case equation (37) is of the very simple form

$$\varphi_t - \frac{1}{3}(\varphi^3)_x - (\varphi^2)_y = 0, \qquad (38)$$

where $\varphi(x, y, t)$ is a scalar field.

6. An important feature of the eigenfunction's equations is that these equations maintain some properties of the linear systems from which they have been derived. For instance, equation (5) for the KdV eigenfunction possesses the plane wave solution $\Psi_0 = e^{\lambda x - 4\lambda^3 t}$. Moreover, the rather peculiar linear superposition principle takes place for equation (5); namely, if Ψ_1 and Ψ_2 are the two distinct solutions of equation (5) obeying the additional constraint $\Psi_1^{-1}\Psi_{1xx} = \Psi_2^{-1}\Psi_{2xx}$, then the linear superposition $a\Psi_1 + b\Psi_2$ where a and b are arbitrary constants is the solution of equation (5) too. This property is an obvious consequence of the equivalence of equation (5) to the linear system (4).

By the same reason the solutions of the eigenfunction equations are given also by the solutions of the certain singular integral equations. These equations are, in fact, nothing but the corresponding inverse problem equations for the linear system (1)—(2). For example, the KdV eigenfunction χ obeys the linear integral equation (see, e. g. [1—3])

$$\chi(x, t, \lambda) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\lambda' \frac{R(\lambda')\,\chi(x, t, -\lambda')\, e^{2i\lambda' x - 8i\lambda'^3 t}}{\lambda' - \lambda + i0},$$

where $R(\lambda)$ is an arbitrary function. So the solutions of the nonlinear eigenfunction equation (7) is given by the solution of this linear equation.

The similar equivalence takes place also for the other equations considered above. More general corresponding singular integral equations can be found in the framework of the direct linearizing transform method (see e. g. [9, 10]).

7. Thus in all cases considered the nonlinear equations for the soliton eigenfunctions which are derived from the corresponding auxiliary linear systems are the equations integrable by the IST method too. One can easily add the other similar examples. It is a general phenomenon.

This general phenomenon can be treated as follows. Let one has the linear system (1)—(2) which contains both the potential U and the eigenfunction Ψ. The elimination of the eigenfunction Ψ from this system gives rise to the nonlinear equation for the potential U which is integrable by the IST method. On the other hand, the elimination of the potential U from (1)—(2) leads to the nonlinear equation for the eigenfunction Ψ which is integrable by the IST method too. So, the integrable equation for the potential U and the integrable equation for the eigenfunction Ψ can be treated as the irreducible integrable forms of the basic mixed and reducible linear system (1) — (2).

In a similar manner one can treat also the different transformations associated with the integrable equations. For instance, the Darboux transformations are the invariance transformations for the mixed linear system (1) — (2) while its irreducible forms are the usual auto-Backlund transformations for the irreducible integrable equations for the potentials and eigenfunctions. This problem and also the interrelations between the conservation laws, symmetries, Hamiltonian structures, recursion operators for the equations for the eigenfunctions and equations generated by them will be considered elsewhere.

Note also that the equations generated by the eigenfunction equation, e. g. the KdV equation and mKdV equation [5], NLS equation and Heisenberg ferromagnet model equation [3], KP equation and mKP equation [5], DS equation and the Ishimori equation [10, 11], equation (31) and equation (35) are the equations which are gauge equivalent to each other. The corresponding generating equation (the equation for eigenfunction) coincides, in fact, with the equation for the corresponding gauge function.

In conclusion we emphasize the fact that the nonlinear integrable eigenfunction equations are the principal chiral fields type equations. In virtue of this, it seems, they should admit a direct algebra-geometric treatment in an orbit framework with an appropriate infinite-dimensional local Lie groups.

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**Интегрируемость нелинейных уравнений
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