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ON THE RADIATIVE RECOMBINATION OF THE ELECTRONS

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ABSTRACT

A process of the radiative recombination of the nonrelativistic electron with a hydrogen-like ion is considered. A simple expression is derived for the total cross section of the process with the use of the analytic properties of the electron Green function in a Coulomb field. The dipole approximation is used.

In the present paper, the radiative transition of an electron from the continuous spectrum state to the discrete spectrum state (the radiative recombination) is considered. This process is of a great importance in plasma physics, astrophysics etc.

The review of the early papers devoted to the problem under consideration can be found in Ref. [1]. The numerical calculations of the cross sections had been a rather complicated problem, especially for the total cross section calculation. The essential step has been made in Ref. [2], where the relatively simple expression has been obtained for the transition on the fixed energy level. In our paper, a simple formula for the total cross section is derived for the radiative recombination of a nonrelativistic electron with a hydrogen-like ion.

We will carry out the calculation in the dipole approximation which is valid if $Z\alpha \ll 1$ ($Z|e|$ is the ion charge, $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant, e is the electron charge). In this case the total cross section σ_{rec} depends on the parameter $\xi = Ze^2/\hbar v$, where v is the initial electron velocity.

Let us start with the well-known formula for σ_{rec} (see e. g. Ref. [3]):

$$\sigma_{rec} = \frac{4}{3} \frac{e^2}{\hbar v c^3} \sum_n |\vec{x}_{ni}|^2 \omega_n^3 \qquad (1)$$

where the summation is performed with respect to the bound states. In eq. (1) $\vec{x}_{ni}$ is a matrix element of $\vec{x}$ between the initial state (ψ_i) and the final state (ψ_n) of an electron, $\hbar\omega_n = E - E_n$, $E = mv^2/2$ is the initial electron energy, E_n is the final electron energy and m is the electron mass. Evidently, the cross section σ_{rec} does not depend on the velocity direction $\vec{\lambda}=\vec{v}/v$ of an initial electron. This fact allows us to transform the expression (1) for the cross section σ_{rec} in the following way. Firstly, we multiply both sides of (1) by $d\vec{\lambda}/4\pi$ and take the integral over the angles of the unit vector $\vec{\lambda}$. Then, it is convenient to use the standard electron Green function $G(\vec{x}, \vec{x}'|\varepsilon)$ in the Coulomb field (see Ref. [3]). As it is known, the function G has a cut along the real axis from 0 то ∞, in the complex plane ε, which corresponds to the continuous spectrum. It also has poles at $\varepsilon<0$, corresponding to a discrete spectrum. In virtue of that, for the functions ψ_i of the continuous spectrum one obtains

$$\int \frac{d\vec{\lambda}}{4\pi} \psi_i(\vec{x})\, \psi_i^*(\vec{x}') = \frac{i\pi\hbar^3}{vm^2}\, \delta G(\vec{x}, \vec{x}'|E)\,, \tag{2}$$

where $\delta G(\vec{x}, \vec{x}'|E) = G(\vec{x}, \vec{x}'|E+i0) - G(\vec{x}, \vec{x}'|E-i0)$ is the discontinuity of the Green function on the cut. For the function ψ_n of the discrete spectrum, we get

$$\sum_n \psi_n(\vec{x}')\, \psi_n^*(\vec{x})\, (E-E_n)^3 = -i \int_{C_1} \frac{d\varepsilon}{2\pi} G(\vec{x}', \vec{x}|\varepsilon)\, (E-\varepsilon)^3\,. \tag{3}$$

Here the contour C_1 of the integration with respect to ε encircles all the poles (see Figure). With the help of the relations (2) and (3), we represent the cross section σ_{rec} in the following form:

$$\sigma_{rec} = \frac{2}{3}\alpha(mvc)^{-2} \int_{C_1} d\varepsilon\, (E-\varepsilon)^3 \int d\vec{x}\, d\vec{x}'(\vec{x}\vec{x}')\, \delta G(\vec{x}, \vec{x}'|E)\, G(\vec{x}', x|\varepsilon)\,. \tag{4}$$

Using the analytic properties of the Green function, it is possible to deform the contour of the integration with respect to ε in (4) in such a way that it will encircle a cut (contour C_2, see Figure). The singularity at $\varepsilon=E$ should be taken into account. The reason of the arising of this singularity will be explained below.

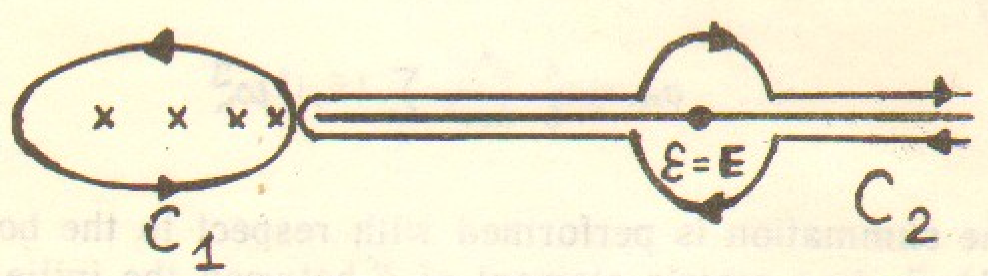

Fig. 1. The contours of integration with respect to ε in eq. (4). The crosses correspond to the poles of the discrete spectrum.

Let us consider now the radiative transition of an electron to the state of a continuous spectrum (bremsstrahlung). Acting in the same way as in the derivation of eq. (4), we obtain for a corresponding cross section σ_b:

$$\frac{d\sigma_b}{d\varepsilon} = -\frac{2}{3}\alpha(mvc)^{-2}(E-\varepsilon)^3 \int d\vec{x}\, d\vec{x}'\, \delta G(\vec{x}, \vec{x}'|\varepsilon)\, \delta G(\vec{x}', \vec{x}|E)\, (\vec{x}\vec{x}')\,, \tag{5}$$

where ε is the energy of a final electron. The cross section σ_b, for the radiation by the nonrealistic electron in the field of a hydrogen-like ion, is well-known (see e. g. Ref. [3]):

$$\frac{d\sigma_b}{d\varepsilon} = \frac{4}{3}\alpha\left(\frac{\hbar}{mc}\right)^2 \frac{\pi^2\xi^2 e^{\pi(\xi'-\xi)}}{\mathrm{Sh}(\pi\xi)\,\mathrm{Sh}(\pi\xi')}\, \eta\frac{d}{d\eta}\, |F(\eta)|^2 (E-\varepsilon)^{-1}\,, \tag{6}$$

where $\xi = Ze^2/\hbar v$, $\xi' = Ze^2/\hbar v'$, $\varepsilon = m(v')^2/2$, $\eta = -4vv'/(v-v')^2$ and the function $F(\eta)$ is the hypergeometric function: $F(\eta) = F(i\xi, i\xi'; 1; \eta)$. It is evident that $\varepsilon<E$ in eq. (6). In the limit $\varepsilon\to E$, we get from eq. (6) the following asymptotic form:

$$\frac{d\sigma_b}{d\varepsilon} = \frac{16}{3}\alpha\,\xi^2 \left(\frac{\hbar}{mc}\right)^2 \frac{\ln\left(\frac{v+v'}{v-v'}\right) + \psi(1) - (\psi(1+i\xi) + \psi(1-i\xi))/2}{E-\varepsilon}\,, \tag{7}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$. It is easy to see now that the singularity at $\varepsilon\to E$ is the usual singularity connected with the soft photon emission.

Let us return to the consideration of the radiative recombination. After the transition from the contour C_1 to the contour C_2 in eq. (4), one can represent σ_{rec} as a sum of two quantity: $\sigma_{rec} = \sigma_1 + \sigma_2$. The quantity σ_1 is given by the formula (4), where the integral with respect to ε is taken in the principal value sense and $G(\vec{x}', \vec{x}|\varepsilon)$ has to be substituted by $\delta G(\vec{x}', \vec{x}|\varepsilon)$. The quantity σ_2 is given by eq. (4), where the integration is performed along the infinitesimal circle around the point $\varepsilon=E$. Comparing eqs. (4) with (5) and taking into account antisymmetry of $\frac{d\sigma_b}{d\varepsilon}$ (5) with respect to substitutions $\varepsilon\leftrightarrow E$, we obtain σ_1 with the use of eq. (6). To calculate σ_2, one has to construct a function for which the discontinuity on the cut in neighbourhood of the point $\varepsilon=E$ to be given by expression (7). It is not difficult to do that, using the well-known relation $\psi(x-1) - \psi(x+1) = \pi\,\mathrm{ctg}\,(\pi x) - 1/x$ and the analytic properties of logarithm. Making the stated calculations, we get finally the following expression for the total cross section of the radiative recombination:

$$\sigma_{rec} = -\frac{8\pi^2}{3}\alpha\,\xi^2\left(\frac{\hbar}{mc}\right)^2\left\{\text{V.P.}\int_0^\infty \frac{d\varepsilon}{E-\varepsilon}\,\frac{\exp\left[\pi Ze^2\left|\frac{1}{v}-\frac{1}{v'}\right|/\hbar\right]}{2\,\text{Sh}(\pi\xi)\,\text{Sh}(\pi\xi')}\,\eta\frac{d}{d\eta}\,|F(\eta)|^2 + \right.$$

$$\left. + 1 + \text{cth}\,\pi\xi - \frac{1}{\pi\xi}\right\}. \qquad (8)$$

The notations used in eq. (8) are defined after eq. (6).

Let us discuss now the asymptotic form of the cross section σ_{rec}. In the case of $\xi \gg 1$, the main contribution to σ_{rec} comes from the following region of integration with respect to ε in (8): $E \ll \varepsilon \ll E\xi^2$. In this region $\eta\frac{d}{d\eta}|F(\eta)^2 \approx \exp(2\pi\xi')/\pi\sqrt{3}$. Substituting this asymptotic into (8) and taking the integral, with logarithmic accuracy, we obtain

$$\sigma_{rec} \approx \frac{32\pi}{3\sqrt{3}}\alpha\xi^2\left(\frac{\hbar}{mc}\right)^2 \ln\xi\,. \qquad (9)$$

The asymptotic form (9) is in agreement with the result adduced in Ref. [2]. In the case of $\xi \ll 1$, it is convenient to devide the region of integration with respect to ε in two parts: from 0 to ε_0 and from ε_0 to ∞, where we choose ε_0 such that $E\xi^2 \ll \varepsilon_0 \ll E$. In the first region $\xi \ll 1$, $\xi' \sim 1$, $\eta \ll 1$. In the second region $\xi, \xi' \ll 1$, $\eta \sim 1$. Making the corresponding expansion, taking the integrals and then adding both contributions, we obtain

$$\sigma_{rec} = \frac{128\pi}{3}\alpha\xi^5\left(\frac{\hbar}{mc}\right)^2 \zeta(3) \qquad (10)$$

that coincides with the well-known result for the total cross section in the case of $\xi \ll 1$ (see e. g. Ref. [3]). In eq. (10) $\zeta(x)$ is the Riemann function.

The method for the calculation of the radiative recombination cross section proposed here seems to be very useful for some other cases as well.

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