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SPINNING PARTICLE IN A GRAVITATIONAL FIELD

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НОВОСИБИРСК

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ABSTRACT

The equations of motion of a spinning particle in an external gravitational field are derived in a simple and general way. The extra force that makes the motion a nongeodesic one is a gravitational analogue of the Lorenz force. Arguments are given that fix the general form of the wave equation for particles of arbitrary spin in electromagnetic and gravitational fields.

1. The motion of a spinning relativistic particle in an external gravitational field was for the first time considered many years ago by Papapetrou [1]. Using for the derivation of the motion equations the method of Fock [2] he has shown that a spinning particle does not move along a geodesic line. Analogous results have been obtained later by means of Grassman variables by Barducci, Casalbuoni and Lusanna [3] and by Ravndal [4] for a spin 1/2 particle in a gravitational field.

In the present work the motion equations for a spinning particle are obtained in a simple and general way. Here a remarkable analogy becomes obvious between the motion of a charged particle in an electromagnetic field and the motion of a spinning particle in a gravitational one. The equations obtained are valid for an arbitrary value of the particle internal angular momentum. At the spin 1/2 they coincide with the equations found in Refs [3, 4].

The close connection between the classical and quantum-mechanical consideration in our approach allows one to get within it the general form of the relativistic wave equation for a particle of arbitrary spin in an external field. For an integer spin the equation found coincides with that proposed earlier by Christensen and Duff [5].

2. It is well known that the canonical momentum $p_\mu = i\hbar\partial_\mu$ enters a relativistic wave equation for a particle in external electromagnetic and gravitational fields through the combination $\Pi_\mu = i\hbar\partial_\mu - eA_\mu - \hbar\Sigma^{ab}\Gamma_{\mu,ab}$. Here e is the particle charge, A_μ is the electromagnetic vector-potential, $\Gamma_{\mu,ab} = -\Gamma_{\mu,ba}$ is the spin-connection

of a gravitational field, Σ^{ab} is the generator of the Lorenz group for the representation to which belongs the wave function ψ of the particle considered. Greek and Latin indices are world and tetrade ones correspondingly. Anticipating the future limiting transition to classical mechanics, we keep explicit for the time being the Planck constant $\hbar$. Note that at the second application of the operator Π_μ it contains alongside with ∂_μ the Christoffel symbol. But being multiplied by $\hbar$ it acquires uncompensated small factor and its presence is therefore inessential for further consideration.

The Heisenberg equations of motion can be obtained with the covariant Hamiltonian

$$H_0 = -g^{\mu\nu}\Pi_\mu\Pi_\nu. \tag{1}$$

The accepted metrics signature is $+---$. Below we shall discuss how one should include consistently into this Hamiltonian the terms that contain explicitly the electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the Riemann tensor

$$R_{\mu\nu ab} = -2[\partial_\mu \Gamma_{\nu,ab} - \partial_\nu \Gamma_{\mu,ab} + 2(\Gamma_{\mu,ca}\Gamma_{\nu,}{}^{c}{}_{b} - \Gamma_{\nu,ca}\Gamma_{\mu,}{}^{c}{}_{b})].$$

For the transition to the classical limit let us present as usually the wave function as $\psi \sim \exp(iS/\hbar)$. Then $p_\mu\psi = i\hbar\partial_\mu\psi \simeq (-\partial_\mu S)\psi$ so that in p_μ the quantum small factor $\hbar$ disappears. It is absent also in the electromagnetic term $-eA_\mu$ entering Π_μ. But the gravitational contribution $-\hbar\Sigma^{ab}\Gamma_{\mu,ab}$ into Π_μ, generally speaking, vanishes in the classical limit $\hbar \to 0$. We shall assume however that the particle spin is so large that the tensor of its internal angular momentum $S^{ab} = \hbar\Sigma^{ab}$ possesses the classical limit. So, as the classical Hamiltonian we use as before, for the time being expression (1) with $\Pi_\mu = p_\mu - eA_\mu - S^{ab}\Gamma_{\mu,ab}$. The classical Poisson brackets are defined in a standard way:

$$\{p_\mu, x^\nu\} = -\delta^\nu_\mu, \tag{2}$$

$$\{S^{ab}, S^{cd}\} = \eta^{ac}S^{bd} + \eta^{bd}S^{ac} - \eta^{ad}S^{bc} - \eta^{bc}S^{ad}; \tag{3}$$

here $\eta^{ac} = \mathrm{diag}(1, -1, -1, -1)$ is the flat metrics. The motion equations are found now easily:

$$\frac{dx^\alpha}{ds} = \{H_0, x^\alpha\} = 2g^{\alpha\beta}\Pi_\beta; \tag{4}$$

$$\frac{d^2x^\alpha}{ds^2} = \left\{H_0, \frac{dx^\alpha}{ds}\right\} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} + 2eF^\alpha{}_\beta\frac{dx^\beta}{ds} - S^{ab}R^\alpha{}_{\beta ab}\frac{dx^\beta}{ds}. \tag{5}$$

While the term with the Christoffel symbol in the right-hand-side of eq. (5) arises due to nonvanishing Poisson brackets $\{g^{\alpha\beta}, \Pi_\gamma\}$, two other terms in the force are due to nonvanishing Poisson brackets $\{\Pi_\alpha, \Pi_\beta\}$. The mentioned eq. (5) can be rewritten evidently in the covariant form as well:

$$\frac{D\dot{x}^\alpha}{Ds} = 2eF^\alpha{}_\beta\dot{x}^\beta - S^{ab}R^\alpha{}_{\beta ab}\dot{x}^\beta. \tag{6}$$

Since the dimensionality of Hamiltonian (1) is energy squared, the dimensionality of the conjugated variable s is unusual one. Going over to the proper time τ by means of the relation $s = \tau/2m$, we rewrite eqs (4), (6) in a more customary way:

$$\dot{x}^\alpha = \frac{1}{m}g^{\alpha\beta}\Pi_\beta, \tag{4a}$$

$$\frac{D\dot{x}^\alpha}{D\tau} = \frac{e}{m}F^\alpha{}_\beta\dot{x}^\beta - \frac{S^{ab}}{2m}R^\alpha{}_{\beta ab}\dot{x}^\beta. \tag{6a}$$

It should be noted that the quantity $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ still is an integral of the motion equations (5), (6), (6a) in total agreement with the evident condition $g_{\alpha\beta}\dfrac{dx^\alpha}{d\tau}\dfrac{dx^\beta}{d\tau} = 1$.

Note also that in the case of massless particles, with $d\tau = 0$ on the trajectory, one should use eqs (4)—(6) (naturally, at $e = 0$ since there are no massless charged particles), s being some parameter varying along the trajectory (see, e. g. Ref. [6]).

In the particular case of spin 1/2 the obtained motion equations coincide with those found in Refs [3, 4]. However, just in this case account for the terms with the Riemann tensor in eqs. (5), (6), (6a) is of somewhat conditional character. The quantum small factor $\hbar$ is not compensated here so the effect discussed could hardly be singled out from other quantum effects.

The last term in the right-hand-side of eqs (5), (6), (6a) is the new force that makes the spinning body to deviate from a geodesic line. Our derivation makes its nature quite obvious. Like the Lorenz force it arises due to the nonvanishing Poisson brackets $\{\Pi_\alpha, \Pi_\beta\}$, or in the quantum case due to noncommutativity of Π_α and Π_β. The correspondence here is evident: $eF_{\alpha\beta} \leftrightarrow -\frac{1}{2}S^{ab}R_{\alpha\beta ab}$, the internal angular momentum S^{ab} is the analogue of the charge e, the

Riemann tensor is the analogue of the electromagnetic field strength. The discussed new force could be called the gravitational Lorenz force.

3. Using the Poisson brackets (3), one can easily derive from Hamiltonian (1) the motion equation for the tensor of internal angular momentum. In the variable s it is

$$\dot{S}^{ab}=-2\dot{x}^{\mu}\,\Gamma_{\mu,ef}\,(\eta^{ae}\,S^{bf}-\eta^{be}\,S^{af})\,. \tag{7}$$

Here however the following discrepancy arises. The tensor of internal angular momentum of a particle has in its rest frame space components only. The covariant expression of this fact is evident:

$$\dot{x}_a\,S^{ab}\equiv\dot{x}^{\mu}\,V_{\mu a}\,S^{ab}=0\,; \tag{8}$$

here $V_{\mu a}$ is a tetrade. It can be easily checked however that eqs (5) and (7) in no way guarantee the required constant value of $\dot{x}_a S^{ab}$.

The way out of the contradiction is prompted by the squared form of the Dirac equation in an external electromagnetic field. It consists in the addition to Hamiltonian (1) of the term $eF_{ab}S^{ab}$. Besides, the above pointed analogy $eF_{ab}\leftrightarrow-\frac{1}{2}S^{ab}R_{abcd}$ combined with the symmetry condition $R_{abcd}=R_{cdab}$ prompts the form of one more additional term: $-\frac{1}{4}R_{abcd}S^{ab}S^{cd}$. Arising in this way the correct Hamiltonian

$$H=-\,g^{\mu\nu}\,\Pi_{\mu}\,\Pi_{\nu}+eF_{ab}\,S^{ab}-\frac{1}{4}\,R_{abcd}\,S^{ab}\,S^{cd} \tag{9}$$

leads to the following motion equation for the internal angular momentum

$$\dot{S}^{ab}=-\left(2\dot{x}^{\mu}\,\Gamma_{\mu,ef}+\frac{e}{m}\,F_{ef}-\frac{S^{cd}}{2m}\,R_{efcd}\right)\,(\eta^{ae}\,S^{bf}-\eta^{be}\,S^{af})\,. \tag{10}$$

Now as can be easily checked with eqs (6a) and (10), indeed

$$\frac{d}{d\tau}\,(\dot{x}_a\,S^{ab})=0\,. \tag{11}$$

When calculating the derivative $\frac{d}{d\tau}(\dot{x}_a S^{ab})\equiv\frac{d}{d\tau}(x^{\mu}V_{\mu a}S^{ab})$, one should not forget to differentiate the tetrade $V_{\mu a}$, i. e. to calculate its Poisson bracket with the Hamiltonian. One should take into account also the definition of the spin connection

$$\Gamma_{\mu,ab}=\frac{1}{4}\,(V_{va;\mu}\,V_{b}^{\nu}-V_{vb;\mu}\,V_{a}^{\nu})\,. \tag{12}$$

In virtue of (10) the evident condition $S_{ab}S^{ab}=\text{const}$ is valid also.

Note that simultaneously with the modification of the spin equation, Hamiltonian (9) leads to additional terms proportional to the derivatives of the field strength and Riemann tensor in the equation for $\frac{D\dot{x}^a}{D\tau}$. As to the electromagnetic term arising in this way, it is in essence known for a long time. This is just the force that in the case of neutral particles results in the Stern—Gerlach splitting of a beam in polarizations in an inhomogeneous magnetic field. Here, however, we shall not consider these terms since they are of a higher order in the small ratio of the size of the body to the characteristic lengths at which the fields vary.

Going over to the nonrelativistic limit, one can easily see that the considered interaction $eF_{ab}S^{ab}$ corresponds to the gyromagnetic ratio $g=2$. If this term were taken with an arbitrary factor, then for self-consistency, i. e. for the validity of condition (8), the Hamiltonian should be supplied with one more term, so that the total electromagnetic additional term to Hamiltonian (1) would be

$$\frac{g}{2}\,eF_{ab}\,S^{ab}-\frac{1}{2}\,(g-2)\,\frac{e}{m^2}\,\Pi^a\,F_{ab}\,\Pi_c\,S^{bc}. \tag{13}$$

The presented considerations are in essence the reformulation of the known derivation of the spin equations by Frenkel [7] and Bargman, Michel and Telegdi [8]. An analogous modification of Hamiltonian (9) would be necessary if one changed in it the factor at $R_{abcd}S^{ab}S^{cd}$. Thus, it is clear that the choice of the Hamiltonian of a charged spinning particle in form (9) is anyway the most simple and economical one. Below we shall return once more to this problem.

4. Starting from Hamiltonian (9), one can write down now the general wave equation for a particle with spin in external fields:

$$\left\{ g^{\mu\nu}(i\hbar D_\mu - eA_\mu)\,(i\hbar D_\nu - eA_\nu) - m^2 - \right.$$
$$\left. - e\hbar F_{ab}\Sigma^{ab} + \frac{\hbar}{4} R_{abcd}\Sigma^{ab}\Sigma^{cd} \right\} \psi = 0. \qquad (14)$$

Here D_μ is the covariant derivative containing spin connections and, if necessary, Christoffel symbol. We shall not discuss here consistency conditions on the wave function ψ.

Let us compare (14) with known wave equations, starting from the electromagnetic interaction. In the case of spin 1/2 we have come evidently to the usual squared Dirac equation. If we wished to include into the equation an anomalous magnetic moment, then a new term would appear in it corresponding to the second term in expression (13). Therefore an evident trouble arises: such an interaction containing a mass in the denominator grows with energy and violates the renormalizability of the theory.

In the case of spin 1 the choice $g=2$ corresponds to the Yang—Mills type switching on of the electromagnetic interaction of charged vector bosons. At the Higgs mechanism of the mass generation for charged vector fields such a theory is renormalizable. But even at the hard mass insertion in the nonrenormalizable electrodynamics of vector particles, $g=2$ corresponds to the smallest growth of divergencies. Note by the way that the minimal switching on of the electromagnetic interaction in the Proca formalism for massive vector particles corresponds to the choice $g=1$. Here already the presence of the second term in expression (13) demonstrates explicitly the nonrenormalizability of this theory.

Electrodynamics of higher spins is nonrenormalizable. But here as well the choice $g=2$ would correspond to the smallest growth of divergencies. It should be noted that neither the Rarita—Schwinger equation for spin 3/2 at the minimal switching on of the electromagnetic interaction, nor the Fierz—Pauli formalism for the electrodynamics of spin 2 particles [9] agree with eq. (14) and give $g=2$.

The gravitational interaction is nonrenormalizable for particles of any spin. But in this case as well the analogous arguments fix the factor at $R_{abcd}\Sigma^{ab}\Sigma^{cd}$. Its change as compared to (9) and (14) would demand the introduction of additional terms singular in mass, like the second term in (13), and in this way to an additional growth of divergencies in the theory. From this point of view the choice of Hamiltonian (9) and wave eq. (14) is indeed the best one.

This wave equation corresponds to the «Feynman» gauge for the field ψ. Such an equation for a particle of arbitrary spin was proposed previously in Ref. [5]. For an integer spin eq. (14) agrees with the result of the mentioned paper. (When comparing, one should have in mind different normalization of the spin operators Σ^{ab}.) But for a half-integer spin there is in general case a difference in the numerical factor at $R_{abcd}\Sigma^{ab}\Sigma^{cd}$.

We start from spin 1/2. The squared Dirac equation in a gravitational field looks as follows

$$\left(- g^{\mu\nu} D_\mu D_\nu - m^2 + \frac{1}{4} R \right) \psi = 0; \qquad (15)$$

here and below we put $\hbar = 1$; the electromagnetic interaction is not considered anymore. The equation proposed in Ref. [5] agrees with (15). As to our equation, it leads to the factor 1/8 instead of 1/4 at R. However, just in this case our arguments based on condition (8) fail: the properties of the spin matrices $\Sigma^{ab} = \frac{i}{2}\sigma^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ are such that the term discussed anyway degenerates into the scalar curvature R without any consequences for the spin motion.

For spin 3/2 our equation (14) agrees with that proposed in Ref. [5], the factors at $R_{abcd}\Sigma^{ab}\Sigma^{cd}$ coincide numerically by accident. As to the squared Rarita—Schwinger equation in a gravitational field

$$\left(- g^{\mu\nu} D_\mu D_\nu - m^2 + \frac{1}{4} R \right) \psi^\lambda - \frac{1}{2}\gamma^a\gamma^b R^\lambda_{\varkappa ab}\psi^\varkappa = 0, \qquad (16)$$

in both approaches the main term, with the Riemann tensor, that influences the spin motion, is reproduced. But in both approaches another factor at R arises: again 1/8 instead of 1/4; besides, the additional term $R^\lambda_\varkappa \psi^\varkappa$ with the Ricci tensor arises.

It is important, however, that in the most interesting case of the Einstein spaces where $R^\lambda_\varkappa = 0$ and $R=0$, eq. (14) agrees trivially with eq. (15) and quite nontrivially with (16).

The disagreement with the Dirac equation (15) which exists at $R \neq 0$, can be eliminated by modifying eq. (14) in the following way for half-integer spins:

$$\left(-g^{\mu\nu}D_\mu D_\nu - m^2 + \frac{1}{4}R_{abcd}\Sigma^{ab}\Sigma^{cd} + \frac{1}{8}R\right)\psi = 0. \qquad (17)$$

The disagreement with the Rarita — Schwinger equation is also partly eliminated by it. As to the spin motion, the introduced additional term $\frac{1}{8}R$ in no way influences it.

For higher half-integer spins eq. (17) looks more preferable than that proposed in Ref. [5] in virtue of the above arguments.

A curious situation takes place for integer spins. There eq. (14) (coinciding with the corresponding equation of Ref. [5]) reproduces exactly the equations in the Feynman gauge for a photon and graviton in an external gravitational field

$$-g^{\mu\nu}D_\mu D_\nu A_\lambda + R_\lambda{}^\varkappa A_\varkappa = 0,$$

$$-g^{\mu\nu}D_\mu D_\nu f_{\varkappa\lambda} + R_\varkappa{}^\rho f_{\rho\lambda} + R_\lambda{}^\rho f_{\varkappa\rho} - 2R_\varkappa{}^\rho{}_\lambda{}^\sigma f_{\rho\sigma} = 0. \qquad (19)$$

Not only the actual term with the Riemann tensor in eq. (19) is reproduced, but also all the terms with the Ricci tensor in both equations are.

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