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VACUUM POLARIZATION AND MAGNETIC MOMENT OF A HEAVY NUCLEUS

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Vacuum Polarization and Magnetic Moment of a Heavy Nucleus

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ABSTRACT

The contribution of the vacuum polarization to the magnetic moment of a heavy nucleus is considered. The leading term is obtained exactly in $Z\alpha$, using the electron Green function in the Coulomb field. This term contains the large logarithm of the ratio $\lambda\!\!\!^{-}_e/R$, R is the nucleus radius.

As known, the consideration of some quantum electrodynamics processes in a strong Coulomb field needs to take into account this field exactly in $Z\alpha$ ($Z|e|$ is the charge of the nucleus, $\alpha=e^2=1/137$ is the fine structure constant, e is the electron charge; we set $\hbar=c=1$). For example, the Coulomb corrections substantially decrease the cross section of the Delbrück scattering [1, 2]. The Coulomb corrections are also important at consideration of the vacuum polarization contribution to the Lamb shift in muonic atoms (see [3] and references cited there).

In the present paper we consider the induced magnetic moment of the electron-positron vacuum in the field of a heavy nucleus. The analogous induced magnetic moment contributes to the muon anomalous magnetic moment (see Ref. [4] where its lowest order evaluation has been done numerically). It contains the large logarithm $\ln(m_\mu/m_e)$ (m_μ and m_e are the muon and electron masses, respectively). In the lowest order the coefficient at the logarithm has been obtained analytically in [5]. In Ref. [6] the simple way has been proposed to obtain the logarithmic contribution in the lowest order of the perturbation theory. The muon was considered as a point source of the Coulomb field and the field of the magnetic dipole. As a result, the logarithmically divergent integral arised. This divergence was removed by choosing tne range of radial integration from $\lambda\!\!\!^{-}_\mu=1/m_\mu$ to $\lambda\!\!\!^{-}_e=1/m_e$.

The corresponding large logarithm $\ln(\lambda\!\!\!^{-}_e/R)$ (R is the nucleus radius) is also appear in the problem under consideration. In our paper we evaluate the coefficient at this logarithm exactly in the parameter $Z\alpha$. For this purpose we use the electron Green function in the Coulomb field.

Let us pass to the calculations. Outside a nucleus its magnetic moment $\vec{\mu}$ creates the magnetic field with the potential $\vec{A}=\vec{\mu}\times\vec{r}/r^4$. This magnetic field induces the vacuum current of electrons:

$$\vec{j}=-ie\int\frac{d\varepsilon}{2\pi}\mathrm{Tr}\,[\vec{\gamma}\,G(\vec{r},\vec{r}\,|\,\varepsilon)] \qquad (1)$$

where $G(\vec{r},\vec{r}\,'\,|\,\varepsilon)$ is the electron Green function, which we present as follows:

$$G(\vec{r},\vec{r}\,'\,|\,\varepsilon)=\langle\vec{r}\,|\,\frac{1}{\gamma_0(\varepsilon+Z\alpha/r)-\vec{\gamma}(\vec{p}-e\vec{A})-m}\,|\,\vec{r}\,'\rangle \qquad (2)$$

where γ_μ are the Dirac matrices. According to the Feynman rules, the contour of integration over the energy ε in (1) goes from $-\infty$ to $+\infty$ below the real axis in the left half-plane of the variable ε and above the axis in the right one. The magnetic moment due to the vacuum current (1) is

$$\vec{M}=\frac{1}{2}\int d\vec{r}\,\vec{r}\times\vec{j}. \qquad (3)$$

This moment is directed along $\vec{\mu}$: $\vec{M}=g\vec{\mu}$. Expanding the Green function $G(\vec{r},\vec{r}\,|\,\varepsilon)$ with respect to $\vec{A}$ and taking the linear term we get from (1) and (3) the following expression for the coefficient g:

$$g=\frac{i\alpha}{12\pi}\int d\varepsilon\int\frac{d\vec{r}}{r^3}\,\mathrm{Tr}\,\langle\vec{r}\,|\,\frac{1}{\hat{\mathscr{P}}-m}\,\vec{r}\times\vec{\gamma}\,\frac{1}{\hat{\mathscr{P}}-m}\,\vec{r}\times\vec{\gamma}\,|\,\vec{r}\,\rangle\,, \qquad (4)$$

where $\hat{\mathscr{P}}=\gamma_0(\varepsilon+Z\alpha/r)-\vec{\gamma}\,\vec{p}$. It is well known that the first term of the expansion with respect to $Z\alpha$ of the renormalized quantity g is proportional to $(Z\alpha)^2$. Therefore we have to subtract from the integrand for g in (4) the value of this integrand at $Z=0$. In the following such a subtraction is assumed to be made and we take it into account in the explicit form in the final result. After this subtraction we, nevertheless, have to regularize the integral in (4) since it diverges logarithmically at small distances. We perform that choosing the limit of integration over r to be equal to the nucleus radius R. The further calculations are carried out with the logarithmic accuracy. Therefore one can set the electron mass m_e in (4) to be equal to zero, cutting off the large distance radial integration at λ_e. Making in such a way we obtain the coefficient at the logarithm exactly in $Z\alpha$.

Representing $1/\hat{\mathscr{P}}$ as $\hat{\mathscr{P}}/(\hat{\mathscr{P}})^2$, it is easy to rewrite the formula (4) as follows:

$$g=\frac{i\alpha}{12\pi}\int d\varepsilon\int\frac{d\vec{r}}{r^3}\,\mathrm{Tr}\Big\{2r^2\,\langle\vec{r}\,|\,\frac{1}{\hat{\mathscr{P}}^2}\,|\,\vec{r}\,\rangle+\langle\vec{r}\,|(2\vec{j}\,^2-\vec{\Sigma}\,\vec{l}\,)\,\frac{1}{(\hat{\mathscr{P}}^2)^2}\,|\,\vec{r}\,\rangle+$$
$$+\langle\vec{r}\,|\Big(\frac{3}{2}\vec{n}\,(\vec{\Sigma}\,\vec{n})-\vec{\Sigma}\Big)\frac{1}{\hat{\mathscr{P}}^2}\vec{\Sigma}\frac{1}{\hat{\mathscr{P}}^2}\,|\,\vec{r}\,\rangle\Big\}\,, \qquad (5)$$

where $\vec{l}$ is the orbital angular momentum operator, $\vec{j}=\vec{l}+\vec{\Sigma}/2$, $\vec{\Sigma}=\gamma_0\gamma_5\vec{\gamma}$. Using the analytic properties of the Green function we deform the contour of integration over ε in (5) so that it coincides finally with the imaginary axis. In the paper [7] the integral representation is derived for the electron Green function in the Coulomb field, which is valid in the whole complex ε plane. This representation is very convenient for applications. With the help of the formula (16) of Ref. [7] we get the following expression for $D(\vec{r},\vec{r}_1\,|\,\varepsilon)=\langle\vec{r}\,|\,1/\hat{\mathscr{P}}^2\,|\,\vec{r}_1\,\rangle$ at $\varepsilon=iE$:

$$D(\vec{r},\vec{r}_1\,|\,\pm i\,|E|)=-\frac{1}{4\pi rr_1\,|E|}\int_0^\infty ds\,\exp\{\pm 2iZ\alpha s-|E|\,(r+r_1)\,\mathrm{cth}\,(s)\}\times$$
$$\times\sum_{l=1}^{\infty}\Big\{(1-\vec{\gamma}\,\vec{n}\cdot\vec{\gamma}\,\vec{n}_1)\,B\,\frac{y}{2}\,I'_{2\nu}(y)+$$
$$+[(1+\vec{\gamma}\,\vec{n}\cdot\vec{\gamma}\,\vec{n}_1)\,lA+iZ\alpha\gamma_0\,\vec{\gamma}\,(\vec{n}+\vec{n}_1)B]\,I_{2\nu}(y)\Big\}\,. \qquad (6)$$

Here $I_{2\nu}(y)$ is the modified Bessel function of the first kind, $\vec{n}=\vec{r}/r$, $\vec{n}_1=\vec{r}_1/r_1$, $y=2|E|\sqrt{rr_1}/\mathrm{sh}\,(s)$, $x=\vec{n}\,\vec{n}_1$, $A(x)=\frac{d}{dx}(P_l(x)+P_{l-1}(x))$, $B(x)=\frac{d}{dx}(P_l(x)-P_{l-1}(x))$, P_l are the Legendre polynomials, $\nu=\sqrt{l^2-(Z\alpha)^2}$. Proceeding from the integration over r to that over $r|E|$, one can easily perform the integration over E which gives the logarithm mentioned above. It is convenient to represent (5) in the following form:

$$g=\frac{2\alpha}{3\pi}\ln\Big(\frac{\lambda_e}{R}\Big)\cdot f(Z\alpha)\,. \qquad (7)$$

It follows from the result of Ref. [5] that $f(Z\alpha)\to(Z\alpha)^2$ when $Z\alpha\to 0$. Using the representation (6) it is not difficult to obtain the result for the first term in braces in (5):

$$f_1=\sum_{l=1}^{\infty}\left[2l\left(\psi(l)-\mathrm{Re}\,\psi(\nu+iZ\alpha)\right)+1-\frac{\nu}{l}\right] \tag{8}$$

where

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)\,.$$

To evaluate the contribution f_2 of the second and the third terms we represent the matrix element $\langle\vec r\,|\,1/(\hat{\mathscr{P}}^2)\,\Sigma\,1/(\hat{\mathscr{P}}^2)\,|\,\vec r\,\rangle$ as

$$\int d\vec r_1\,\langle\vec r\,|\,1/(\hat{\mathscr{P}}^2)\,|\vec r_1\rangle\,\Sigma\,\langle\vec r_1\,|\,1/(\hat{\mathscr{P}}^2)\,|\vec r\,\rangle\,,$$

and similarly for $\langle\vec r\,|\,1/(\hat{\mathscr{P}}^2)^2\,|\vec r\,\rangle$. Then we take trace over γ-matrices and integrate over directions $\vec n$ and $\vec n_1$ using the standard relations for the Legendre polynomials. As a result, the double sum with respect to l_1, l_2 contains only the terms with $l_1=l_2$ («diagonal transitions») and $l_1=l_2\pm1$ («non-diagonal transitions»). We get:

$$f_2=\sum_{l_1,l_2=1}^{\infty}\sum_{\sigma_{1,2}=\pm1}\iint_0^{\infty}\frac{ds\,dt}{\mathrm{sh}(s)\,\mathrm{sh}(t)}\cos(2Z\alpha T)\iint_0^{\infty}\frac{dr\,r_1\,dr_1}{r^2}\times$$

$$\times\exp\left(-\frac{(r+r_1)\,\mathrm{sh}(T)}{\mathrm{sh}(s)\,\mathrm{sh}(t)}\right)I_{2\nu_1+\sigma_1}(y_1)\,I_{2\nu_2+\sigma_2}(y_2)\,G\,, \tag{9}$$

where
$y_1=2\sqrt{rr_1}/\mathrm{sh}(s)$, $y_2=2\sqrt{rr_2}/\mathrm{sh}(t)$, $T=s+t$, $\nu_{1,2}=\sqrt{l_{1,2}^2-(Z\alpha)^2}$,
and

$$G=\frac{\delta_{l_1,l_2}}{\nu_1^2}\frac{4l_1^3}{4l_1^2-1}\left[l_1^2(1-\nu_1\sigma_1-2\nu_1^2)\,\delta_{\sigma_1,\sigma_2}-(Z\alpha)^2\delta_{-\sigma_1,\sigma_2}\right]+$$

$$+\frac{\delta_{l_2,l_1+1}}{2}\frac{l_1l_2}{l_1+l_2}\left[1+\frac{l_1\sigma_1}{\nu_1}-\frac{l_2\sigma_2}{\nu_2}-\frac{l_1l_2-3(Z\alpha)^2}{\nu_1\nu_2}\sigma_1\sigma_2\right]. \tag{10}$$

We used in (9) the symmetry with respect to the permutation $l_1\leftrightarrow l_2$. We would like to make several comments about the way of the integration, since this integration is not trivial.

First we consider the contribution f_{21} of the diagonal transitions. In this case all the integrals can be taken analytically. For this purpose we integrate with respect to the variable r_1 and then with respect to r, using the relations ([8], pp. 321, 303):

$$\int_0^{\infty}dx\,xe^{-cx^2}I_\alpha(ax)\,I_\alpha(bx)=\frac{1}{2c}\,I_\alpha\left(\frac{ab}{2c}\right)\exp\left(\frac{a^2+b^2}{4c}\right),$$

$$\int_0^{\infty}\frac{dx}{x}e^{-px}I_\alpha(x)=\frac{\left(p-\sqrt{p^2-1}\,\right)^{\alpha}}{\alpha}, \tag{11}$$

and the recurrence relations for the Bessel functions. Introducing the variables $T=s+t$ and $\tau=s-t$ and integrating with respect to τ and then with respect to T, we finally get

$$f_{21}=4(Z\alpha)^2\sum_{l=1}^{\infty}\frac{l^5}{\nu^2(4l^2-1)}\left[\frac{\mathrm{Re}\,\psi'(\nu+iZ\alpha)}{2\nu-1}+\frac{\mathrm{Re}\,\psi'(\nu+1+iZ\alpha)}{2\nu+1}-\frac{1}{l^2}\right] \tag{12}$$

where $\psi'(x)=(d/dx)\,\psi(x)$. The evaluation of the contribution f_{22} of the non-diagonal transitions is more cumbersome. We introduce variables $T=s+t$, $y=\mathrm{sh}(s)/\mathrm{sh}(T)$, $\rho=\sqrt{rr_1}$ and $u=\sqrt{r/r_1}$. After that we take the integral over u and then over T, using the relation ([8], p.358)

$$\int_0^{\infty}K_0\left(\sqrt{a^2+b^2+2ab\,\mathrm{ch}(T)}\,\right)\mathrm{ch}(\mu T)\,dT=K_\mu(a)\,K_\mu(b)\,. \tag{13}$$

Here $K_\mu(x)$ is a modified Bessel function of the third kind. Taking the integral over ρ and y we represent f_{22} in the form

$$f_{22}=\frac{1}{2}\sum_{l_1=1}^{\infty}\frac{l_1l_2}{l_1+l_2}\int_0^1\frac{dx}{x^2}\left(\frac{2}{x}-1\right)\{[\Phi'(\nu_1,x)-l_1\Phi(\nu_1,x)]\times$$

$$\times[\Phi'(\nu_2,x)+l_2\Phi(\nu_2,x)]+3(Z\alpha)^2\,\Phi(\nu_1,x)\,\Phi(\nu_2,x)-\Phi_0^2(x)\}\,, \tag{14}$$

where

$$\Phi(\nu,x)=x^{\nu}\,\frac{\Gamma(\nu+iZ\alpha)\,\Gamma(\nu-iZ\alpha)}{\Gamma(2\nu+1)}\,F(\nu+iZ\alpha,\nu-iZ\alpha;2\nu+1;x)\,,$$

$F(a,\,b;\,c;\,x)$ is the hypergeometric function,

$$\Phi'=\frac{\partial}{\partial x}\Phi,\quad l_2=l_1+1,\quad \nu_{1,2}=\sqrt{l_{1,2}^2-(Z\alpha)^2},$$

$$\Phi_0(x) = x^{l_2} \frac{(l_1!)^2}{(l_1+l_2)!} F(l_2, l_2; 2l_2; x) .$$

Remind, that the function $f(Z\alpha)$ entering eq. (7) equals $f_1 + f_{21} + f_{22}$ (see (8), (12) and (14)). Thus we obtain the value of the induced magnetic moment in the field of a nucleus.

Let us discuss the results obtained. Figure shows the dependence of the ratio $f(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$. The contribution of the non-diagonal transitions to this ratio proves out to be less than 3 per cent and it varies very slowly with respect to $Z\alpha$. At the same time, as one can see from Figure, the ratio under discussion increases rapidly in the vicinity of the point $Z\alpha = \sqrt{3}/2$. The origin of such

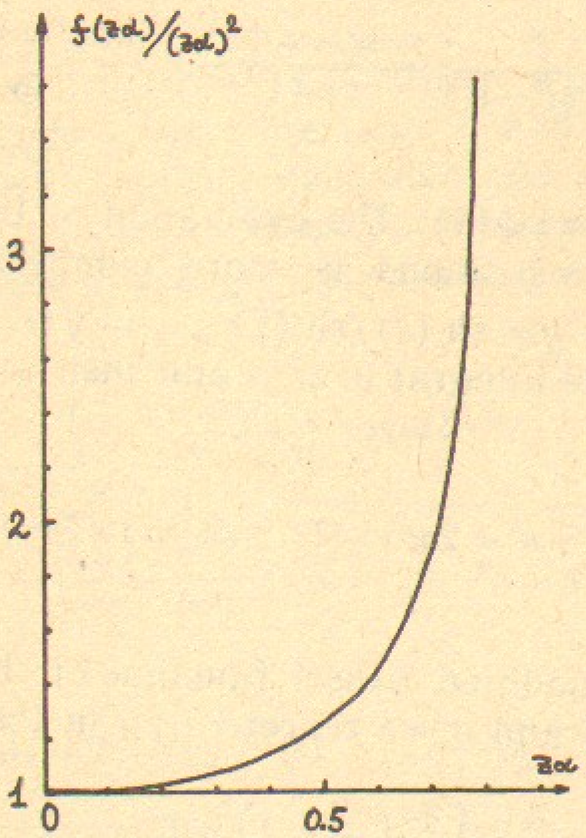

Fig. The dependence of $f(Z\alpha)/(Z\alpha)^2$ on $Z\alpha$.

behavior is connected with the presence of the pole in the expression for f_{21} (12) at $\nu = 1/2$, which corresponds to $l = 1$, $Z\alpha = \sqrt{3}/2$. It is known that such pole exists in the matrix element of the hyperfine interaction potential calculated with the Dirac wavefunction in the Coulomb field (see, e. g., Ref. [9]). It arises due to the strong singularity of the vector potential $\vec{A}$ at small distances. It is clear that in the vicinity of the point $Z\alpha = \sqrt{3}/2$ the finite size of a nucleus should be taken into account more accurately. One can verify that the formula (12) is valid at $2\nu - 1 > 1/\ln(\lambdabar_e/R)$.

Note that, as it follows from our numerical calculations,

$$f(Z\alpha)/(Z\alpha)^2 \approx 1 + 0.657\,(Z\alpha)^2$$

at $Z\alpha \to 0$. This result determines the coefficient at the logarithm $\ln(m_\mu/m_e)$ in the vacuum polarization contribution of the order of α^5 to the muon anomalous magnetic moment.

The possibility of the experimental observation of the phenomenon under discussion requires a special investigation. However, we would like to say some words concerning this problem. As it follows from our consideration, the main contribution to the induced magnetic moment is determined by the range of radial integration from R to $\lambdabar_e$. Hence the mostly appropriate object for the experimental observation of the phenomenon is, presumably, a heavy muonic atom. In such atoms the typical size of a low-lying state wavefunction is $1/(Z\alpha m_\mu) \ll \lambdabar_e$. Measuring the ratio of a hyperfine interval between high energy levels and that between low-lying ones, one can exclude the quantity of a bare magnetic moment μ. One has to do this since the theoretical accuracy of a bare magnetic moment calculations is too low. It is worthy to note that in order to compare the theoretical and experimental results, some other effects should be taken into account, e. g., the influence of the vacuum polarization on the muon wavefunction, Coulomb corrections to the muon magnetic formfactor and so on.

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