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OPERATOR APPROACH TO THE CALCULATION OF THE NONRELATIVISTIC AMPLITUDES IN A COULOMB FIELD

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НОВОСИБИРСК

Operator Approach to the Calculation of the Nonrelativistic Amplitudes in a Coulomb Field

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ABSTRACT

A new method for the calculation of the nonrelativistic amplitudes in a Coulomb field is proposed. The method is based on an operator representation of the Green function for a charged particle with a subsequent disentanglement of operator expressions. The dynamical algebra O(4, 2) of the hydrogen atom is used. The calculation of the bremsstrahlung cross-section is performed.

It is known that, in certain cases, the use of the operator technique essentially simplifies the problem of the calculation of various amplitudes in an external field. The characteristic feature of these cases is the existence of the closed algebra of operators connected with the problem under consideration. An intensive application of the operator methods has been initiated by Schwinger's paper [1]. The important development of these methods has been made in Refs [2—4] for the case of a homogeneous external electromagnetic field, and in Refs [5, 6] for the case of a plane wave. In Ref. [2] Schwinger has formulated a method of calculating the mass operator in a homogeneous field using the example of particles of spin zero. In Refs [3—6] the operator diagram technique has been developed, and within the framework of this technique the electron mass operator and the photon polarization operator have been obtained. The method has been generalized to the case of a plane wave moving along a magnetic field in Ref. [7].

The case of a Coulomb field requires a special consideration. As it has been shown in Refs [8—11], the Schrödinger equation in a Coulomb field possesses a dynamical group O(4, 2). A dynamical group of the radial Schrödinger equation in a Coulomb field is a group O(2, 1) A review of numerous papers devoted to the discussion of the dynamical groups can be found in Refs [12, 13].

Using different realizations of the generators of groups, one can obtain the Green functions of wave equations. In Ref. [14] the algebra O(2, 1) has been used to find the Green function of the Dirac equation in operator form. In Ref. [15], with the help of this algebra, a new integral representation for the Green function of a charged particle moving in a Coulomb field has been obtained. This integral representation is valid in the intire complex plane of the energy.

The method of Green functions is a very convenient to obtain various amplitudes in an external field. When the expression for the amplitude contains several Green functions, the use of the operator technique allows one to avoid the integration with respect to the great number of the variables. Instead of this integration the problem arises of the transformation of operator expressions.

In the present paper, we propose the method for the calculation of the nonrelativistic amplitudes in a Coulomb field with the help of the dynamical algebra O(4,2) of the hydrogen atom. To demonstrate our method, we apply it to the calculation of the photon radiation (bremsstrahlung) cross-section σ_b.

We shall carry out the calculation in the dipole approximation which is valid if $Z\alpha \ll 1$ (the nuclear charge is $Z|e|$ where $e=-|e|$ is the electron charge, $\alpha=e^2=1/137$ is the fine structure constant, we set $\hbar=c=1$). Let us start with the well-known formula for $d\sigma_b$ (see e. g. Ref. [16]):

$$d\sigma_b = \frac{4}{3}\frac{\alpha\omega^3}{v}|\vec{x}_{fi}|^2\frac{d\vec{p}'}{(2\pi)^3}, \tag{1}$$

here v is the incoming electron velocity, v' is the outgoing electron velocity, $\vec{p}'=m\vec{v}'$, m is the electron mass, ω is the photon frequency, $\vec{x}_{fi}$ is a matrix element of the operator $\vec{x}$ between the initial (ψ_i) and the final (ψ_f) states of a continuous spectrum. After the integration of $d\sigma_b$ with respect to the velocity directions of the outgoing electron, the cross-section does not depend on the velocity direction of the incoming electron ($\vec{\lambda}=\vec{v}/|\vec{v}|$). Therefore, one can multiply both sides of eq. (1) by $d\vec{\lambda}/4\pi$ and take the integral over the angles of unit vector $\vec{\lambda}$. Let us now consider the Green function $G(\vec{x},\vec{x}'|E)$ of the electron in a Coulomb field. As is known, the function G has, in the complex plane E, a cut along the real axis from 0 to ∞, which corresponds to the continuous spectrum. It also has poles, corresponding to a discrete spectrum, at $E<0$. Using these analytic properties and standard definition of the Green function (see e. g. Ref. [16]), one obtains for the functions $\psi_{\vec{\lambda}}$ of the continuous spectrum with the energy E:

$$\int\frac{d\vec{\lambda}}{4\pi}\psi_{\vec{\lambda}}(\vec{x})\,\psi^*_{\vec{\lambda}}(\vec{x}') = \frac{i\pi}{vm^2}\,\delta G(\vec{x},\vec{x}'|E). \tag{2}$$

Here $\delta G(\vec{x},\vec{x}'|E)=G(\vec{x},\vec{x}'|E+i0)-G(\vec{x},\vec{x}'|E-i0)$ is the discontinuity of the Green function on the cut. Relation (2) allows one to represent $d\sigma_b$ as follows:

$$\frac{d\sigma_b}{d\omega} = -\frac{2}{3}\frac{\alpha\omega^3}{(mv)^2}\iint d\vec{x}\,d\vec{x}'(\vec{x}\vec{x}')\,\delta G(\vec{x}',\vec{x}|\varepsilon)\,\delta G(\vec{x},\vec{x}'|E), \tag{3}$$

where $E=mv^2/2$, $\varepsilon=m(v')^2/2$, $\omega=E-\varepsilon$. We obtain the input expression for application of the operator method. Let us represent the Green function $G(\vec{x},\vec{x}'|E)$ in the form

$$G(\vec{x},\vec{x}'|E)=\langle\vec{x}|\left(E+\frac{Z\alpha}{r}-\frac{p^2}{2m}\right)^{-1}|\vec{x}'\rangle=\langle\vec{x}|\left(Er-\frac{rp^2}{2m}+Z\alpha\right)^{-1}r\,|\vec{x}'\rangle. \tag{4}$$

Making the exponential parametrization, we get:

$$G(\vec{x},\vec{x}'|E)=-i\int_0^\infty ds\,e^{iZ\alpha s}\langle\vec{x}|\exp\left[is\left(Er-\frac{rp^2}{2m}\right)\right]r\,|\vec{x}'\rangle. \tag{5}$$

It has been shown in Ref. [15] that formula (5) gives the analytic continuation of the Green function in the upper half-plane of the variable E. To obtain the analytic continuation to the lower half-plane, we have to take the integral with respect to the parameter s from 0 to $-\infty$. So, one has for the discontinuity of the Green function:

$$\delta G(\vec{x},\vec{x}'|E)=-i\int_{-\infty}^{\infty} ds\,e^{iZ\alpha s}\langle\vec{x}|\exp\left[is\left(Er-\frac{rp^2}{2m}\right)\right]r\,|\vec{x}'\rangle. \tag{6}$$

Using formula (6) and the completeness relation, we get from eq. (3):

$$\frac{d\sigma_b}{d\omega}=\frac{2}{3}\frac{\alpha\omega^3}{(mv)^2}\iint_{-\infty}^{\infty} ds_1\,ds_2\,e^{iZ\alpha(s_1+s_2)}\int d\vec{x}\langle\vec{x}|\,r\vec{x}\times$$

$$\times\ \exp\left[is_1\left(\varepsilon r-\frac{rp^2}{2m}\right)\right]r\vec{x}\exp\left[is_2\left(Er-\frac{rp^2}{2m}\right)\right]|\vec{x}\rangle. \tag{7}$$

For the purpose of further transformations it is convenient to use the relation

$$\left[\frac{p^2}{2m}-\frac{Z\alpha}{r},\ \vec{x}\right]=-\frac{i\vec{p}}{m}$$

and to write eq. (7) as follows:

$$\frac{d\sigma_b}{d\omega}=i\frac{2}{3}\frac{\alpha\omega^2}{m^3v^2}\iint\limits_{-\infty}^{\infty} ds_1\, ds_2\, e^{iZ\alpha(s_1+s_2)}\int d\vec{x}\,\Theta, \tag{8}$$

where the function Θ is

$$\Theta=\langle\vec{x}|\ r\vec{x}\exp\left[is_1\left(\varepsilon r-\frac{rp^2}{2m}\right)\right]r\vec{p}\exp\left[is_2\left(Er-\frac{rp^2}{2m}\right)\right]|\vec{x}\rangle. \tag{9}$$

Now we pass to the transformation of the operator expression in (9). Let us consider the following fifteen operators:

$$L_{ij}=\varepsilon_{ijk}l_k, \quad L_{0i}=(\vec{x}\vec{p}-i)\,p_i-x_ip^2/2-x_i/2,$$
$$L_{4i}=(\vec{x}\vec{p}-i)\,p_i-x_ip^2/2+x_i/2, \quad L_{5i}=-rp_i, \tag{10}$$
$$L_{04}=\vec{x}\vec{p}-i, \quad L_{50}=(rp^2+r)/2, \quad L_{54}=(rp^2-r)/2,$$

where $\vec{l}$ is the orbital angular momentum operator, and i, $j=1, 2, 3$. These operators satisfy the commutation relations $(L_{\mu\nu}=-L_{\nu\mu})$:

$$[L_{\mu\nu}, L_{\rho\sigma}]=i(g_{\nu\rho}L_{\mu\sigma}+g_{\mu\sigma}L_{\nu\rho}-g_{\mu\rho}L_{\nu\sigma}-g_{\nu\sigma}L_{\mu\rho}). \tag{11}$$

Here $g_{\mu\nu}=0$ if $\mu\neq\nu$, $g_{00}=g_{55}=-1$, $g_{44}=g_{ii}=1$. Eq. (11) shows that $L_{\mu\nu}$ generate the algebra O(4, 2) (see Ref. [12]). The operators L_{04}, L_{50}, L_{54} generate the algebra O(2, 1). Using the commutation relations of the algebra O(2, 1), one can obtain (see eq. (15) in Ref. [15]):

$$\exp\left[is_1\left(\varepsilon r-\frac{rp^2}{2m}\right)\right]=\exp[ikr\,\mathrm{th}(t_1)]\times$$
$$\times\exp[-2iL_{04}\ln(\mathrm{ch}(t_1))]\exp\left[-\frac{irp^2}{k}\mathrm{th}(t_1)\right],$$

$$\exp\left[is_2\left(Er-\frac{rp^2}{2m}\right)\right]=\exp\left[-\frac{irp^2}{\varkappa}\mathrm{th}(t_2)\right]\times$$
$$\times\exp[2iL_{04}\ln(ch(t_2))]\exp[i\varkappa r\,\mathrm{th}(t_2)], \tag{12}$$

where $t_1=\sqrt{\frac{\varepsilon}{2m}}s_1$, $t_2=\sqrt{\frac{E}{2m}}s_2$, $k=\sqrt{2m\varepsilon}$, $\varkappa=\sqrt{2mE}$. Substituting (12) in (7) and taking into account the obvious relation $\exp(-iaL_{04})\varphi(r)=e^{-a}\varphi(re^{-a})$, we get for the function Θ:

$$\Theta=r\,\mathrm{ch}^{-4}(t_2)\exp[ir(k\,\mathrm{th}(t_1)+\varkappa\,\mathrm{th}(t_2))]\left\langle\frac{\vec{x}}{\mathrm{ch}^2(t_1)}\right|\vec{x}\vec{B}\left|\frac{\vec{x}}{\mathrm{ch}^2(t_2)}\right\rangle, \tag{13}$$

where

$$\vec{B}=\exp\left[-\frac{i}{k}\,\mathrm{th}(t_1)\,rp^2\right]r\vec{p}\exp\left[-\frac{i}{\varkappa}\,\mathrm{th}(t_2)\,rp^2\right]. \tag{14}$$

Now we have to use the commutation relations of the algebra O(4, 2). With the use of (11), it is easy to find the following representation for the operator $\vec{B}$ (14):

$$\vec{B}=\left\{r\vec{p}-\frac{2}{k}\,\mathrm{th}(t_1)\,[(\vec{x}\vec{p}-i)\,\vec{p}-\vec{x}p^2/2]\right\}\times$$
$$\times\exp\left[-irp^2\left(\frac{\mathrm{th}(t_1)}{k}+\frac{\mathrm{th}(t_2)}{\varkappa}\right)\right]. \tag{15}$$

Taking eq. (15) into account, we obtain the expression for the matrix element $\langle\vec{R}|\vec{x}\vec{B}|\vec{R}_1\rangle$:

$$\langle\vec{R}|\vec{x}\cdot\vec{B}|\vec{R}_1\rangle=\left[-iR^2\frac{\partial}{\partial R}+\frac{2}{k}\,\mathrm{th}(t_1)\left(R\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+i\frac{R}{2}\frac{\partial}{\partial\gamma}\right)\right]\times$$
$$\times\langle\vec{R}|\,e^{-i\gamma rp^2}\,|\vec{R}_1\rangle, \tag{16}$$

where $\gamma=\frac{\mathrm{th}(t_1)}{k}+\frac{\mathrm{th}(t_2)}{\varkappa}$. So the problem is reduced to the calculation of the matrix element $\langle\vec{R}|\exp(-i\gamma rp^2)|\vec{R}_1\rangle$. This matrix element can be calculated with the help of the algebra O(2, 1). Acting in the same way as in Ref. [15], one has:

$$\langle\vec{R}|\,e^{-i\gamma rp^2}\,|\vec{R}_1\rangle=-\frac{1}{4\pi R_1\gamma^2}\exp[i(R+R_1)/\gamma]\,J_0\left(\sqrt{2(RR_1+\vec{R}\vec{R}_1)}\,/\gamma\right), \tag{17}$$

where $J_0(x)$ is the Bessel function. Substituting (17) in (16), differentiating and using the result obtained, we get from eq. (13):

$$\Theta=\frac{r^2\exp\left\{ir\left[2+\left(\frac{k}{\varkappa}+\frac{\varkappa}{k}\right)\mathrm{th}(t_1)\,\mathrm{th}(t_2)\right]\gamma^{-1}\right\}}{4\pi\gamma^4(\mathrm{ch}(t_1)\,\mathrm{ch}(t_2))^3}\left[\left(\frac{\mathrm{sh}(t_1)}{k\,\mathrm{ch}(t_2)}-\frac{\mathrm{sh}(t_2)}{\varkappa\,\mathrm{ch}(t_1)}\right)\times\right.$$
$$\left.\times J_0(y)+i\left(\frac{\mathrm{th}(t_1)}{k}-\frac{\mathrm{th}(t_2)}{\varkappa}\right)J_1(y)\right]. \tag{18}$$

Here $y=2r[\gamma\,\mathrm{ch}(t_1)\,\mathrm{ch}(t_2)]^{-1}$. Thus, we have calculated all matrix element under consideration. Then we substitute the expression for Θ (18) in (8). Our remaining problem is to take the integrals with respect to the variables s_1, s_2 and r. The integration with respect to angles of vector $\vec{x}$ is trivial. Performing the integration by parts with respect to the variables s_1, s_2 and r, it is not difficult to represent the expression for the cross-section $d\sigma_b/d\omega$ in the following form:

$$\frac{d\sigma_b}{d\omega}=-\frac{4}{3}\frac{\alpha\omega(Z\alpha m)^2}{k^2\varkappa^4}\iint_0^\infty dT\,d\tau\cos(aT)\left(\operatorname{ch}(T)-\operatorname{ch}(\tau)\right)\times$$

$$\times\int_0^\infty r\,dr\,J_1(r)\sin(r\psi+b\tau)\,, \qquad (19)$$

where $a=Z\alpha m(\varkappa+k)/\varkappa k$, $b=Z\alpha m(\varkappa-k)/\varkappa k$, $\psi=A\operatorname{ch}(T)-B\operatorname{ch}(\tau)$, $B=(\varkappa-k)^2/4\varkappa k$, $A=B+1=(\varkappa+k)^2/4\varkappa k$. In eq. (19) we have changed over to the variables $T=t_1+t_2$, $\tau=t_1-t_2$ and have used the follows properties of the Bessel functions: $J_0(-x)=J_0(x)$ and $J_1(x)=-J_1(-x)$.

Let us now proceed to the calculation of the integrals. The integral with respect to the variable T is expressed via Hankel functions. Taking the integral, one has

$$\frac{d\sigma_b}{d\omega}=-\frac{2\pi\alpha\omega(Z\alpha m)^2}{3k^2\varkappa^4}\operatorname{Im}\frac{\partial}{\partial B}\times$$

$$\times\int_0^\infty dr\,J_1(r)\,H_{ia}^{(1)}(r(B+1))\int_0^\infty d\tau\exp\left[i(b\tau-Br\operatorname{ch}(\tau))\right]\,, \qquad (20)$$

where $H_{ia}^{(1)}(x)$ is the Hankel function of the first kind, standard difinition of the Hankel functions has used. Then we use the relation (Ref. [17], Vol.2, eq. (32), section 7.3)

$$\int_0^\infty \exp\left[i(b\tau-x\operatorname{ch}(\tau))\right]=\frac{i}{\operatorname{sh}(\pi b)}\left[\int_0^\infty dt\operatorname{ch}(b\tau)\,e^{ix\operatorname{ch}(\tau)}-\pi e^{\pi b/2}J_{-ib}(x)\right]. \qquad (21)$$

Let us substitute this representation for the integral in eq. (20). In the expression obtained one can deform the contour of the integration over r so that the integral will be extended from 0 to $i\infty$ ($r\to ir$). After that the contribution to the integral in (20) of the first term in (21) is pure real. As a result, this contribution to the cross-section vanishes. Making the stated transformations, one obtains

$$\frac{d\sigma}{d\omega}=-\frac{4\alpha\omega(Z\alpha m)^2}{3k^2\varkappa^4}\,e^{\pi b}\frac{\partial}{\partial B}\int_0^\infty dr\,I_1(r)\,K_{ia}(r(B+1))\,K_{ib}(rB)\,, \qquad (22)$$

where $I_1(x)$ and $K_\nu(x)$ are the modified Bessel functions of the first and third kinds respectively. We have used the relations $K_\nu(x)=K_{-\nu}(x)$, $I_{ib}(x)-I_{-ib}(x)=-2i\operatorname{sh}(b\pi)K_{ib}(x)/\pi$. Then we take the integral (22) with respect to the variable r with the help of eq. (1), p.399 in Ref. [18] and relations for the hypergeometric functions of two variables. Taking the integral and differentiating with respect to B, we get finally the following result:

$$\frac{d\sigma_b}{d\omega}=\frac{4\alpha\pi^2\xi^2\exp\left[\pi(\xi'-\xi)\right]}{3m^2\omega\operatorname{sh}(\pi\xi)\operatorname{sh}(\pi\xi')}\,\eta\frac{d}{d\eta}\left|F(\eta)\right|^2, \qquad (23)$$

where $\xi=mZ\alpha/\varkappa$, $\xi'=mZ\alpha/k$, $\eta=-4\varkappa k(\varkappa-k)^{-2}$, the function $F(\eta)$ is the hypergeometric function: $F(\eta)=F(i\xi,i\xi';1;\eta)$. Our result (23) coincides with the well-known result for the bremsstrahlung cross-section of nonrelativistic particles in a Coulomb field (see e. g. [16]).

Quite similar our operator method can be applied to the calculation of other amplitudes in a Coulomb field. It follows from the consideration presented above that the method discussed is especially effective for the calculation of the expressions which contain several Green functions. One more Green function in the initial expression we are interesting in leads only to one more integration with respect to the parameter in the final expression. So, the method developed in the present paper seems to be very useful for the solution of different problems in a Coulomb field.

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