ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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ADIABATIC THEORY OF NONLINEAR ELECTRON CYCLOTRON RESONANCE HEATING

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НОВОСИБИРСК

Adiabatic Theory of Nonlinear
Electron Cyclotron Resonance Heating

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ABSTRACT

Plasma heating at electron cyclotron frequency by an ordinary wave propagating at right angle to unidirectional magnetic field is treated. Injected microwave power is assumed to be so large that relativistic change of electron gyrofrequency during one flight through the wave beam is much greater than inverse time of flight. The electron motion in the wave field is described using Hamiltonian formalism in adiabatic approximation. It is shown that energy coupling from the wave to electrons is due to a bifurcation of electron trajectory which results in a jump of the adiabatic invariant. The probability of bifurcational transition from one trajectory to another is calculated analytically and is used for the estimation of the beam power absorbed in plasma.

Адиабатическая теория нелинейного
электронного циклотронного нагрева плазмы

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АННОТАЦИЯ

Рассматривается нагрев плазмы на электронной циклотронной частоте с помощью обыкновенной волны, распространяющейся под прямым углом к магнитному полю. Предполагается, что вводимая в плазму мощность настолько велика, что релятивистское изменение циклотронной частоты электрона за один пролет через волновой пучок значительно больше обратного времени пролета. Движение электрона в поле волны описывается с помощью гамильтонова формализма и при указанном условии является адиабатическим. Показано, что передача энергии от волны электрону обусловлена бифуркацией траектории движения электрона, сопровождающейся скачкообразным изменением адиабатического инварианта. Аналитически вычислена вероятность бифуркационного перехода с одной траектории на другую. Полученный результат использован для вычисления мощности, поглощенной электронами из микроволнового пучка.

1. INTRODUCTION

Plasma heating with microwave radiation is widely used in many fusion experiments. For relatively small injected power the relevant physical processes can often be described by a linear theory. Wave propagation and absorption within the framework of this theory are now well understood [1, 2]. However, using high-power pulsed microwave sources such as free-electron lasers [3] inevitably leads to nonlinear effects.

In this paper we continue studying nonlinear dynamics of electrons interacting with an ordinary wave beam propagating at right angle to unidirectional magnetic field [4]. We assume that 1) the frequency of the wave ω is close to the electron gyrofrequency ω_H, $|\omega-\omega_H|\ll\omega$; 2) the electron gyroradius ρ_H is small compared with the wavelength, $k\rho_H\ll 1$; 3) relativistic effects are small, $v\ll c$; and, at last, 4) the width of the beam l in the direction of the magnetic field is relatively large so that the relativistic shift of the gyrofrequency $\Delta\omega_H\sim\omega_H(\Delta\mu B/mc^2)$ due to the increase of transverse energy μB is much greater than the inverse time of flight v_z/l. The last condition is characteristic of the adiabatic regime in which there exists an adiabatic invariant J determining a single-valued dependance of an electron energy on the wave amplitude [4].

Depending on the ratio $\Delta\mu B/T_e$ one can distinguish between two subregimes: a strongly nonlinear one, when $\Delta\mu B\gg T_e$, and a weakly nonlinear regime, when $\Delta\mu B\ll T_e$. The first regime has been studied in Ref. [4]. Here we develop a general theory of adiabatic cyclotron heating and apply it to the weakly nonlinear regime.

2. NONLINEAR HAMILTONIAN

Interaction of an electron with an ordinary wave propagating at right angle to the direction of magnetic field can be described by the following Hamiltonian*)

$$\mathscr{H}(I,\psi)=\Omega I-\frac{1}{2}I^2+a\sqrt{I}\cos\psi \qquad (1)$$

with the equations of motion

$$\frac{dI}{d\tau}=-\frac{\partial\mathscr{H}}{\partial\psi},\qquad \frac{d\psi}{d\tau}=\frac{\partial\mathscr{H}}{\partial I} \qquad (2)$$

Throughout this paper we use units such that $|e|=m=c=1$ and the following variables: the action $I=\mu B$ and the «slow» phase $\psi=\varphi-\omega t$ canonically conjugate to I (φ is the gyrophase), relative frequency mismatch $\Omega=(\omega_H-\omega)/\omega_H$, dimensionless wave amplitude $a=(kv_z/\sqrt{2}\,\omega)E/B$ (where E is the amplitude of the electric field in the wave), dimensionless time $\tau=\omega_H t$, and the gyrofrequency $\omega_H=B(1-v_z^2/2)$ of particle with $I=0$. For the sake of simplicity a is assumed to be positive. Negative values of a can be transformed into the positive ones by changing the definition of the phase $\psi\to\psi+\pi$.

For the constant wave amplitude, phase points move along lines of the constant value of $\mathscr{H}$ in the phase plane I, ψ. Their trajectories are shown in Fig. 1 for different values of Ω and a. Depending on the magnitude of a, there may be one or three fixed points $\dot{I}=0$, $\dot{\psi}=0$. The first one, a stable center 1, is placed at $\psi=0$. It exists for all values of a. Another couple of fixed points, stable center 2 and unstable hyperbolic point 3, are placed at $\psi=\pi$. They appear only if Ω is positive and a is not large, $a<4(\Omega/3)^{3/2}$.

Choose the z coordinate in the direction of the magnetic field. Taking into account finite width of the microwave beam in this direction, one concludes that an electron passing through the beam along a field line feels the amplitude a slowly changing in time.

*) Details of the Hamiltonian description of electron motion in ordinary wave field can be found in Ref. [4], Appendix 1.

The dependence of a on τ for a given particle is obtained by substituting its coordinate $z=v_z\tau/\omega_H$ into the function $a(z)$. In this case, the Hamiltonian (1) is no more an integral of motion and, hence, phase trajectories (now defined at a given z as lines of constant $\mathscr{H}$) also change in time. Transition of the particle from one trajectory to another is governed by conservation of an adiabatic invariant J,

$$J=\oint I\,d\psi. \qquad (3)$$

The quantity (3) is conserved provided the period $2\pi/\Delta\omega$ of phase motion is much smaller then characteristic time of the amplitude changing

$$\Delta\omega\gg\frac{v_z}{l}. \qquad (4)$$

From the conservation of J it follows that after crossing the beam a particle comes back to its initial trajectory and, hence, does not change its energy. However, this is true only if $\Omega<0$. For positive Ω, where the hyperbolic point 3 exists, the requirement (4) breaks if the trajectory passes through the separatrix (which is the trajectory coming through the hyperbolic point). In this case, the adiabatic invariant changes by finite value. The magnitudes of J just before and after the crossing of the separatrix are equal to the corresponding areas in I, ψ plane. So we define areas S_1 and S_2 of the regions bounded by the separatrix and containing fixed points *1* and *2* respectively. We will also consider the region S_3, lying over the separatrix (see Fig. 1). The change ΔS_3 of the area of this region is evidently equal to $-(\Delta S_1+\Delta S_2)$.

Let's calculate now the flux of particles from one region into another. Note, first, that regions *2* and *3* have no common border except for the only point *2*. That is why the particle flux q_{23} between these regions is equal to zero. We will show now that the fluxes q_{12} and q_{13} from the region *1* into the regions *2* and *3*, respectively, are

$$q_{1j}=f(\mathscr{H}_c)\,\dot{S}_j,\qquad j=2,3, \qquad (5)$$

where $\mathscr{H}_c$ is the value of the Hamiltonian at the separatrix, and $f(\mathscr{H})$ is the distribution function of particles in the phase plane; in the adiabaic approximation the later is ergodic and depends on $\mathscr{H}$ only.

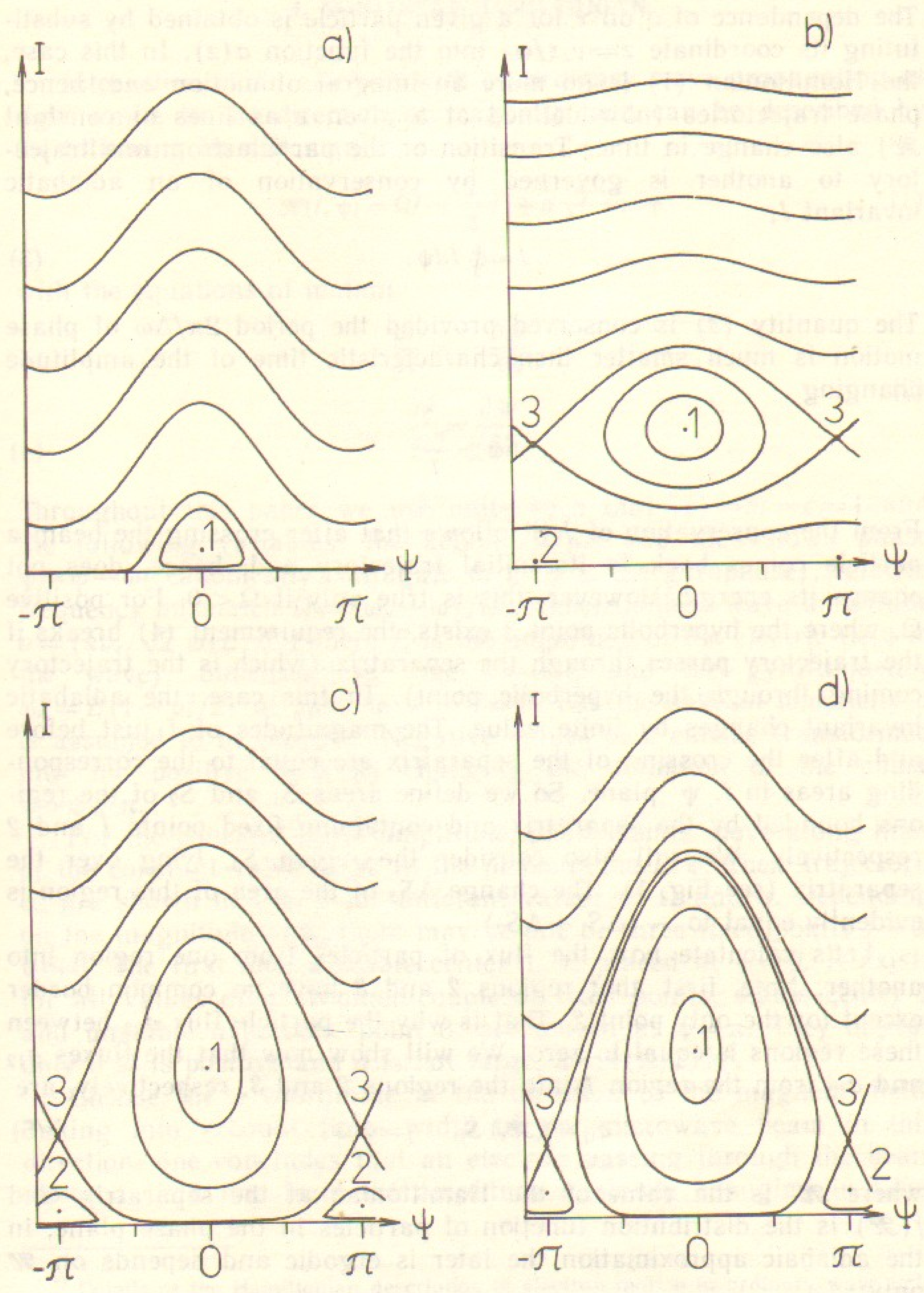

Fig. 1.

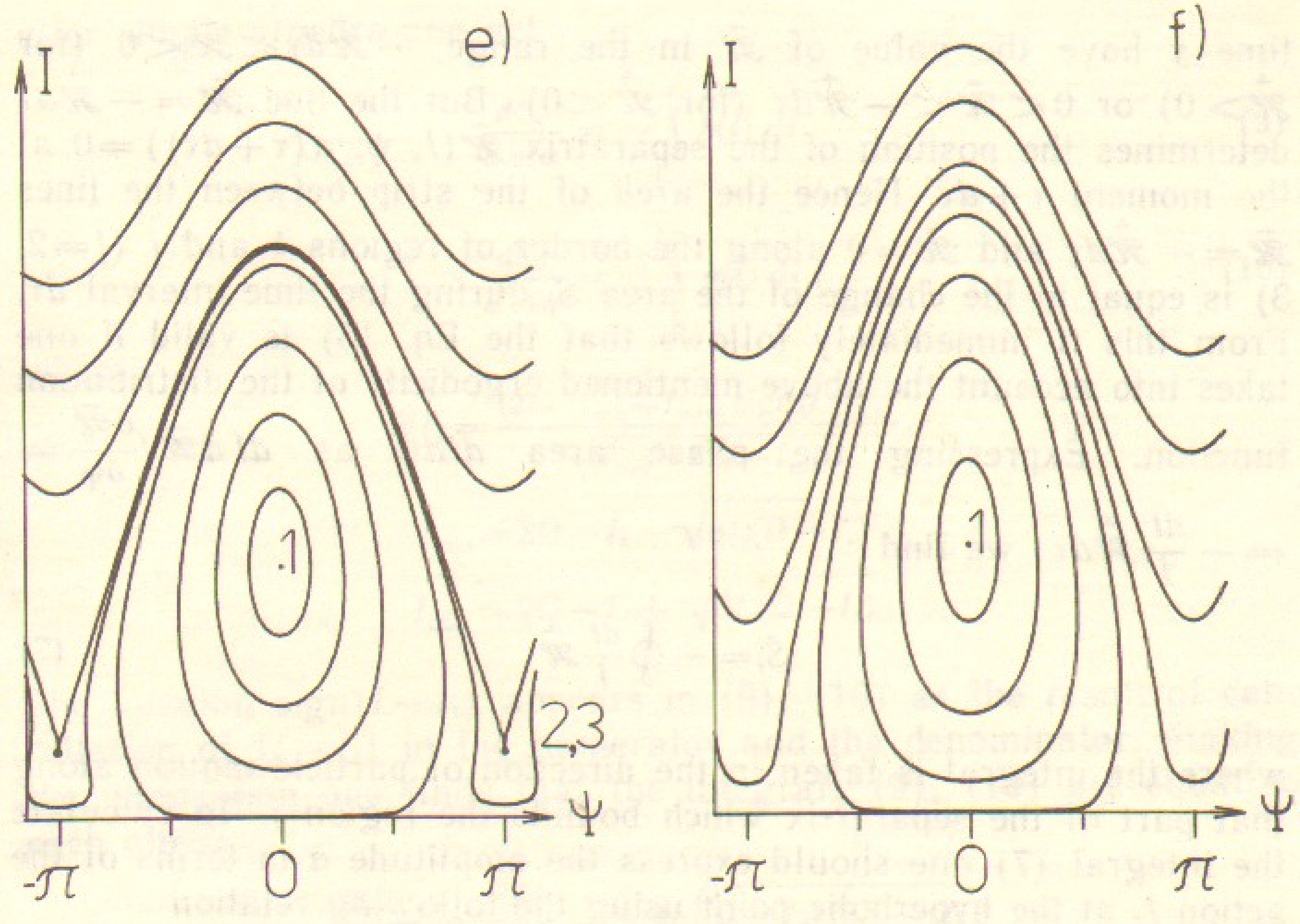

Fig. 1. Phase trajectories for different values of Ω, a: $a - \Omega<0$, $a=|\Omega|^{3/2}$, the trajectories are qualitavely the same for other values of a; the other pictures correspond to positive Ω; $b - a=0.1\Omega^{3/2}$; $c - a=0.5\Omega^{3/2}$; $d - a=0.65\Omega^{3/2}$; $e - a=4(\Omega/3)^{3/2}=0.77\Omega^{3/2}$, the fixed points *2*, *3* merge; $f - a=\Omega^{3/2}$.

For what follows it is useful to redefine the Hamiltonian in such a way that it be zero at the separatrix, positive in the region *1* and negative in the regions *2* and *3*:

$$\bar{\mathcal{H}}(I,\psi;a(\tau)) \equiv \mathcal{H}(I,\psi;a(\tau)) - \mathcal{H}_c(a(\tau))\,. \qquad (6)$$

The direction in which a particle crosses the separatrix is determined by the sign of the derivative

$$\dot{\bar{\mathcal{H}}} = \dot{a}\frac{\partial}{\partial a}(\mathcal{H}-\mathcal{H}_c)\,.$$

For $\dot{\bar{\mathcal{H}}}>0$, particles in a thin layer near the separatrix enter the region *1* from regions *2* and *3*, for $\dot{\bar{\mathcal{H}}}<0$ particles move in the opposite direction. For a differentially small time interval $d\tau$ the separatrix is crossed by the particles which at a given instant of time τ have the value of $\bar{\mathscr{H}}$ in the range $-\dot{\mathscr{H}}d\tau<\bar{\mathscr{H}}<0$ (for $\dot{\mathscr{H}}>0$) or $0<\bar{\mathscr{H}}<-\dot{\mathscr{H}}d\tau$ (for $\dot{\mathscr{H}}<0$). But the line $\bar{\mathscr{H}}=-\dot{\mathscr{H}}d\tau$ determines the position of the separatrix $\bar{\mathscr{H}}(I,\ \psi;\ a(\tau+d\tau))=0$ at the moment $\tau+d\tau$. Hence the area of the strip between the lines $\bar{\mathscr{H}}=-\dot{\mathscr{H}}d\tau$ and $\bar{\mathscr{H}}=0$ along the border of regions 1 and j ($j=2$, 3) is equal to the change of the area S_j during the time interval $d\tau$. From this it immediately follows that the Eq. (5) is valid if one takes into account the above mentioned ergodicity of the distribution function. Expressing the phase area $dId\psi$ as $dI\,d\bar{\mathscr{H}}/\frac{\partial\bar{\mathscr{H}}}{\partial\psi}=$ $=-\frac{dI}{\dot{I}}\dot{\mathscr{H}}d\tau$, we find

$$\dot{S}_j=-\oint\frac{dI}{\dot{I}}\dot{\mathscr{H}},\tag{7}$$

where the integral is taken in the direction of particle motion along that part of the separatrix which bounds the region j. To calculate the integral (7) one should express the amplitude a in terms of the action I_c at the hyperbolic point using the following relation

$$a=2\sqrt{I_c}\,(\Omega-I_c)\tag{8}$$

which follows directly from the equation $\dot{\psi}=0$ at $\psi=\pi$. Note that the hyperbolic point exists only when I_c lies in the range $\Omega/3<I_c<\Omega$. At $I_c=\Omega/3$ it merges with the stable centre (see Fig. 1e) and both disappear, whereas at $I_c=\Omega$ the amplitude a vanishes. Using parametrization (8) it is easy to show that

$$\mathscr{H}_c=\frac{3}{2}I_c^2-\Omega I_c,$$

$$\dot{\mathscr{H}}=\dot{a}(\sqrt{I}\ \cos\psi+\sqrt{I_c}\ )$$

and that

$$\sqrt{I}\cos\psi=\frac{\frac{3}{2}I_c^2+\frac{1}{2}I^2-\Omega(I_c+I)}{2\sqrt{I_c}\,(\Omega-I_c)}$$

along the separatrix. Using

$$\dot{I}=a\sqrt{I}\ \sin\psi$$

after simple algebra one get

$$\frac{dS_2}{d\ln a}=-\int_{I_{\min}}^{I_c}h(I)\,dI,\tag{9}$$

$$\frac{dS_3}{d\ln a}=-\int_{I_c}^{I_{\max}}h(I)\,dI,\tag{10}$$

where

$$h(I)=\frac{(2I-I_{\min}-I_{\max})\,\mathrm{sign}\,(I-I_c)}{\sqrt{(I_{\max}-I)(I-I_{\min})}},$$

$$I_{\min}=2\Omega-I_c-\sqrt{8I_c(\Omega-I_c)},$$

$$I_{\max}=2\Omega-I_c+\sqrt{8I_c(\Omega-I_c)}.$$

The function $\mathrm{sign}(I-I_c)$ appears in (9), (10) as the result of cancellation of $|I-I_c|$ in the numerator and the denominator. Making the integration one finds that the integrals (9), (10) are equal to each other:

$$\frac{dS_2}{d\ln a}=\frac{dS_3}{d\ln a}=-4\sqrt{(\Omega-I_c)\,(3I_c-\Omega)}.$$

Hence

$$\frac{dS_1}{dI_c}=-2\frac{dS_2}{dI_c}=-\frac{4}{I_c}\,\frac{(3I_c-\Omega)^{3/2}}{(\Omega-I_c)^{1/2}}.\tag{11}$$

Integrating (11) yields

$$S_1(I_c)=4\pi\Omega-2S_2(I_c),$$
$$S_2(I_c)=4\Omega\,\mathrm{arctg}\sqrt{\frac{3I_c-\Omega}{\Omega-I_c}}-6\sqrt{(\Omega-I_c)(3I_c-\Omega)}.\tag{12}$$

Expressing I_c through a from (8) one can find S_1 and S_1+S_2 as functions of the amplitude a. These dependances are shown in Fig. 2.

Consider now an electron passing through the beam. Suppose that before entering the beam it had an energy I_0 and corresponding value of $J_0=2\pi I_0$. At the beam front, the amplitude a slowly grows from 0 up to a maximum value $a_{\max}$. The particle will cross the separatrix at the moment when $J_0=S_2$ (provided $J_0<2\pi\Omega$) or when $J_0 = S_1 + S_2$ (if $2\pi\Omega < J_0 < 4\pi\Omega$). The amplitude a and the parameter I_c at this moment we denote by a_* and I_* respectively (see Fig. 2). The value I_* is determined from the following equation

$$|J_0 - 2\pi\Omega| = 2\pi\Omega - S_2(I_*) . \qquad (13)$$

If $a_{max} < a = 2\sqrt{I_*}(\Omega - I_*)$ or $J_0 > 4\pi\Omega$, the particle does not cross the serapatrix and the adiabatic invariant is conserved, $\Delta J = 0$. For $a_{max} > a_*$ (and $J_0 < 4\pi\Omega$) the particle initially moving in the region *2* or *3* is trapped (with 100 per cent probability) into the region *1* which is expanded as the amplitude a increases.

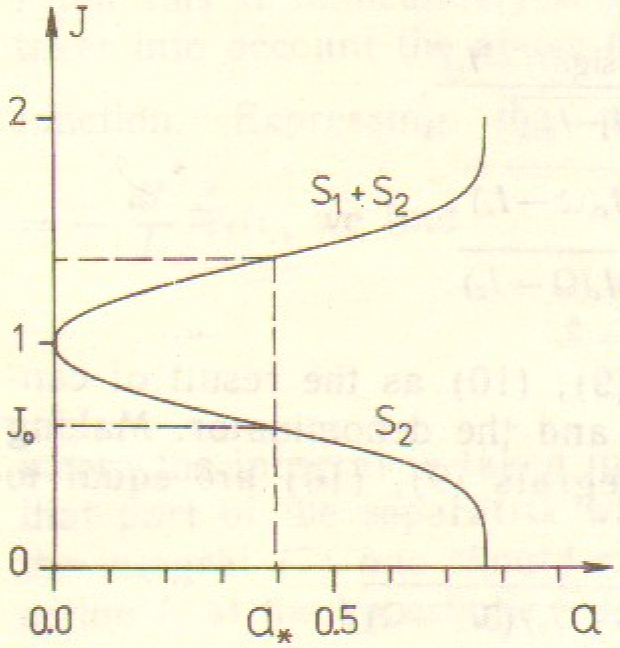

Fig. 2. Plots of areas S_1 and $S_1 + S_2$. The point of intersection of the curve with the line $J = J_0 = \text{const}$ corresponds to the crossing of the separatrix.

At the back front the amplitude a decreases from a_{max} down to 0. The particle under consideration crosses the separatrix once again at $a = a_*$. Now it passes from first region into the second or third one. Since $\dot{S}_2 = \dot{S}_3$ the probabilities to be trapped into any of these regions are equal to each other and, hence, equal to 50 per cent*). If the particle returns to the region it started from, its value of J and energy is not changed. But if it is trapped into the other region its adiabatic invariant changes by $\Delta J = 4\pi\Omega - 2J_0$ and hence its energy at final state (outside the beam) is changed by $\Delta I = \Delta J / 2\pi$ as compared with the initial energy I_0. Averaging over the equal probability of both final states one finds the change in particle energy:

$$\overline{\Delta I} = \begin{cases} \Omega - I_0 & \text{if } \; I_0 < 2\Omega \; \text{ and } \; a_{max} > a_*, \\ 0 & \text{if } \; I_0 > 2\Omega \; \text{ or } \; a_{max} < a_*. \end{cases} \qquad (14)$$

The equations (8), (11) — (14) give a complete solution to the problem of adiabatic crossing of the separatrix for the Hamiltonian (1).

*) This conclusion does not contradict to the results of Refs [5, 6] where trapping into the region *2* was due to the changing of the magnetic field along the field line.

Having calculated the mean energy change for a single electron after passing through the beam we can now calculate the amount of absorption. Because of magnetic field inhomogeneity in the direction of beam propagation the width of the resonant zone is finite. To obtain analytical results we constrain ourselves to the case when the power absorbed in this zone is small compared with the beam power. In this case, the wave amplitude is almost constant in the direction of beam propagation. Choose this direction as the x-axis.

The absorbed power is equal to the amount of energy brought out of the beam in unit time by electrons streaming along the magnetic field. Every electron brings out the energy ΔI on the average which depends on electron initial velocity because $v_\perp$ and v_z enters I_0 and a. In a single-transit regime the electron distribution function is Maxwellian

$$f(v_\perp, v_\parallel) = \frac{n}{(2\pi T_e)^{3/2}} \exp\left(-\frac{v_\perp^2 + v_z^2}{2T_e}\right)$$

with given density n and temperature T_e. The energy flux that the electrons bring out of the beam is

$$Q = 2\int_0^\infty v_z \, dv_z \int_0^\infty 2\pi v_\perp f \, \overline{\Delta I} \; dv_\perp , \qquad (15)$$

where the factor 2 takes into account the contribution of the particles with negative v_z. Though the upper limits of integration in (15) are formally infinite, $\overline{\Delta I}$ vanishes outside the interval

$$\Omega(1 - \xi) \leqslant \frac{1}{2} v_\perp^2 \leqslant \Omega(1 + \xi) ,$$

where ξ is a function of v_z. For

$$v_z < \left(\frac{32}{27}\right)^{1/2} \frac{B\omega}{kE_{max}} \qquad (16)$$

this function is implicitly given by the formulae

$$\xi = 1 - \frac{S_2(I_*)}{2\pi\Omega}, \qquad v_z = 2^{3/2} I_*^{1/2} (\Omega - I_*) \frac{B\omega}{kE_{max}}$$

(this corresponds to the interval $\Omega/3 < I_* < \Omega$), and

$$\xi = 1$$

for v_z greater than the right hand side of (16). The energy flux Q as well as ΔI depends on the frequency mismatch Ω and on the maximum amplitude E_{max} at a given field line but it does not depend on the wave profile along the magnetic field. In the small vicinity of the resonant point, $|x| \ll B/|\nabla B| \equiv L$, the magnetic field B can be considered as a linear function of x:

$$B = B_0\left(1 + \frac{x}{L}\right),$$

so that $\Omega = x/L - v_z^2/2$. Putting this into (15) and integrating along x one finds the power absorbed in plasma per unit length in the y direction. Using the variables

$$\beta = \frac{\Omega}{T_e}, \quad \zeta = \frac{I_*}{\Omega}$$

the result takes the form

$$\int dx\, Q = -\sqrt{\frac{2}{\pi}}\; n T_e^{5/2} L \mathscr{F}\left(\frac{2B\omega T_e}{kE_{max}}\right), \tag{17}$$

where

$$\mathscr{F}(v) = 2\int_0^\infty d\beta\, e^{-\beta}\left\{ v^2\beta^3 \int_{1/3}^{1} d\zeta (1-\zeta)(3\zeta-1)\, \exp\left[-v^2\beta^3\zeta(1-\zeta)^2\right] \times \right.$$

$$\left. \times (\alpha\, \mathrm{ch}\, \alpha - \mathrm{sh}\, \alpha) + \exp\left[-\frac{4}{27}v^2\beta^3\right](\beta\, \mathrm{ch}\, \beta - \mathrm{sh}\, \beta)\right\},$$

$$\alpha = \beta\left[1 - \frac{2}{\pi}\mathrm{arctg}\sqrt{\frac{3\zeta-1}{1-\zeta}} + \frac{3}{\pi}\sqrt{(1-\zeta)(3\zeta-1)}\right].$$

In the limit $2B\omega T_e \ll kE_{max}$ Eq. (17) reduces to the result of Ref. [4]:

$$\int dx\, Q = -\sqrt{\frac{2}{\pi}}\; \frac{9}{2^{11/3}}\, \Gamma\left(\frac{5}{4}\right) n L T_e^{7/6}\left(\frac{kE_{max}}{B\omega}\right)^{3/2}.$$

In the opposite limit, $2B\omega T_e \gg kE_{max}$, we find

$$\int dx\, Q = \frac{2^{15/2}}{2\pi^{7/2}}\, \Gamma^2\left(\frac{7}{4}\right) n L T_e\left(\frac{kE_{max}}{B\omega}\right)^{3/2}.$$

For a given beam profile one can calculate all power absorbed by plasma

$$P_{ab} = \int dy \int dx\, Q.$$

If, for example, E_{max} has a Gaussian profile,

$$E_{max} = E_0 \exp\left(-\frac{y^2}{l^2}\right),$$

then in the above mentioned limits we get

$$P_{ab} = 0{,}78 n T_e^{7/6} L l \left(\frac{kE_0}{B\omega}\right)^{4/3},$$

$$P_{ab} = 1{,}56 n T_e L l \left(\frac{kE_0}{B\omega}\right)^{3/2},$$

respectively.

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