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TORONS, CHIRAL SYMMETRY BREAKING AND U(1) PROBLEM IN σ-MODEL AND IN GAUGE THEORIES. I. σ-MODEL

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I. σ-MODEL.

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ABSTRACT

A novel class of self-dual solutions in σ-models and in SU(2) gauge theories is considered. The solution is defined on manifold with boundary, it has topological charge Q = 1/2. The contribution of the corresponding fluctuations to chiral condensate is calculated. This contribution has finite nonzero value.

The APS (Atiyah, Patodi, Singer) theorem for a manifold with a boundary is discussed for the O(3) σ model. The necessity of imposing non-local boundary conditions for the Dirac operator is explained. The toron effects in supersymmetric 2d O(3) σ model and 4d supersymmetric gluodinamics (SYM) may be reconducted to fermionic zero modes (ZM). In the gauge theories (QCD and SQCD for example), containing the fields in the fundamental representation (guarks), the situation is guite different. The unbound resonances of the continuum at $\lambda \to 0$ play a crucial rôle in this case.

The contribution of toron configurations to chiral condensates $\langle \bar{\varphi}\varphi \rangle$, $\langle \lambda^2 \rangle$ in SQCD is calculated and it is consistent with the Konishi anomaly. For the fermion coudensate in QCD (with $N_f = N_c = 2$) we find $\langle \bar{\psi}\psi \rangle = -\pi^2 \exp\{5/12\} 2^4 \Lambda^3$, $\Lambda^3 = M_0^3 g^{-4} e^{-\frac{4\pi^2}{g^2}}$.

The U(1) problem and θ periodicity puzzle in QCD are also discussed.

1. Introduction.

At this time the best-known example of nonperturbative fluctuations is the instanton [1,2]. The integral nature of the topological charge Q is in that case related to the compactification of the space to a sphere, i.e. with the identification of all infinitely distant points. The choice of other boundary conditions could result in fractional topological charges. In particular, in gluodinamics with the SU(N) gauge group, the introduction of so-called twisted boundary conditions [3] permits new solutions of the classical eguations-torons [4,5] with Q = K/N, K = 0,1... and with action $S = (8\pi^2/g^2 N)K$.

What physical effects arise due to fluctuations with fractional Q? These effects appear most glaringly in supersymmetric variants of a theory. In particular, in supersymmetric Yaug-Mills theory (SYM) with SU(2) gauge group torons ensure spontaneous breaking of discrete chiral symmetry. Indeed, the model possesses naive U(1) chiral symmetry with respect to the transformations $\lambda^a \to \exp(i\alpha)\lambda^a$ (λ^a is gluino field), which is broken by the anomaly $\partial_\mu a_\mu \sim G\tilde{G}$. However, under this transformation the discrete symmetry $Z_2 \times Z_2$ is conserved. As was shown some years ago, the 't Hooft's torons [4] generate the condensate $\langle \lambda^2 \rangle$ and break this symmetry down to Z_2 [6]. However, the standard guasiclassical calculations of $\langle \lambda^2 \rangle$ [6], based on the solution [4], are unreliable because solution [4], defined in a box of sizer L_μ, exists only if the ratios of the sizes L_μ satisfy certain relations, and constant $g(L \to \infty) \to \infty$ becomes too large for the correctness of the guasiclassical calculations. Moreover, the introduction of fields in the fundamental (rather then andjoint) representation is rather difficult because of special (twisted) boundary conditions. Therefore the 't Hooft's solution can be considered as only an illustrative example with fractional charge.

Nevertheless, we believe that solutions with a fractional number may play an important role in the theory, but these solutions should be formulated in some way other than the 't Hooft's solution. As will be shown below, our solution can be formulated in a manifold with boundary and may be understood as the point defect. This solution admits the introduction of the fields in the fundamental representation. So, our solution can be used for consideration of interesting physical theories (like QCD) as like as of SYM.

Let us remind that instantons give zero contribution to $\langle\lambda^2\rangle$ in SU(2) SYM and can ensure nonzero values only for the correlator $\langle\lambda^2(x),\lambda^2(0)\rangle$ [7,8]. Indeed, in this model there are four fermion zero modes per an instanton (Z.M.). It means that one instanton transition is always accompained by emission of four fermion fields and thus $\langle\lambda^2(x),\lambda^2(0)\rangle\neq 0$. By clustering, this relation implies a non-vanishing $\langle\lambda^2\rangle$ condensate, in agreement with the value of the Witten index [9], which equals two and in agreement with rigorous results of Ret [10]. However $\langle\lambda^2\rangle_{instanton}=0$ because we have four (not two) zero modes. It is obvious that we would get $\langle\lambda^2\rangle\neq 0$ for solutions, which have 2 Z.M. But the toron solution with Q = 1/2 has just two ZM So, it is the main reason for considering fractional Q in supersymmetric theories. In ordinary (nonsupersymmetric) theories (like QCD) the solution with fractional topological number should play an important role (in the solutiou of the U(1) problem and θ-periodicity puzzle) too, as was shown some years ago [11,12].

The paper is organized as follows. In Sec.2 O(3) σ model is formulated in terms of various fields: the unit vector field n^a, a=1,2,3, $n^a n^a=1$; the complex field φ, the unit complex spinor u_α, α = 1,2, $u^+u=1$. The various formulations help to understand different aspects of the configurations with fractional Q. In this Section the toron solution is formulated and the interpretation of this solution as the point defect is discussed. In Sec.3 the problems of guantum fluctuations, surrounding the classical solution and the requirements for selection of "right" modes are discussed. The APS theorem for manifold with a boundary is formulated and the necessity of imposing nonlocal boundary conditions is explained. In. Sec.4 the stability of toron configuration with Q = 1/2 is proved and the corresponding contribution to $\langle\bar\psi\psi\rangle$ is calculated. Then, in sections 5 and 6 the corresponding construction is generalized to gauge theories. For this purpose the self-dual equation for gauge theories will be formulated on the language analogous to

Cauchy-Riemann condition for the O(3) σ model. The contribution of the toron configuration to gluino condensate $\langle\lambda^2\rangle$ in SYM. is calculated. In Sec. 7,8 the very important question, concerning the introduction of the fields in the fundamental representation (guarks) to gauge theories is considered. Well-understood supersymmetric QCD (SQCD) model presents a perfect theoretical laboratory to help understand the role of fundamental representation fields in gauge theories. In this case a lot of various results is known (dependence of condensates on m, and g, the Konishi anomaly and so on...). Our approuch is in agreement with these general results.

After that, in Sections 9-11 we pass to the analysis of toron calculation in QCD. The contribution of the corresponding configurations to chiral condensate is calculated and it equals to $\langle\bar\psi\psi\rangle=-\pi^2\exp\{5/12\}2^4\Lambda^3$ at $N_f=N_c=2$. It is our main result. The nonzero value for $\langle\bar\psi\psi\rangle$ is obtained because of quasizero modes of quarks.

This property of the spectrum of Dirac operator is the important feature of configurations with fractional topological number. In Sec.10 the pseudoscalar correlation function is considered and the pion pole is obtained in agreement with the Goldstone theorem. In Sec.11 the U(1) problem, the anomalous Ward identities and θ-periodicity puzzle are considered from the view point of toron calculation. As is well-known, in any consistent mechnism for chiral breaking all these problems must be solved in automatic way. The calculation of the corresponding correlators will provide a valuable consistency check for our approuch.

Thus, in Sections 2-4 we consider a simple 2d O(3) σ model as an illustration of general approuch to fractional Q. Then, in sections 5-8 we test our calculations in supersymmetric theories, where many different results are knowh independently. Gaining some experience with toron calculation in a simple models we consider in Sections 9-11 the theory of hadrons, QCD.

*

2. Torons in O(3) σ model.

Before describing the toron solution we discuss the duality eguations and the Lagrangian for the ordinary (not supersymmetric) O(3) σ model. The modification due to introduction of fermions will be considered later. The action, the topological charge and the equations of duality have the following form in terms of the fields n^a [1] :

$$S = \frac{1}{4f}\int d^2x \,(\partial_\mu n^a)^2 ; \quad n^a n^a = 1;\ a = 1,2,3;\ \mu = 1,2$$
$$Q = \frac{1}{8\pi}\int d^2x\, \epsilon^{abc}\epsilon_{\mu\nu}\, n^a \partial_\mu n^b \partial_\nu n^c \tag{1}$$
$$\partial_\mu n^a = -\epsilon^{abc} n^b \epsilon_{\mu\nu}\partial_\nu n^c .$$

Here f is the bare coupling constant.

To avoid the complications due to the constraint $n^a n^a = 1$ one often introduce (see, e.g. the review, Ref. [13]) in place of the three fields n^a which live on the unit sphere, two independent fields, φ_1 and φ_2 , by means of streographic projection:

$$\varphi_1 = \frac{n_1}{1+n_3} \qquad \varphi_2 = \frac{n_2}{1+n_3} \tag{2}$$
$$n_1 = \frac{2\varphi_1}{1+\varphi_1^2+\varphi_2^2} \qquad n_2 = \frac{2\varphi_2}{1+\varphi_1^2+\varphi_2^2} \qquad n_3 = \frac{1-\varphi_1^2-\varphi_2^2}{1+\varphi_1^2+\varphi_2^2}$$

Next one combines φ_1 and φ_2 into one complex field $\varphi = \varphi_1 + i\varphi_2$ and introduces the complex variable $z = x_1 + i x_2$ and then reformulates Eqs(1) as follows:

$$S = \frac{2}{f}\int \frac{d^2x}{(1+\bar\varphi\varphi)^2}\left(\left|\frac{\partial\varphi}{\partial z}\right|^2 + \left|\frac{\partial\varphi}{\partial \bar z}\right|^2\right) \tag{3}$$

$$Q = \frac{1}{\pi}\int \frac{d^2x}{(1+\bar\varphi\varphi)^2}\left(\left|\frac{\partial\varphi}{\partial z}\right|^2 - \left|\frac{\partial\varphi}{\partial \bar z}\right|^2\right)$$
$$\frac{\partial\varphi_{cl}}{\partial\bar z} = 0, \quad z = x_1 + i x_2, \quad \bar z = x_1 - i x_2, \quad \varphi = \varphi_1 + i\varphi_2, \quad \bar\varphi = \varphi_1 - i\varphi_2$$
$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \quad \bar\partial = \frac{1}{2}(\partial_1 + i\partial_2) = \partial/\partial\bar z .$$

*We keep the term "toron", introduced in Ref.[4], for self-dual solution in the O(3) σ model and in gauge theories (Sec.5). as well. By this means we emphasize the fact, that the solution minimizes the action and carries the topological Q=1/2, i.e. possesses all of characteristics ascribed to the toron [4]. We keep the term "toron" despite the fact that our solution is formulated in principle in another way than in Ret.[4].

As can be seen from Eq.(3), the duality equations have their simplest form in the φ -language. So, the any analytical function $G_{cl.}(Z)$ is the solution of the duality equations (3).

Regarding topological ideas, it turns out that for the closest analogy with gauge theories we need another formulation of the O(3) σ model in which local gauge invariance is present.

Namely, we define the action, the topological charge and the equations of duality of CP^1-theory, equivalent to the O(3) σ model, as follows [14]:

$$S = \frac{1}{f}\int d^2x\, |D_\mu n_\alpha|^2, \quad D_\mu = \partial_\mu + iA_\mu, \quad A_\mu = -i\bar n_\alpha \partial_\mu n_\alpha, \ \mu = 1,2$$
$$Q = \frac{1}{4\pi}\int d^2x\, \epsilon_{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \bar n_\alpha n_\alpha = 1, \ \alpha = 1,2 \tag{4}$$
$$(D_\mu n_\alpha)_{cl.} = i\epsilon_{\mu\nu}(D_\nu n_\alpha)_{cl.}$$

Here n_α is a two - component complex spinor, A_μ is an auxiliary gauge field. In terms of (4) local gauge invariance has the obvious form:

$$n'_\alpha = e^{i\theta} n_\alpha , \quad A'_\mu = A_\mu + \partial_\mu\theta \tag{5}$$

Equivalence with the original formulation is verified with the help of the relations

$$n^a = \bar n_\alpha \sigma^a_{\alpha\beta} n_\beta , \quad \varphi = \frac{n_2}{n_1} , \quad n_\alpha = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{6}$$

where σ^a are the usual Pauli matrices. We note that the duality equations are written in the spinor language very simply:

$$\frac{\partial}{\partial\bar z}\left(\frac{n_2}{n_1}\right) = 0 \tag{7}$$

That's why the classical solutions in the spinor language

$$(n_\alpha)_{cl.} = \frac{P_\alpha(z)}{|P|} , \quad |P|^2 = \bar P_\alpha P_\alpha \tag{8}$$

are determined by any analytical function $P_\alpha(Z)$.

As can be seen from defemition of Q, Eq.(4), the topological charge is determined by the change of spinor phase around the large countour:

$$Q = \frac{1}{4\pi}\int d^2x\, \epsilon_{\mu\nu}F_{\mu\nu} = \frac{1}{2\pi}\oint_{|x|=\infty} A_\mu dx_\mu = \frac{1}{2\pi}\oint \partial\theta \tag{9}$$

$$n(x\to\infty) = n_0 \exp(i\theta), \quad n_0 = \text{const.}, \quad A_\mu(x\to\infty) = \partial_\mu \theta$$

The standard instanton solution in this language takes the form:

$$\begin{aligned} \rho_\alpha(z) &= u_\alpha \cdot \rho + v_\alpha (z-a) \qquad & \bar{v}_\alpha \bar{u}_\alpha = 0 \\ |\rho| &= \left(|\rho|^2 + |z-a|^2\right)^{1/2} \qquad & \bar{v}_\alpha v_\alpha = \bar{u}_\alpha u_\alpha = 1 \end{aligned} \tag{10}$$

Here v_α and u_α are the unit constant spinors; "ρ" and "a" are 4 free parameters associated with translation and scale invariance. Because $n_2/n_1 = p_2/p_1$ depends only on Z, according to (7), the duality equation is satisfied. Further, after traversing a large contour the spinor acquires the phase 2π, which in accordance with (9) corresponds to Q = 1.

We pass now to the analysis of toron solution. It is not hard to see that the action is invariant not only with respect to global SU(2) transformations, but also with respect to local U(1) transformations (5). However, the group transformation is not simply SU(2) x U(1), but G = SU(2) x U(1)/Z_2, so that $\pi_1(G) \sim Z_2$. This last circumstance is connected with the fact that a simultaneous transformation from the SU(2) group of the form $\exp(i\pi\sigma_3)$ and rotation $\exp(i\pi)$ by angle π from the U(1) group leaves the form of the fields unchanged. Consequently the corresponding transformations should be identified with unity. This means, in turn, that the theory admits Q = 1/2 and consequently (as will be seen below) multivalued functions appear in the description of classical solution. A geometric interpretation of this fact is given below.

As is easily verified, the toron solution with Q = 1/2, is a double - valued function. Indeed, as was discussed above, we admit a larger class of solutions. Namely, upon completion of the coutour of large radius, we allow the appearance of an overall factor $\exp(i\pi\sigma_3) = -1$. Taking into account that a factor (-1) arises due to analytic functions of the type $z^{1/2}$, we arrive at the following form of toron solution :

$$n_{cl.} = \lim_{a\to b} \left(|z-a| + |z-b|\right)^{-1/2} \begin{pmatrix} (z-a)^{1/2} \\ (z-b)^{1/2} \end{pmatrix}; \quad \varphi_{cl.} = \lim_{a\to b} \left(\frac{z-b}{z-a}\right)^{1/2} \tag{11}$$

This solution is defined on two Riemann sheets; real physical space corresponds to but one of them. Further, it is easily seen, that the duality equation $\bar{\partial}(n_2/n_1) = 0$ is automatically satisfied.

We note further that the solution (11) is defined in the limit $a \to b$. In terms of the field $\varphi = n_2/n_1$, Eq.(6), the solution (11) corresponds to the function $\varphi = \lim_{a\to b} \left(\frac{z-b}{z-a}\right)^{1/2}$, with a cut, tending to zero as $a \to b$, i.e. in terms of the field φ the limit $a \to b$ means reestablishment of the single - valuedness on one physical sheet.

If one sets a=b from the very beginning, then

$$n_{cl.} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp\left(\frac{i\theta}{2}\right) \tag{12}$$

corresponding, accorging to Eq.(5), to the gauge rotation of the empty vacuum solution. At first sight this suggests that such a solution can not lead to physical effects. As will be shown below this is not so. In particulary, in supersymmetric O(3) σ model the solution (11) ensures nonzero value of the chiral condensate. Analogous behaviour arises in the calculation of the 't Hooft's toron contribution to the gluino condensate in supersymmetric gluodynamics. Although to the toron solution [4] corresponds a field strength $G_{\mu\nu} \sim L^{-2}$, which tends to zero everywhere in the system with increasing size, $L \to \infty$, the condensate turns out to be finite [6].

We return to the analysis of the solution (11). To this end, instead of Eq.(11), corresponding to the boundary condition (12) ($\varphi = n_2/n_1 \to 1$) as $|z| \to \infty$, we consider the solution:

$$\varphi_{cl.} = \lim_{\Delta\to 0} \left(\frac{\Delta}{z-a}\right)^{1/2}, \quad n_{cl.} = \lim_{\Delta\to 0} \frac{1}{\sqrt{|\Delta| + |z-a|}} \begin{pmatrix} (z-a)^{1/2} \\ \Delta^{1/2} \end{pmatrix} \tag{13}$$

satisfying the standard boundary conditions $\varphi(z\to\infty) = 0$ and differing from the original by an overall rotation.

The toron solution (13) is defined on manifold with boundary, fig.1. We can to cousider the conformal mapping

$$z = \tilde{z}^2 \tag{14}$$

In this case, physical space is half-plane with boundary $\operatorname{Im} \tilde{z} = 0$, fig.2. If we consider the conformal mapping

$$\omega = R\,\frac{\Delta + i\tilde{z}}{\Delta - i\tilde{z}} \tag{15}$$

from half-plane to the disk with radius R, fig.3, the physical space corresponds to this disk R. We may then define the theory on a disk R and take the limit $R \to \infty$ on the final stage of calculation. The toron solution in this interpretation is smeared over all space with size $R \to \infty$ and in this sense remind to us the 't Hooft's solution [4]. On the other hand, we may define the theory on the exterior of small circle Δ, fig.4, and take the limit $\Delta \to 0$ on the final stage of calculation. In this language toron is the point defect at $\Delta \to 0$. In any case, the toron action equals $S_{cl} = \pi/f$ and does not depend on dimensional parameters R, Δ due to the classical conformal symmetry.

Thus, the description of the toron solution with fractional topological number on manifold with boundary is not so hard. The only problems which arise are: Is this solution stable? Is the toron contribution to $\langle\bar\varphi\varphi\rangle$ finite after the limit $\Delta \to 0$ ($R \to \infty$)? To answer these questions, we have to calculate the toron measure and condensate (Sectious 3,4). However, before calculating the toron measure, we would like to note, that the solution with any fractional number can be described on manifold with boundary. But only Q = 1/2 is stable at the quantum level. (for SU(2) group). Just for this value the correct renormalization-group dependence is restored.

There is an alternative point of view on solution with fractional number. It is connected with the consideration of the orbifolds [15] or ordinary non-singular manifolds [16]. We wouldn't discuss this question in a more details below because our starting manifold, fig.1, is a more suitable one from the technical point of view. However we would like to note that the parametr $\Delta \to 0$ which presents in the defenition of toron solution (13), is interpreted as regularization ("blowing up" in literature) of the fixed points of the orbifold (see Appendix of ref.[15]). The analjsis of toron solution on a disk R, fig 3, is discussed in Appendix A.of this paper.

In conclusion of this section I would like to describe the beautiful analogy between the toron solutions and dislcocations in solid state physics. If we denote by $u_i(x)$ a displacement of atom placed at the point x of the crystal, and by L

a loop surrounding the point x_0 (the place of dislocation) then

$$\oint du_i = \oint \frac{\partial u_i}{\partial x_k} dx_k = -b_i \qquad (16)$$

Here b_i is a lattice vector. (see, e.g. [17]). For such $U_i(x)$ it is said that we have a dislocation located at x_0. This means that the multivalued functions appear in the description of the dislocation. Therefore, we have to account for jumps in $U_i(x)$. However the physical values (the tensors of strength and deformations) are single-valued. Usually, for description of dislocations the certain fictious δ-like singularities are introdused. They ensure the necessary jump in $U_i(x)$ [17].

In the present toron calculation we prefer to describe the solution in two Riemann sheets without introdusing fictitious δ-like singularities. The gauge invariant values (in particular, the action density) must be single-valued. Namely this requirements single out the value Q = 1/2 as compared to other fractional values (see next Section).

It is obviously, that the situation, which described above for the dislocation, corresponds to line defect which is determined by vector b_i (16). In our case the vector analogous to b is orthogonal to physical space and so the toron defect is the point defect with finite full action.

Really, the evaluation of the action (3) in the φ-field terms for the solution (13) is guite simple:

$$\varphi_{cl} = \left(\frac{\Delta}{z-a}\right)^{1/2}, \quad S_{cl} = \frac{2}{f}\int \frac{d^2x}{(1+\bar\varphi\varphi)^2}\left|\frac{\partial\varphi}{\partial z}\right|^2 = \frac{2\pi}{f}Q = \qquad (17)$$

$$= \frac{|\Delta|}{2f}\int d^2x \frac{1}{|z-a|(|\Delta|+|z-a|)^2} = \frac{\pi\Delta}{f}\int_0^\infty \frac{\rho\, d\rho}{\rho(\Delta+\rho)^2} = \frac{\pi}{f}$$

$$S_{cl} = \pi/f$$

As expected, the classical action has decreased in comparison with instanton value by a factor 2.

The analogy between the toron solution and dislocation allow us to interpret the Δ-regularization from the lattice point of view. In this case, as is well-known [18], in order to account for the periodicity of the action one has to introduce a set of vortices into the system (it is just our point-like singularities). To describe this singularities directly in the continuum theory one needs some prosedure of regulazation. In particular , we can consider the lattice regulazation or some generalization of starting theory*.

If we use the lattice regularization, then a cut in the complex plane is appear in continuum limit with the branch point at the position of the vortex [18]. It corresponds exactly to our fig.1.

We prefer to use the Δ -regularization, which preservs the duality equation at finite value of Δ (not only in the limit $\Delta \to 0$).

3. Evaluation of toron measure.

We pass now to the analysis of quantum fluctuations, surrounding the classical solution (13). As usual, for the quasiclassical calculation it is necessary to expand the field $n_\alpha(x)$ in the neighborhood of $n_\alpha^{cl.}(x)$ (13), keeping only the quadratic terms in the action. The resultant bilinear form reduces to the following expression:

$$
\begin{gathered}
S = S_{cl.} + \int d^2x \, \overline{\delta n_\alpha} \, M_{\alpha\beta} \, \delta n_\beta \\
-M_{\alpha\beta} = |\rho| \, \partial \frac{1}{|\rho|^2} \left(\delta_{\alpha\beta} - \frac{\rho_\alpha \bar{\rho}_\beta}{|\rho|^2} \right) \bar{\partial} \, |\rho| \qquad (18) \\
\overline{\delta n_\alpha} \, \rho_\alpha = 0 \quad ; \quad n_\alpha = (n_\alpha)_{cl.} + \delta n_\alpha \quad ; \quad \rho_\alpha = u_\alpha \Delta^{1/2} + v_\alpha (z-a)^{1/2}
\end{gathered}
$$

Here δn_α is the small quantum fluctuation and $\rho_\alpha(z)$ is the classical toron solution (13). The supplementary condition $\overline{\delta n_\alpha} \, \rho_\alpha(z) = 0$ is due to the constraint $\bar{n}_\alpha n_\alpha = 1$. This supplementary requirement can be satisfied with the help of the vector t_α , which is orthogonal to the classical solution [23] :

$$
\begin{gathered}
\delta n_\alpha = t_\alpha F(z,\bar{z}), \quad \alpha = 1,2, \quad t_\alpha = \frac{v_\alpha \bar{\Delta}^{1/2} - u_\alpha \bar{z}^{1/2}}{|\rho|} \qquad (19) \\
\bar{t}_\alpha t_\alpha = 1, \quad \bar{t}_\alpha \rho_\alpha = \bar{t}_\alpha (n_\alpha)_{cl.} = 0 .
\end{gathered}
$$

*For example, the compact QED can be obtain from Non--Abelian **gauge** theory. All heavy fields can be viewed as a regularization of photon theory, which replaces the lattice regularization [18,19].

Without loss of generality we set a=0, $\Delta = 1$. In the final relations the corresponding dependence can be easily restored by demensional consideration.

For the further analysis the following change of variables is crucial:

$$
\xi = \frac{|z|-1}{|z|+1}, \quad \varphi = \operatorname{arctg} \frac{x_1}{x_2}, \quad -1 \le \xi \le 1, \quad 0 \le \varphi < 2\pi \qquad (20)
$$

The meaning of ξ , φ is obvious - they are corresponding coordinates of the sphere obtained with the help compactification of the complex plane Z with boundary (fig.1) into the sphere S with the cut joining the north and south poles. In terms of the variables ξ , φ the bilinear form (18) becomes the well-known equation for d-functions. Really, substituting the expressions (19), (20) to Eq.(18), we obtain the following bilinear form:

$$
\begin{gathered}
S = S_{cl.} + \int_{-1}^{1} d\xi \int_{0}^{2\pi} d\varphi \, \bar{F} \Big\{ -(1-\xi^2)\frac{\partial^2}{\partial \xi^2} + 2\xi \frac{\partial}{\partial \xi} - \qquad (21) \\
- \frac{1}{1-\xi^2} \Big[4\frac{\partial^2}{\partial \varphi^2} - 4i(1+\xi)\frac{\partial}{\partial \varphi} - 2(1+\xi) - 2 \Big] \Big\} F .
\end{gathered}
$$

Let us note that the standard requirement consists of single-valuedness of the physical modes on the manifold without boundary (sphere). Now it is not the case because our manifold has a boundary, fig.1, and lines $\varphi = 0$ and $\varphi = 2\pi$ are not identified. However, as can be seen from the expression (13) the toron solution is defined on the two Riemann sheets with corresponding identification. That's why the polar angle dependence must be the following:

$$
F \sim \exp\{ i m \varphi/2 \}, \quad m = 0, 1 \ldots \qquad (22)
$$

As will be shown below, exactly this behaviour ensure the regular solutions. The dependence (22) can be arrived in another way. We can require the regularity of the modes F and this requirement is satisfied precisely by Eq.(22). Taking the eq.(22) into account we obtain the following equation for the eigenvalues:

$$
\Big\{ -(1-\xi^2)\frac{\partial^2}{\partial \xi^2} + 2\xi \frac{\partial}{\partial \xi} + \frac{1}{1-\xi^2} \Big[(m^2 - 2m + 2) - \xi(2m-2) \Big] - 2 \Big\} F = \lambda_F F \qquad (23)
$$

The regular solutions of Eq.(23) are well known (see, e.g.[24]) and look as follows:

$$F \sim \exp\{im\varphi/2\}\, d^{\,j+1}_{-m,1}(z)\,, \quad \lambda_F = j(j+3)\,; \quad m, j = 0, 1 \ldots \tag{24}$$

As usually, similar calculations should be carried out for the vacuum field, i.e. for $(n_\alpha)_{cl.} = V_\alpha$ (for definiteness). In that case the equation for the eigenvalues is determined by the standard angular momentum operator:

$$\begin{gathered}\left[-(1-z^2)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z} + \frac{m^2}{1-z^2}\right]\delta n_\alpha = \lambda\,\delta n_\alpha \\ \delta n_\alpha = u_\alpha \exp\left\{\frac{im\varphi}{2}\right\} P_{jm}(z) \\ \lambda = j(j+1)\,, \quad j = 0, 1 \ldots\,, \qquad \overline{\delta n_\alpha}\,(n_\alpha)_{cl.} = \bar{v}_\alpha u_\alpha = 0\,.\end{gathered} \tag{25}$$

Anticipating events, we also formulate equations for fermionic modes in susy O(3) σ model. In the quasiclassical calculation the addition to the action is determined by the expression [20]:

$$\begin{gathered}\Delta S_{f.} = \int d^2x\, \bar{\Psi}_\alpha L_{\alpha\beta} \Psi_\beta \qquad \bar{\Psi}_\alpha (n_\alpha)_{cl.} = 0 \\ L_{\alpha\beta} = i\left(\delta_{\alpha\beta} - \frac{P_\alpha \bar{P}_\beta}{|P|^2}\right)\begin{pmatrix} 0 & |P|\partial\left(\frac{1}{|P|}\right) \\ \frac{1}{|P|}\bar{\partial}|P| & 0 \end{pmatrix}.\end{gathered} \tag{26}$$

Here $\Psi_\alpha = \begin{pmatrix}\psi^1_\alpha \\ \psi^2_\alpha\end{pmatrix}$ is two - component spinor, belonging to the fundamental representation of the global SU(2) group.

We shall not try to solve the eq.(26), instead it will be demonstrated that each bosonic mode (with $\lambda \neq 0$) is necessarily accompanied by two degenerate fermionic modes. Taking into account the structure $\Psi_\alpha \sim t_\alpha$ orthogonal to the classical solution $(n_\alpha)_{cl.}$, we obtain the following equation for the fermion eigenfunctions:

$$\Psi_\alpha = t_\alpha \begin{pmatrix} F_1 \\ iF_2 \end{pmatrix} \qquad \begin{aligned} -|P|^2\partial\left(\frac{1}{|P|^2}F_2\right) &= \lambda F_1 \\ \frac{1}{|P|^2}\bar{\partial}\,|P|^2 F_1 &= \lambda F_2 \end{aligned} \tag{27}$$

It has two solutions- this fact immediately stems from eq.(23). Indeed, the first solution

$$F_1 \sim F\,, \quad F_2 \sim \frac{1}{\sqrt{\lambda_F}}\frac{1}{|P|^2}\left(\bar{\partial}\,|P|^2 F_1\right)$$

corresponds to $\lambda = +\sqrt{\lambda_F}$ while the second one

$$F_1 \sim F\,, \quad F_2 \sim -\frac{1}{\sqrt{\lambda_F}}\frac{1}{|P|^2}\bar{\partial}|P|^2 F$$

corresponds to $\lambda = -\sqrt{\lambda_F}$. The two-fold degeneracy of the fermion modes is the consequence of the γ_5 invariance of the model; the boson-fermion degeneracy reflects the supersymmetry. Hence no surprise that the fermion contribution coincides with the boson one and they cancel each other. As a result, all non-zero modes combine to give unity in functional integral. Up to now we considered the equation for the eigenvalues only and our discussion of eigenfunction (24) was independent of supplementary requirements applicable to them because of existence of boundary. But it is obviously that only some of them must be taken into account. Indeed, the different eigenfunctions(24) are nonorthogonal to each other on one Riemann sheet (on the physical space with boundary) and they will be mutually orthogonal only on two sheets. The another explanation is following: since eigenvalues of λ for toron (24) and instanton solution [23] coincide, the degree of degeneracy must be different in toron and instanton cases*. So, only some (not all) eigenfunctions must be taken into account. It is obviously that all these problems are connected with the boundary of our manifold.

Now, we want to formulate the criterion for selection of modes (24), which must be accounted. Besides that, we consider the zero modes only due to the cancellation (of fermion and boson non-zero modes) which is mentioned above.

The simplest way to understand the requirements applicable to modes is to consider the O(3) σ model in terms of unconstrained φ-field (3). In this case any supplementary requirements on the modes $\delta\varphi$, due to constraint are absent, and φ -field describes exactly the physical degree of freedom (the local gauge invariance in this language is absent). It is well-known [13] that the normalized zero modes satisfy the equa-

*If it would not be so, the toron and instanton mesures would coincide and we would obtain the wrong answer.

tion, which is just the Cauchy-Riemann condition:

$$\bar{\partial}(\delta\varphi_0) = 0 \qquad \int d^2x \frac{|\delta\varphi_0|^2}{(1+\bar{\varphi}_{cl.}\varphi_{cl.})^2} = 1. \tag{28}$$

However, an arbitrary analytical function is not yet a zero mode; only the functions also satisfying the finitness condition [25]

$$\left.\frac{|\delta\varphi_0|^2}{(1+\bar{\varphi}_{cl.}\varphi_{cl.})^2}\right|_{z\to 0} = const. \tag{29}$$

are acceptable.

In particular, for instanton this requirment is satisfied precisely by two complex modes (four real) [13]:

$$\delta\varphi_0 \sim z^{-1} \quad ; \quad \delta\varphi_0 \sim z^{-2} \tag{30}$$

in according with 4 (2 complex) free parameters (10) ρ , a , associated with the collective coordinates of instanton. We also note that the normalization integral (28) diverges logarithmically for the mode $\delta\varphi_0 \sim z^{-1}$ for large Z. However as noted in Ref.[25], this fact has no effect on the physical content of the theory. We shall run into analogous behavior also in the case of torons. Usually, the infrared regularization is achieved by introduction of the factor [20-22] Ω :

$$\Omega = \left(1+\frac{x^2}{R^2}\right)^{-1} \quad ; \quad \int\frac{|\delta\varphi_0|^2\,\Omega^2\,d^2x}{(1+\bar{\varphi}_{cl.}\varphi_{cl.})^2} = 1. \tag{31}$$

In this case two extra normalizable zero modes $\delta\varphi_0 \sim 1$ would arise. However, similtaneously the vacuum amplitude would acquire the same two zero modes. Since the instanton amplitude is always normalized to the vacuum amplitude (see above), the effect of the two extra zero modes would cancel out. Consequently the number of nontrivial zero modes in the instanton equals (4+2)-2 = 4 as before (30). The number of nontrivial femion zero modes in SUSY O(3) σ model in instanton field coinsides with the number of boson ones and eguals four. This is because the fermion zero modes satisfy the same equations (28). Moreover, the number of the fermion zero modes is known beforehand, from consideration of the axial anomaly and equals 4. (two complex zero modes).

Collecting all factors together we obtain the following expression for instanton density in supersymmetric O(3) σ model [13, 20] :

$$Z_{inst.} \sim exp\{-\tfrac{2\pi}{f}\}\, M_0^4\, d^2\rho\, d^2a\, \frac{d^2\varepsilon_1}{M_0}\, \frac{d^2\varepsilon}{M_0} \tag{32}$$

In obtaining (32) we took into account the fact that the nonzero modes contributions cancel between bosons and fermions. The factor exp $(-{}^{2\pi}/_f)$ is connected with the classical action; d^2ad^2p corresponds to integration over the four collective variables, mentioned above; M_0^4 is the regulator contribution, corresponding to these four zero modes. Further, each complex fermion zero mode is accompanied by the corresponding collective integral $d^2\varepsilon$ and regulator contribution M_0^{-1}. As was expected, there appears in Eq.(32) the renormalization - invariant combunation:

$$m^2 = M_0^2\, exp\{-{}^{2\pi}/_{f(M_0)}\}. \tag{33}$$

We have on purpose analyzed in detail the instanton zero modes and the requirements applicable to them. In the next the corresponding criteria will help us choose the "correct" zero modes in the case of the toron.

We return to the analysis of the zero modes in the toron background (13). In this case zero modes satisfy the Eq.(28) as well. However, the only function satisfying finitness condition (29) is:

$$\delta\varphi_0 \sim z^{-1} \tag{34}$$

in according with two free parameters "a"(13), associated with translations.*

Only this function is acceptable. Although this mode, like in the case of the instanton, is logarithmucally divergent for $z\to\infty$, this fact has no bearing on the physical content of the theory (see Eq.(31) and corresponding discussion).

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*We remind that the parameter Δ is the regulator but in does not has a sence of collective coordinate. The analogous situation is seen in 't Hooft's solution [4], where parameters L_μ (the sizes of the box) are only regulators but are not connected with the collective coordinates.

In the fermionic sector, as explained above, the number of nontrivial modes coinsides with boson number and equals two in the toron field.This number - two (one comlex mode) - is in agreement with axial anomaly condition and is known beforehand. Indeed, the decreasing of the action (and topological number) by two times, we see the multiplicity of admissible zero modes decreased twice as well as it should be.

If we were interested in the O(3) σ model only, this would be the end of the story, because we can now find the toron mesure and calculate the chiral condensate $\langle \bar\psi\psi \rangle$. The result turns out to be finite (see next Section), because the toron solution with Q = 1/2 changes the chiral charge by two units and has two zero modes. Therefore the corresponding vacuum transition is necessarily accompanied by the production of $\bar\psi\psi$ pair as it will be demonstrated by explicit colculation.

However, our main purpose is to find the criterion for selection of modes in the formulation with local gauge invariance and fictious degree of freedom.

For empty space the zero modes correspond to j = 0 (25) and have the multiplicity g = 2 (two real modes). For the toron, zero modes correspond to j = 0, m = ,0,1,2 (24). However, only two of them (m = 0,2, g = 4 real modes) are orthogonal to each other on the physical space:

$$\delta n_\alpha^{(1)} \sim t_\alpha d^1_{11} \sim t_\alpha (1-\xi) \qquad (35,a)$$

$$\delta n_\alpha^{(2)} \sim t_\alpha d^1_{-11} \sim t_\alpha (1+\xi) \exp(i\varphi) \qquad (35,b)$$

$$\int_{-1}^{1} d\xi \int_0^{2\pi} d\varphi \, \delta \bar n_\alpha^{(1)} \delta n_\alpha^{(2)} = 0, \quad \bar t_\alpha (n_\alpha)_{cl.} = 0$$

(On the two Riemann sheets three complex modes (m = 0,1,2) are orthogonal).

Consequently, we choose from the solutions (24) the orthonormal set of eigenfunctions. Roughly speaking, for large j this reduces the degree of degeneracy by the factor 2 as compared to the instanton case, and this fact automatically ensures the correct renormalization - group dependence, see Ref. [15]. The another requirement of reqularity is satisfied by solutions (24) in the form of d-functions.

Collecting all results together we obtain that the number of nontrivial zero modes in the toron field equals 4-2 = 2 (for the instanton 6-2 = 4, see discussion after Eq.(31)) in agreement with the number obtained before and in according with two free parameters associated with translations.

The forms of zero modes in φ -lauguage and in n_α -language are in agreement with each other. It is can be checked with the help of Eq.(6). Upon substitution of $n_\alpha = (n_\alpha)_{cl.} + \delta n_\alpha$ and $\varphi = \varphi_{cl.} + \delta\varphi$ we arrive at the following connection between the modes δn_α in terms n_α -field (4) and the modes $\delta\varphi$ in terms of the φ -field (3).

$$\delta\varphi = \frac{\delta n_2 (n_1)_{cl.} - \delta n_1 (n_2)_{cl.}}{(n_1)^2_{cl.}} \qquad (36)$$

When we substitute explicit expression for (δn_α) from (35,a) we obtain $\delta\varphi_0 \sim 1$ and from (35,b) we obtain $\delta\varphi_0 \sim z^{-1}$ in agreement with the results obtained before.

Our problem, however, was not the deriviation of toron zero modes by yet another method. Rather we wanted to demonstrate the important criterion for selection of modes, which must be accounted in the formulation with local gauge invariance. Namely, the mode $\delta\varphi_0 \sim z^{-1}$ in φ -language is single-valued, but the same mode (35) is not single-valued (because of the factor $\bar z^{1/2}$ in t_α). However the gauge - invariant value $\delta\bar n_\alpha \cdot \delta n_\alpha$ is single-valued.

The lesson from this is as follows: only gauge - invariant values in the theories with extra degree of freedom must be singlevalued. Namely these modes have a physical sense. The fermion modes in n_α -language can be obtained by the same way and they satisfy to the same requirement.

Moreover, this result can be understood from the different view-point, namely, from APS-theorem for manifold with boundary[26]. The necessity of imposing non-local boundary conditions for calculation of zero-modes number is well known [26,27]. The global boundary conditions must be imposed so that Dirac operator $L_{\alpha\beta}$ (26) is self-adjoint operator on manifold with boundary. Since $L_{\alpha\beta}$ is the first-order differential operator, the self-adjoint condition has the following form for arbitrary ψ^1_α, ψ^2_α [26,27] :

$$\psi_\alpha = \begin{pmatrix} \psi_\alpha^1 \\ \psi_\alpha^2 \end{pmatrix} \qquad \int_\gamma dy\, \bar\psi_\alpha^1 \psi_\alpha^2 + H.C. = 0. \tag{37}$$

Here $\int_\gamma dy$ is the integral over the boundary. In particular, for manifold of the fig.1*, the global boundary condition has the form:

$$\int_0^\infty dr \left(\bar\psi_\alpha^1 \psi_\alpha^2\right)\Big|_{\varphi=2\pi} - \int_0^\infty dr \left(\bar\psi_\alpha^1 \psi_\alpha^2\right)\Big|_{\varphi=0} + H.C. = 0. \tag{38}$$

If the gauge- in-variant value $\bar\psi_\alpha \psi_\alpha$ on the upper edge of the cut coinsides with the value on the lower edge, then the Eq.(38) is satisfied. But precisely this requirement is formulated above for the gauge-invariant values from different viewpoint.

Thus, from the various points of view (APS index theorem, the analysis of CP^1 model in terms of the unconctrained φ -field) the criteria for the choose of the "correct" modes in the toron field was formulated. It turns out that these requirements can be satisfied only for Q = 1/2. Besides, a few questions, such as, the single-valuedness of the gauge-invariant values, the self - adjointness of the operator of (18); the orthonormality of the set of eigenfunctions are interconnected in the formulation of theory on a manifold with boundary. Moreover, the configuration with Q = 1/2 is stable at quantum level, as will be shown below.

4. The stability of the toron solution and the calculation of chiral condensate in SUSY O(3) σ model.

In this Section we prove that the toron configuration with Q = 1/2 is stable under quantum fluctuations and toron topological charge is conserved. With the above consideration taken into account (concerning the stability of the solution, the number of zero modes and so on...) the toron measure will be obtained and the correponding contribution to $\langle\bar\psi\psi\rangle$ will be calculated. The degenerate ground states arising from non--anomalous discrete chiral symmetry (see introduction) will be

* The analysis on the manifold of the fig.3 (disk R) is discussed in Appendix A.

discussed. As is well known, instanton calculations yield the average vacuum expectation value of a product of operators over these degenerate vacua [28]. The toron calculations yield the non-vanishing value of $\bar\psi\psi$ itself.

We return to the analysis of the definition of topological charge Q(4) and rewrite it in the tollowing form:

$$Q = \frac{1}{4\pi}\int d^2x\, \epsilon_{\mu\nu} F_{\mu\nu} = -\frac{i}{2\pi}\int d^2x\, \overline{D_\mu n}\, D_\nu n\, \epsilon_{\mu\nu} = \tag{39}$$
$$= \frac{1}{\pi}\int d^2x \left(|Dn|^2 - |\bar D n|^2\right); \quad D \equiv \frac{1}{2}(D_1 - iD_2)$$

Upon substitution of

$$n_\alpha = \frac{\rho_\alpha}{|\rho|}\sqrt{1 - |\delta n_\alpha|^2} + \delta n_\alpha \tag{40}$$
$$\bar n_\alpha n_\alpha = 1, \quad \rho_\alpha \delta\bar n_\alpha = 0$$

where $\rho_\alpha(z)$ is classical solution (13) and δn_α is any quantum mode orthogonal to the toron solution, we arrive with the help (39) at the following expression for δQ :

$$Q = 1/2 + \delta Q\ ; \quad \delta Q \sim \int d^2x \Big\{ -\bar\partial\Big[\frac{\delta n_\alpha}{|\rho|}\partial(|\rho|\delta\bar n_\alpha)\Big] - \tag{41}$$
$$-\partial\Big[\frac{\delta n_\alpha}{|\rho|}\bar\partial(\delta\bar n_\alpha |\rho|)\Big] + 2\partial\big[\delta n_\alpha \bar\partial(\delta\bar n_\alpha)\big]\Big\}.$$

As it should be, the δQ is a full derivative and hence reduces to a surface integral. The latter in turn vanishes for the modes which satisfy to the requirements discussed above. Indeed, because $\delta n_\alpha \sim t_\alpha F$ (19) the δQ can be rewritten in the following form:

$$\delta Q \sim \int d^2x \Big\{ -\bar\partial\Big[\frac{F}{|\rho|}\partial(|\rho|\bar F)\Big] - \partial\Big[\frac{F}{|\rho|}\bar\partial(\bar F|\rho|)\Big] + 2\partial(F\bar\partial\bar F)\Big\}. \tag{42}$$

Now it is obviously, that the $\delta Q = 0$ because nonsingle-valued structure $\sim t_\alpha \sim \sqrt{z}$ is absent in the expression (42) and functions F are well-defined.

We pass to the discussion of toron measure of supersymmetric O(3) σ model. As is known, supersymmetric models differ conveniently from ordinary ones in that only zero modes need be considered. In the bosonic and fermionic sectors we found two non-trivial modes, written in the form of one complex mode (34). Collecting all factors together we obtain the following expression for the toron density in the model:

$$Z_{toron.} \sim M_0^2 d^2a \, \frac{d^2\varepsilon}{M_0} \exp\left\{-\frac{\pi}{f(M_0)}\right\} = m\, d^2a\, d^2\varepsilon \qquad (43)$$

Here the factor $M_0^2 d^2a$ is due to the single complex bosonic zero mode; d^2a is the corresponding integral over the collective variable; the factor $d^2\varepsilon/M_0$ is connected with the single complex fermion zero mode; lastly, exp $(-\pi/f)$ is the contribution of the classical toron action. As in the case of instanton (32), the expression (43) for the toron mesure has precisely the renormalization - invariant form. It is easy to trace this phenomenon: while the action is decreased by a factor two, the number of zero modes is decreased by the same factor which exactly restored the correct renormalization - invariant relation (33).

Now all is ready for the calculation of the chiral condensate $\langle\bar\psi\psi\rangle$ in the model. Substituting in place of ψ their zero modes, and recalling that integration over the collective fermionic variables exactly satisfies $\int \varepsilon^2 d^2\varepsilon = 1$, we verify that

$$\langle Q_5=2|\bar\psi\psi|Q_5=0\rangle = mC \int d^2a\, \bar\psi_0(x-a)\psi_0(x-a) = mC \qquad (44)$$

where "c" is the calculable numerical constant. In the last step we used the value of normalization integral.

We note that the instanton can only ensure a nonvanishing value for the correlator $\langle\bar\psi\psi(x), \bar\psi\psi(0)\rangle$ [13,20], in accordance with the fact that the solution with Q = 1 changes the chiral charge ΔQ_5 by four units. The toron solution with Q = 1/2 changes the chiral charge by two units and has two zero modes. Therefore the corresponding vacuum transition is necessarily accompined by the production of a $\bar\psi\psi$ pair, as the explicit calculation of (44) also demonstrated. Because the transition amplitude (44) is nonzero and because the toron tunneling process changes the chiral charge Q_5 by 2 units, the true physical states $|\Omega_i\rangle$ must be the superposition of the states $|Q_5=0\rangle$, $|Q_5=2\rangle$ [29;9, page 297] :

$$|\Omega_0\rangle = \frac{1}{\sqrt2}\left(|Q_5=0\rangle + |Q_5=2\rangle\right) \qquad (45)$$

$$|\Omega_1\rangle = \frac{1}{\sqrt2}\left(|Q_5=0\rangle - |Q_5=2\rangle\right)$$

The construction of vacuum states $|\Omega_i\rangle$ from $|Q_5\rangle$ eigenstates is very similar to the standard $|\theta\rangle \sim \sum \exp(in\theta)|n\rangle$ vacuum consideration, but now there is only a finite number of vacua.

Now, the quantum numbers of vacuum states $|Q_5\rangle$ are suitable so that appropriate linear combinations of the states $|\Omega_i\rangle$ (45) would be the two vacua of spontaneously broken discrete chiral symmetry:

$$\langle\Omega_k|\bar\psi_L\psi_R|\Omega_i\rangle = \exp(i\pi k)\langle Q_5=2|\bar\psi_L\psi_R|Q_5=0\rangle = mC\exp(i\pi k) \qquad (46)$$

As is well known [9,13], the nonvanishing of the condensate (46) indicates spontaneous breaking of discrete chiral symmetry: $\psi \to \pm\gamma_5\psi$ which does not take place in any order of perturbation theory. This agrees with the value of the Witten index which equals two [9]. We would like to remind that the instanton calculation gives an average over these vacua [28]:

$$G(x) = \langle\bar\psi\psi(x), \bar\psi\psi(0)\rangle_{inst.} = \frac{1}{2}\sum_{k=0}^{1}\left[\langle\Omega_k|\bar\psi\psi|\Omega_k\rangle\right]^2 \qquad (47)$$

Moreover, each vacuum gives one and the same contribution to (47). That's why the nonvanishing result in instanton calculation can be obtained only for special form of the Green function, in particular for (47). The result (46) in this case can be obtained by the exctraction of the square root.

Let us discuss now the θ -dependence of the $\langle\bar\psi\psi\rangle$ condensate. The θ -term in the lagrangian has the form:

$$\mathcal{L}_\theta = \frac{1}{4\pi}\theta\, \epsilon_{\mu\nu}F_{\mu\nu} \qquad (48)$$

Let us rescale now the field ψ and introduce a new field

$$\psi'_{R,L} = e^{\pm i\alpha}\psi_{R,L}$$

Then due to the anomaly relation

$$\partial_\mu a_\mu = 4\cdot\frac{1}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}, \quad a_\mu = \bar\psi\gamma_\mu\gamma_5\psi$$

the θ term in the lograngian for ψ' will disappear provided that $\alpha = -\theta/4$. Hence, the θ -dependence of the starting condensate (46) is:

$$\langle\bar\psi_L\psi_R\rangle_\theta = \exp(i\theta/2)\langle\bar\psi_L\psi_R\rangle_{\theta=0} \qquad (49)$$

Now, the θ evolution from $\theta = 0$ to $\theta = 2\pi$ according to the law (49) renumbers two degenerate states (46) with spontaneous breaking of discrete chiral symmetry. An analogous situation has been observed previously in QCD with N_f flavours

[11,12] and in SYM in Ref. [10] and will be discussed in the next sections.

In conclusion of this section we would like to note, that the corresponding construction can also be generalized to CP^{N-1} theories. For supersymmetric variant of the CP^{N-1} theory one can also calculate the $\langle\bar{\psi}\psi\rangle$ condensate in complete analogy with the calculations discussed above in the O(3) σ model [30]. Moreover, it can be shown (by the method wich will be discribed for gauge theories in Sections 8,9) that the constant C (46) equals one in agreement with the instanton results [20]. Anticipating events we also note that the analogous calculations in gauge theories (SYM, SQCD) do not coincide with instanton results [28] and difference is equal to $\sqrt{4/5}$ (see Sect.8).

Appendex A

The main goal of this Appendix is the formulation of the toron solution in the CP^1 theory on a disk. The disk, as the manifold with boundary, is intensively discuss in the literature [27] for analysis APS theorem [26]. So this manifold is well understood and our describtion is the particular example only of the general approach [27].

We find that the introducing of the nonlocal boundary condition ensure the existence of the exactly one zero complex mode in agreement with the result of section 4.

We begin from the consideration of Dirac operator (26). Let us write the starting action in the following form:

$$\int dz\, d\bar{z}\, (\bar{F}_1 \bar{F}_2) \begin{pmatrix} 0 & L^+ \\ L & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \int dz\, d\bar{z}\, (\bar{F}_1 L^+ F_2 + \bar{F}_2 L F_1) \qquad (A1)$$

$$L = \bar{\partial} + \bar{\partial} \ln |P|^2 , \quad L^+ = -\partial + \partial \ln |P|^2 , \quad |P|^2 = 1 + |z|$$

In this case it is more convenient to study the eg.(A 1) by considering the another complex variable ω (ω is the coordinates of the disc, fig. 2) and defining a new spinor ($f_1 f_2$):

$$\omega = R \frac{1 + i\tilde{z}}{1 - i\tilde{z}} \quad z = \tilde{z}^2 , \quad f_1 = F_1 \left(\frac{dz}{d\omega} \right)^{1/2} , \quad f_2 = F_2 \left(\frac{d\bar{z}}{d\bar{\omega}} \right)^{1/2}$$

It can be easily verified that the action (A1) can be rewritten in a new variables in a standard Dirac form:

$$\int d^2\omega \left(\bar{f}_1 L_\omega^+ f_2 + \bar{f}_2 L_\omega f_1 \right) , \quad |P|^2 = 1 + \left| \frac{\omega - R}{\omega + R} \right|^2 \qquad (A2)$$

$$L_\omega^+ = -\partial_\omega + \partial_\omega \ln |P|^2 , \quad L_\omega = \bar{\partial}_\omega + \bar{\partial}_\omega \ln |P|^2$$

$$\omega = u + iv = r \exp(i\alpha) \quad 0 \le \alpha < 2\pi , \quad 0 \le r \le R , \quad \partial_\omega = \frac{1}{2}(\partial_u - i\partial_v)$$

with gauge field A_μ, $\mu = u, v$:

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu \ln |P|^2 , \quad A_u - iA_v = 2i \partial_\omega \ln |P|^2 \qquad (A3)$$

Let us note that the form (A2) still unacceptable for the using general relations of Ref. [27] because gauge fields does not satisfy of the "radial gauge", where

$$A_u = \frac{v}{r} A , \qquad A_v = -\frac{u}{r} A$$

The gauge transformation with the function Λ :

$$e^{i\Lambda} = \frac{\omega - R}{\omega + R}, \quad A'_\mu = A_\mu - \partial_\mu \Lambda, \quad f'_{1,2} = e^{i\Lambda} f_{1,2} \tag{A4}$$

provide us the form which just what is needed:

$$A'_u = \frac{v}{z} A, \quad A'_v = -\frac{u}{z} A, \quad A = \frac{2z}{z^2 + R^2}, \quad z^2 = u^2 + v^2 \tag{A5}$$

The number of zero (complex) modes in this case is defined by general relation of Ref.[27] and eguals:

$$n_+ - n_- = \left[z A(z) \right]_{z=R} = 1 \tag{A6}$$

where [x] denotes the largest integer less than or equal to x. As it should be we have exactly one complex zero mode at any R, in agreement with result of Section 4.

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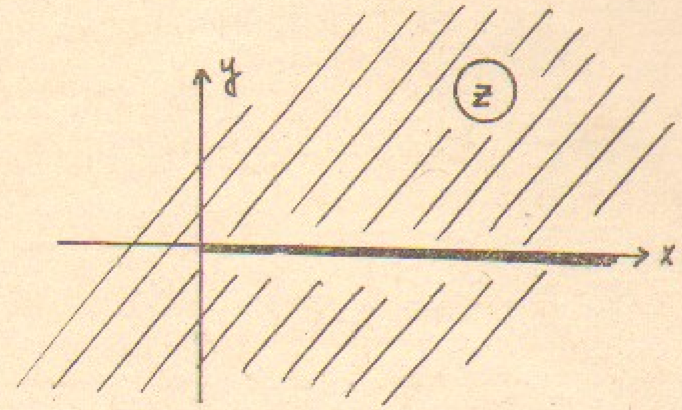

Fig. 1

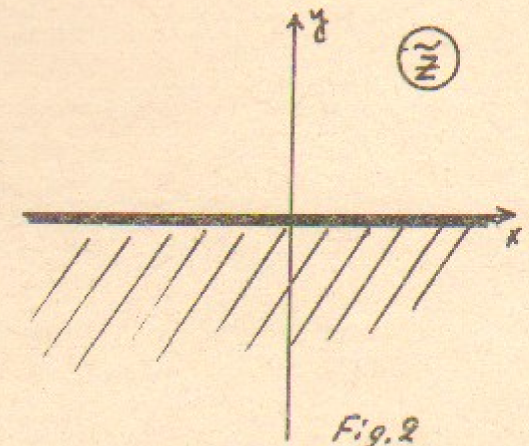

Fig.2

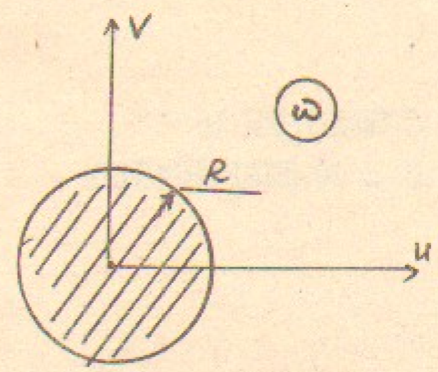

Fig. 3.

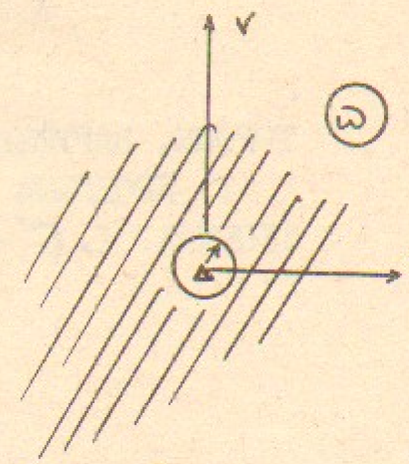

Fig.4

А.Р.Житницкий

ТОРОНЫ, НАРУШЕНИЕ КИРАЛЬНОЙ СИММЕТРИИ И (I) ПРОБЛЕМА В σ-МОДЕЛЯХ И В КАЛИБРОВОЧНЫХ ТЕОРИЯХ. I. σ-МОДЕЛИ

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