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A New Way of Instanton—Antiinstanton Interaction Description. The Nonlinear O(3)-Sigma Model Example

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ABSTRACT

To find the instanton—antiinstanton potential a method of calculation of the functional integral near the configurations, which are only approximate solutions of equation of motion, is developed. The collective coordinates are chosen for only that modes which can't be integrated in Gauss approximation. Instanton—antiinstanton pair is a name for a configuration minimizing the action in all other «undangerous» directions.

As a practical example the asymptotic of pseudoparticle interaction in nonlinear O(3)-sigma model is found. At large distances the field of instanton—antiinstanton pair turns out to be very different from the sum of fields of two single pseudopaticles. This means, in particular, that to go beyond the dilute instanton gas approximation one can't introduce two-particle interaction only. The opportunity to investigate some other field configurations is discussed (merons).

1. INTRODUCTION

After the instanton discovery in 1975 [1] the question about the role of nontrivial solutions of classical Euclidean equation of motion in quantum field theory attracts a permanent attention. The most important examples of such fluctuations are the instantons in SU(N) Yang—Mills theory, directly associated with strong interaction physics. In this theory an exact N-instanton (N-antiinstanton) solution [2, 3] was found in addition to one-instanton solution [1]. Another popular class of theories looking in many features very like the SU(N) gluodynamics and having also an exact many-instanton solution are the nonlinear O(3)-sigma model [4] and CP(N)-sigma models [5]. In all mentioned theories the N-instanton configuration realizes the absolute minimum of action for a fixed topological class. If we want to go beyond the dilute gas approximation, we must learn to deal with configurations, constructed from a few instantons and antiinstantons. The simplest example is presented by an instanton and antiinstanton, situated at a distance much larger their sizes. This configuration has zero topological charge and, generally speaking, do not satisfy to the equation of motion. The aim of this paper is to find an adequate method to describe the field configurations which minimize the action only approximately. All practical applications are made for nonlinear O(3)-sigma model—the simplest theory we concerned. The experience stored in dealing with this model is believed to be useful for calculation of the Yang—Mills instanton—antiinstanton interaction.

There are many papers devoted to the problem of instanton—antiinstanton potential calculation (see e. g. [6—13]). However, in most of them the shape of instanton—antiinstanton configuration is chosen quite arbitrary. Either two different solutions of classical equation of motion in two separate parts of coordinate space are made equal at the boundary (with break of derivatives), or more or less argued ansatz is used for far separated pseudoparticles.

Among the quoted papers outstands the method of I.I. Balitsky and A.V. Yung [12, 13]. In this method the choice of an infinite dimensional functional space trajectory, describing the gradual moving up of initially far separated pseudoparticles, is really model independent. Nevertheless, the method of paper [12] does not seem good (see detailed discussion in the end of the following section).

Development of the paper proceeds as follows. In section 2 a way to distinguish the collective variables which describe the motion of an instanton and antiinstanton is shown. Calculation of the nonlinear O(3)-sigma model instanton—antiinstanton interaction is examined in section 3. Practically, an origin of only the first nonvanishing term in the long range expansion of the potential $\sim 1/R^2$ may be understood (R-distance between pseudoparticles). Nevertheless, even this result occurs to be unexpectedly interesting. The field of instanton—antiinstanton configuration at distances much larger R turns out to decrease much faster than the fields of pseudoparticles itself, or their sum. Some original screening takes place. Hence the pseudoparticle interaction is not of the two particle form even in the lowest $\sim 1/R^2$ approximation. Let's for example consider an instanton very far distant from an instanton—antiinstanton pair. Because of screening the single pseudoparticle should not interact with the pair. It is to be mentioned, that first attempts to calculate the nonlinear O(3)-sigma model pseudoparticle interaction were perfomed in [7, 8]. The results of these papers differ both from each other and from present paper.

Trying to obtain other corrections to the instanton—antiinstanton potential one encounters the principle problems. Practically, the main criterion for the choice of collective coordinates consists in requirement that the Gauss integral over all other modes should slightly differ from the product of integrals near individual pseudoparticles. Unfortunately, this natural, model independent definition allows to calculate only the first term in the long range expansion of potential.

Some secondary detailes of section 3 are carried out to Appendixes A and B. The problem of Gauss integral calculation near an instanton—antiinstanton configuration, obtained in section 3, is discussed in Appendix C. Finally, an application of the developed method to get the smoothed nonlinear O(3)-sigma model meron solutions is examined in Appendix D.

The principal results and discussion of possible perspectives are shown in the conclusion.

2. GENERAL DESCRIPTION OF THE METHOD

Our principal aim is to calculate an infinite dimensional functional integral

$$Z=\int \exp\left[-\frac{1}{g}S(\varphi)\right]\prod d\varphi. \tag{2.1}$$

Generally accepted way to find Z consists in calculation of the functional integral in Gauss approximation near the minimum of $S(\varphi)$ (the saddle point approximation). To get the minimums one have to find the finite action solutions of the equation of motion $\delta S/\delta\varphi=0$.

As it was pointed out in the introduction in this paper we are interested in the contribution to integral (2.1) from the finite action configurations which are only approximate solutions of Euclidean equation of motion.

The physical theories may be divided in two classes. We face a typical representative of one class when investigate with path integral the problem of the anharmonic oscillator with potential $\lambda(x^2-\eta^2)^2$. In this case among the eigenvectors of operator $\delta^2S/\delta\varphi^2$ in the field of instanton there is one level with zero eigenvalue separated by finite split ($\sim\lambda\eta^2$) from other eigenvectors.

It is evident to suppose that for such theories the operator $\delta^2S/\delta\varphi^2$ in the field of instanton—antiinstanton pair has a few eigenvectors with anomalously small eigenvalues. In other words, we suppose that there exists a certain segment of the functional space where an expansion

$$\varphi=\varphi_0+\sum_1^\infty |n\rangle a_n,$$

$$S(\varphi)=S(\varphi_0)+\sum_1^\infty \lambda_k a_k+\sum_1^N \frac{\varepsilon_n a_n^2}{2}+\sum_{N+1}^\infty \frac{\omega_j^2 a_j^2}{2}+O(a^3) \qquad (2.2)$$

is valid, and besides $|\varepsilon_n| \ll \omega_j^2$. Here N is the doubled number of zero modes for a pseudoparticle. All parameters of the expansion (2.2) — $S(\varphi_0)$, λ_k, ε_n, ω_j are slow functions of the point φ_0.

It is easy to show, that in the part of functional integral, where the expansion (2.2) is valid a surface of the function $S(\varphi)$ looks like a valley sloping very gently. Due to relatively big value of ω_j^2 the integration over a_j $(j > N)$ may be performed in the Gauss approximation. This direction corresponds to the motion across the valley. It seems quite natural for the pair instanton — antiinstanton to minimize the action in all a_j with $j > N$. On the other hand to integrate along the a_n $(n \leqslant N)$ the Gauss approximation can't be used because of anomalously small values of ε_n, i. e. the collective coordinates are to be introduced. The modes a_n $(n \leqslant N)$ reflect the motion along the valley.

Now it is easy to define the valley bottom — the line corresponding to different relative positions of pair components. The bottom of valley is a set of points where function $S(\varphi)$ has a conditional minimum so that $\delta S/\delta\varphi$ does not equal to zero, but is a linear combination of N eigenvectors of the operator $\delta^2 S/\delta\varphi^2$, which have anomalously small eigenvalues. In other words, the set of equations to find the configuration $\varphi(x)$ for an instanton — antiinstanton pair has a form:

$$\begin{cases} \dfrac{\delta S}{\delta\varphi}=\sum_1^N \lambda_i \psi_i \\ \dfrac{\delta^2 S}{\delta\varphi^2}\,\psi_j=\varepsilon_j \psi_j \end{cases} \qquad (2.3)$$

Here the coefficients λ_i may be done equal to the unit by a choice of functions ψ_i normalization.

It is wellknown that if the equation of motion is fulfiled $(\delta S/\delta\varphi=0)$ any symmetry of the Lagrangian leads to appearance of zero eigenvalue of $\delta^2 S/\delta\varphi^2$. One can easily see that if $\delta S/\delta\varphi \neq 0$ all the ε_j in (2.3) generally speaking differ from zero.

It is interesting, that the value of first variation of action $\delta S/\delta\varphi$ determines practically the valley slope only — not the position of its bottom. The faster action would grow along the valley, the less

should be the role of this part of the functional space in the continual integral.

To introduce the collective coordinates the functional integral (2.1) is to be multiplied by the Faddeev — Popov unit. It seems very natural to use the delta functions to make equal zero the a_j $(j \leqslant N)$ — the eigenvalues of $\delta^2 S/\delta\varphi^2$ having anomalously small eigenvalues:

$$Z=\int \prod_1^N \delta(\langle \varphi-\varphi_0 \,|\, \psi_j\rangle)\, \det \left|\left| \frac{\partial \langle \varphi-\varphi_0 \,|\, \psi_k\rangle}{\partial r_l} \right|\right| \times$$

$$\times \prod_1^N dr_i \exp\left[-\frac{1}{g} S(\varphi)\right] \Pi d\varphi. \qquad (2.4)$$

Here ψ_j are the same eigenfunctions of $\delta^2 S/\delta\varphi^2$ as in (2.3) and r_i is the set of useful collective variables (the positions, orientations and sizes of the pseudoparticles). Such a definition of the collective coordinates makes it most simple to calculate the Gauss integral over the directions across valley (compare e. g. with [12]). One is only to calculate the determinant of the operator $\delta^2 S/\delta\varphi^2$ with N eigenvalues omitted.

Let's discuss what happens in more interesting to us sigma models and Yang — Mills theories. One can see that the expansion (2.2) is not applicable in this case. The continuous spectrum of eigenvalues of the operator $\delta^2 S/\delta\varphi^2$ begins immediately from $\omega^2=0$. That means there is an infinite number of directions near any configuration in the functional space which the action varies along almost linearly. Therefore there are no real valley on the $S(\varphi)$ surface.

Nevertheless we could construct the solutions of equations of (2.3) type which should have a certain physical sense. Involved in all physical quantities is not the value of functional integral (2.1), calculated near the instanton configuration, but only its difference from the functional integral around the trivial vacuum of the theory. Of course, the Gauss integral along the eigenvector of $\delta^2 S/\delta\varphi^2$ in the field of pseudoparticle pair with very small eigenvalue should produce a dangerously big multiplier in Z. But generally speaking, the same multiplier in Z appears when we calculate the functional integral near the vacuum. The collective variables are necessary only if spectra of the operator $\delta^2 S/\delta\varphi^2$ in the instanton — antiinstanton pair field and near the vacuum differ drastically.

As we shall see in the following section there are two types of differencies in the spectra of $\delta^2 S/\delta\varphi^2$ in two pseudoparticles field and in vacuum. We know that in the field of an individual pseudoparticle there are a few zero modes — the localized eigenvectors of $\delta^2 S/\delta\varphi^2$ operator with the equally zero eigenvalues. In section 3 it will be shown that some zero modes in the field of instanton — antiinstanton pair turn into discrete spectrum levels (corresponding eigenvalue becomes negative and small). The rest zero modes turn into quasi-stable levels (there appears a resonance type behaviour of the scattering phase shift).

We are only to understand what collective coordinates will remove this «dangerous» propeties in the $\delta^2 S/\delta\varphi^2$ spectrum. If we discuss the zero modes which turn into a bound states the equations (2.3) and (2.4) don't change. Detailed discussion of the collective modes associated with quasi-stable levels determination will be done in the following section for a concrete theory. Let's show almost without discussion what the equations (2.3) and (2.4) should transform in if the quasi-stable levels are involved. Instead of (2.3) we get:

$$\begin{cases} \dfrac{\delta S}{\delta\varphi} = \sum\limits_i \lambda_i \psi_i + \sum\limits_j \int f_j(\varkappa)\, \psi_\varkappa\, d\varkappa \\ \dfrac{\delta^2 S}{\delta\varphi^2}\psi_i = \varepsilon_i \psi_i \\ \dfrac{\delta^2 S}{\delta\varphi^2}\psi_\varkappa = \varkappa^2 \psi_\varkappa \end{cases} \tag{2.5}$$

Here $\varepsilon_i < 0$, $f_j(\varkappa)$ is a very narrow function in the momentum representation which differs from zero only near the j-th resonance. The function $f_j(\varkappa)$ is chosen so that vector

$$F_j = \int f_j(\varkappa)\, \psi_\varkappa\, d\varkappa \tag{2.6}$$

accepts all dangerous properties of $\delta^2 S/\delta\varphi^2$. In the other words, when we found the eigenvectors of $\delta^2 S/\delta\varphi^2$ which are made orthogonal to F_j, the j-th resonance disappears. In more detailes the choice of $f_j(\varkappa)$ for concrete theory is shown in following section.

Finally (2.4) turns into:

$$Z = \int \prod \delta(\langle \varphi - \varphi_0 | \psi_i \rangle) \prod \delta(\langle \varphi - \varphi_0 | F_j \rangle) \times$$

$$\times \det \left|\left| \frac{\partial \langle \varphi - \varphi_0 | \psi_k, F_k \rangle}{\partial r_l} \right|\right| \prod dr_s \exp\left[-\frac{1}{g} S(\varphi) \right] \prod d\varphi. \tag{2.7}$$

Now F_j is not an eigenvalue of the operator $\delta^2 S/\delta\psi^2$ and the problem of functional integral calculation becomes more complicated. As we shall see below, at least for far separated pseudoparticles the function $f_j(\varkappa)$ is quite narrow and the trouble can be easily avoided (see also the Appendix C).

It is interesting to compare the way to define the instanton — antiinstanton configurations, proposed by Balitsky and Yung [12, 13] with that of this paper. We saw that if the excitation spectrum near the vacuum of theory did not begin directly from zero, the pseudoparticle pair is associated with a well established valley on the map of $S(\varphi)$ function relief. It's intuitively evident that the way of the valley bottom definition is to be local in the functional space (i. e. to say whether a point $\varphi_0(x)$ lies on the valley bottom the only behaviour of $S(\varphi)$ in the vicinity of φ_0 is necessary to know). The equations (2.3) and (2.5) are evidently local in this sense (only two variational derivatives of action are used to determine φ).

On the other hand the definition of the valley bottom used in [12] isn't local at all. Balitsky and Yung describe the pseudoparticles moving up by means of a trajectory which in every point φ_0 coincides with the vector $\delta S/\delta\varphi$ calculated in the same point. But there are an infinite number of such trajectories (at least one goes through the every point of functional space). To make the definition unique the asymptotic behaviour of the trajectory is to be involved. The only trajectory, which in the asymptotic gives two infinitely separated pseudoparticles, is called by the bottom of the valley.

Nevertheless if the well established valley is associated with the pair of pseudoparticles at the map of $S(\varphi)$ function relief the Balitsky — Yung method and that of present paper at least for far separated pseudoparticles produce very close values of instanton — antiinstanton potential. It happens, however, that in the most interesting theories the spectrum of excitations begins from $\omega^2 = 0$. So the assumption itself that the well established valley corresponds to the instanton — antiinstanton configuration is wrong. For such theories the method of the paper [12] for pseudoparticle interaction description is incorrect.

3. THE CALCULATION OF THE NONLINEAR O(3)-SIGMA MODEL INSTANTON—ANTIINSTANTON INTERACTION

The great progress was maid last a few years in the nonlinear O(3)-sigma model investigation. In addition to papers [4, 7, 8] quoted in the introduction in this theory there were calculated the one loop functional integral near one instanton [15, 7] and even near many instanton [16] configurations. Thus, many of the formulas adduced in the beginning of this section may be considered as wellknown.

The action for O(3)-sigma model looks like:

$$S=\int \sum_{i=1,2} (\partial_i \vec{\varphi}\, \partial_i \vec{\varphi})\, d^2x\,,$$
$$\vec{\varphi}=(\varphi_1, \varphi_2, \varphi_3)\,, \qquad (3.1)$$
$$\vec{\varphi}^2=1\,.$$

Functional integral around the point $\vec{\varphi}_0$ takes a form [15]:

$$Z=\int \exp\left[-\frac{1}{g}S(\vec{\varphi})\right]\prod \delta(\vec{\varphi}^2-1)\, d\vec{\varphi}=$$
$$=\int \exp\left\{-\frac{1}{g}\left[S(\vec{\varphi}_0)+2\int \vec{\psi} m \varphi_0 d^2x+\int \vec{\psi} m \vec{\psi} d^2x+O(\psi^3)\right]\right\}\times \qquad (3.2)$$
$$\times\prod \delta(2\vec{\varphi}_0\vec{\psi})\, d\vec{\psi}\,,$$

where

$$m=-\Delta+(\vec{\varphi}_0\,\Delta\,\vec{\varphi}_0)\,.$$

It's known [17] that nonlinear O(3)-sigma model is asymptotically free. Let's discuss only the field configurations which have such a typical momentum that coupling constant g is small enough. One can then assume g to be a fixed value, neglecting its variation from scale to scale.

The equations (2.5) turns into:

$$\begin{cases} -\Delta\vec{\varphi}+(\vec{\varphi}\Delta\vec{\varphi})\,\vec{\varphi}=\sum \lambda_i \vec{\psi}_i+\sum\int f_j(\varkappa)\,\vec{\psi}_\varkappa\, d\varkappa \\ [-\Delta+(\vec{\varphi}\Delta\vec{\varphi})]\,\vec{\psi}=\varepsilon\vec{\psi}+u(\vec{r})\,\vec{\varphi} \end{cases} \qquad (3.3)$$

Here function $u(\vec{r})$ is a Lagrange multiplier making it possible to satisfy the condition $\vec{\varphi}\vec{\psi}=0$ for every point $\vec{r}$. The right hand side of the first equation (3.3) contain the summation over a discrete and

integration over a continuous spectrum. Formulas for $f_j(\varkappa)$ could be found only at the end of this section. Now we can only state that $f_j(\varkappa)$ differs from zero only at small momentum $\varkappa$. The integral over the excitations with small wavelength converge well and Gauss approximation may be applied for its calculation. The need in collective variables may arise for small $\varkappa$ only.

It is convenient to use two complex functions ω and F instead of two vector functions $\vec{\varphi}$ and $\vec{\psi}$ satisfying two additional conditions ($\vec{\varphi}^2=1$ and $\vec{\varphi}\vec{\psi}=0$) so that:

$$\vec{\varphi}=(\varphi_1+i\varphi_2, \varphi_3)=\left(\frac{2\omega}{1+|\omega|^2}, \frac{1-|\omega|^2}{1+|\omega|^2}\right), \qquad (3.4)$$

$$\vec{\psi}=(\psi_1+i\psi_2, \psi_3)=\left(\frac{F-\omega^2F^*}{(1+|\omega|^2)^2}, \frac{-\omega^*F-\omega F^*}{(1+|\omega|^2)^2}\right). \qquad (3.5)$$

Now the conditions $\vec{\varphi}^2=1$ and $\vec{\varphi}\vec{\psi}=0$ are fulfilled automatically. For such notations the action is

$$S=8\int \frac{|\frac{\partial\omega}{\partial z}|^2+|\frac{\partial\omega}{\partial z^*}|^2}{(1+|\omega|^2)^2}\, dx\, dy\,, \qquad (3.6)$$
$$z=x+iy\,. \qquad (3.6)$$

The equations (3.3) after some transformations turn into:

$$-\frac{\partial^2\omega}{\partial z\,\partial z^*}+\frac{2\omega^*}{1+|\omega|^2}\frac{\partial\omega}{\partial z}\frac{\partial\omega}{\partial z^*}=\sum \frac{\lambda_i}{4}F_i+\sum\int \frac{f_j}{4}F_\varkappa\, d\varkappa \qquad (3.7)$$

$$\left\{-\frac{\partial^2}{\partial z\,\partial z^*}+\frac{2\omega^*}{1+|\omega|^2}\left[\frac{\partial\omega}{\partial z}\frac{\partial}{\partial z^*}+\frac{\partial\omega}{\partial z^*}\frac{\partial}{\partial z}+\frac{\partial^2\omega}{\partial z\,\partial z^*}\right]-\right.$$
$$\left.-\frac{6\omega^{*2}}{(1+|\omega|^2)^2}\frac{\partial\omega}{\partial z}\frac{\partial\omega}{\partial z^*}\right\}F+\frac{2}{(1+|\omega|^2)^2}\frac{\partial\omega}{\partial z}\frac{\partial\omega}{\partial z^*}F^*=\frac{\varepsilon}{4}F\,. \qquad (3.8)$$

Solutions of equation (3.8) are conveniently normalized:

$$\int \frac{F_i^*F_k+F_iF_k^*}{2(1+|\omega|^2)^2}\, dx\, dy=\delta_{ik}\,. \qquad (3.9)$$

If we get the right hand side of eq. (3.7) equal to zero we obtain the usual Euclidean equation of motion in terms of ω and z. In particular any analytic (only z dependent) or antianalytic (only z^* dependent) function should be an exact solution of equation of motion. The configuration $\omega=A(z-R_A)^{-1}$ is usually called as an

instanton of size $|A|$ with the center pointed at $\vec{R}_A=(\mathrm{Re}\,R_A,\ \mathrm{Im}\,R_A)$ and $\omega=(B(z-R_B)^{-1})^*$ — an antiinstanton of size $|B|$ centered at $\vec{R}_B=(\mathrm{Re}\,R_B,\ \mathrm{Im}\,R_B)$.

It's evident to search for the solutions of eq. (3.7) in a form:

$$\omega=\frac{A}{z}+\left(\frac{B}{z-R}\right)^*+u \tag{3.10}$$

where $u=u(z,\ z^*)$ is a small correction (we suppose $|A|\ll|R|$ and $|B|\ll|R|$). The magnitude of u can be found easily. For example, the field of antiinstanton in the vicinity of instanton in the first order over $|R|^{-1}$ equals to $(-B/R)^*$. Adding of the constant doesn't change the analytic behaviour of the ω. The first nonanalytic term making the solution not to satisfy the equation of motion is of the order $|R|^{-2}$. So it is evident to suppose that $u\sim|R|^{-2}$ (I use $|A|\sim1$, $|B|\sim1$).

Now let us discuss the equation (3.8). For pure instantonic (antiinstantonic) configuration this equation is also simplified. Thus for analytic ω one gets:

$$\left\{-\frac{\partial^2}{\partial z\,\partial z^*}+\frac{2\omega^*}{1+|\omega|^2}\frac{\partial\omega}{\partial z}\frac{\partial}{\partial z^*}\right\}F=\frac{\varepsilon}{4}F. \tag{3.11}$$

It's easy to show that for $\omega=A/z$ only two of the eq. (3.11) with $\varepsilon=0$ solutions $F\sim1/z$ and $F\sim1/z^2$ decrease at the infinity and can be integrated in the sense of formula (3.9) at $z\Rightarrow0$.

Four functions $1/z$, i/z, $1/z^2$, i/z^2 — are the zero modes of one instanton solution. $1/z^2$ and i/z^2 appear due to translational invariance of the action, $1/z$ and i/z — due to rotational and scale invariance. So one can state that in the right hand side of eq. (3.7) the functions F appears in a form:

$$F=\frac{p}{z}+\frac{q}{(z-R)^*}+\frac{s}{z^2}+\frac{t}{(z-R)^{*2}}+G, \tag{3.12}$$

where p, q, s, t are complex constants and function $G=G(z,\ z^*)$ is small everythere. Moreover if we work in the first order over $|R|^{-2}$ the coefficiients s and t occurs to be zero. Zero modes corresponding to the translation of pseudoparticles in this approximation remain the solutions of (3.8) with $\varepsilon=0$ and only $F\sim z^{-1}$, $(z-R)^{*-1}$ appear in the eq. (3.7) right hand side.

Let us see how our equations behave at $|z|\gg|A|$, $|z-R|\gg|B|$. Now ω is small and one gets:

$$\begin{cases}-\dfrac{\partial^2\omega}{\partial z\,\partial z^*}=\sum\dfrac{\lambda i}{4}F_i+\sum\int\dfrac{f_i}{4}F_\varkappa\,d\varkappa, & (3.13)\\[2ex] -\dfrac{\partial^2F}{\partial z\,\partial z^*}=\dfrac{\varepsilon}{4}F. & (3.14)\end{cases}$$

If to remember that $\dfrac{\partial^2}{\partial z\,\partial z^*}=\dfrac{1}{4}\Delta$ one sees (3.14) is the ordinary Bessel equation. It's clear now that F is a linear combination of functions like $\left(\dfrac{z}{|z|}\right)^n Z_{|n|}(\varkappa|z|)$, where $Z_{|n|}$ is the McDonald function for $\varepsilon=-\varkappa^2$, and $Z_{|n|}$ — Bessel or Neuman function for $\varepsilon=\varkappa^2$. If we remember the formula (3.12) it's easy to write:

$$F=\frac{p}{z}\varkappa|z|\,K_1(\varkappa|z|)+\frac{q}{(z-R)^*}\varkappa|z-R|\,K_1(\varkappa|z-R|), \tag{3.15}$$

$$\varkappa=\sqrt{-\varepsilon}$$

in the discrete spectrum,

$$F=\frac{p}{z}\frac{\pi}{2}\varkappa|z|\,[\cos\varphi_I J_1(\varkappa|z|)-\sin\varphi_I N_1(\varkappa|z|)]+$$

$$+\frac{q}{(z-R)^*}\frac{\pi}{2}\varkappa|z-R|\,[\cos\varphi_A J_1(\varkappa|z-R|)-\sin\varphi_A N_1(\varkappa|z-R|)], \tag{3.16}$$

$$\varkappa=\sqrt{\varepsilon}$$

in the continum. At first sight it seems that the phase shift of a wave scattered on the instanton ($z=0$) and antiinstanton ($z=R$) may differ. In the Appendix A the both terms in (3.16) are shown to have the same phase shift for a reasonable boundary condition at the infinity $\varphi=\varphi_I=\varphi_A$.

It is lucky that with (3.15) and (3.16) one can get the solution of equation (3.13):

$$\omega=\omega_0+\sum\frac{\lambda_i}{\varepsilon_i}F_i+\sum\int\frac{f_i}{\varepsilon_\varkappa}F_\varkappa\,d\varkappa, \tag{3.17}$$

where ω_0 is the homogeneous equation solution $-\dfrac{\partial^2\omega_0}{\partial z\,\partial z^*}=0$. Substituting F (3.15), (3.16) and choosing ω_0 so to fulfil the formula

(3.10) one gets:

$$\omega=\frac{1}{z}\left\{A+\sum\frac{\lambda_i p_i}{\varkappa_i^2}-\sum\int\frac{f_i p_\varkappa}{\varkappa^2}\sin\varphi_\varkappa\,d\varkappa\right\}+\frac{1}{(z-R)^*}\left\{B^*+\right.$$

$$\left.+\sum\frac{\lambda_i q_i}{\varkappa_i^2}-\sum\int\frac{f_i q_\varkappa}{\varkappa^2}\sin\varphi_\varkappa\,d\varkappa\right\}-\sum\frac{\lambda_i F_i}{\varkappa_i^2}+\sum\int\frac{f_i F_\varkappa}{\varkappa^2}d\varkappa\,. \qquad (3.18)$$

It's naturall to suppose that λ_i, $f_i\sim|R|^{-2}$ and $\varkappa\sim|R|^{-1}$. Then we have a surprising result—even for far separated pseudoparticles the asymptotic of the instanton—antiinstanton configuration field differs significantly from the sum of instanton and antiinstanton fields.

The functions ω (3.18) and F (3.15), (3.16) at $|z|\sim|A|$ and $|z-R|\sim|B|$ satisfy the equations (3.7) and (3.8) only in the zero order over pseudoparticle interaction. It happens the expressions for interaction induced correction u (3.10) and G (3.12) at $|z|\sim|A|$ and $|z-R|\sim|B|$ can be found. The procedure of u and G determination is examined in the Appendix B.

It is interesting that except for big parameters $|R/A|$ and $|R/B|$ the big values $\ln|R/A|$ and $\ln|R/B|$ are involved. Everywhere below we shall use only lowest nontrivial approximation over $[\ln|R/A|]^{-1}$, $[\ln|R/B|]^{-1}$.

Comparing the solutions of equation (3.7) to (3.18) at $|A|\ll|z|\ll|R|$ and $|B|\ll|z-R|\ll|R|$ one gets a set of two complex equations which allows to find the values of λ_i and normalization constants for function $f_i(\varkappa)$ (B.5). Comparing the (3.8) solutions in the discrete spectrum to (3.15) one obtains the equations (B.9) for eigenvalues $\varkappa$ and complex quantities p and q. And comparing to (3.16) in the continum one obtains equatins (B.10) which allow to find p, q and phase shift φ as a function of $\varkappa$.

Two complex equations (B.9) give us the $\varkappa^2$ values in discrete spectrum.

$$\varkappa_1^2=\varkappa_2^2=\frac{2}{|R|^2\sqrt{L_A L_B}}. \qquad (3.19)$$

Here $L_A=\ln|R/A|$, $L_B=\ln|R/B|$. Another two solutions have no physical sense ($\varkappa^2<0$).

While we discuss the infinitely distant instanton and antiinstanton the operator $\delta^2S/\delta\varphi^2$ has eight localized eigenfunctions with even zero eigenvalues—$1/z^2$, i/z^2, $1/(z-R)^{*2}$, $i/(z-R)^{*2}$, $1/z$, i/z, $1/(z-R)^*$, $i/(z-R)^*$ corresponding to eight zero modes. After we

take into account the finite distance between pseudoparticles in the first order over $|R|^{-2}$ four zero modes ($\sim z^{-2}$, $(z-R)^{*-2}$) don't change. But from another four zero modes $1/z$, i/z, $1/(z-R)^*$, $i/(z-R)^*$ two become the discrete spectrum levels and two disappears somewhere. So to understand where do zero modes «disappear» we are to analyze the behaviour of the eq. (3.8) solutions in continuous spectrum.

From the equations (B.10) one finds in continuous spectrum

$$\varkappa^2=\frac{2}{|R|^2\sqrt{\left(L_A+\frac{\pi}{2}\operatorname{ctg}\varphi\right)\left(L_B+\frac{\pi}{2}\operatorname{ctg}\varphi\right)}}\,,$$

$$p\sim\alpha R\sqrt{L_B+\frac{\pi}{2}\operatorname{ctg}\varphi}\,, \qquad (3.20)$$

$$q\sim\alpha R^*\sqrt{L_A+\frac{\pi}{2}\operatorname{ctg}\varphi}\,,$$

where α equals to 1 or i according to the double degeneracy of all solutions.

We are interested in that part of $\varphi(\varkappa)$ function there φ varies fast. If $\operatorname{ctg}\varphi\ll L_A$, $\operatorname{ctg}\varphi\ll L_B$ one can rewrite the first of (3.20) formulas

$$\varkappa^2=\varkappa_0^2-2\Gamma\varkappa_0\operatorname{ctg}\varphi\,,$$

$$\varkappa_0^2=\frac{2}{|R|^2\sqrt{L_A L_B}}\,, \qquad (3.21)$$

$$\Gamma=\frac{\pi}{8}\varkappa_0\left(\frac{1}{L_A}+\frac{1}{L_B}\right).$$

Such a phase shift energy dependence, as it is known from quantum mechanics, corresponds to a scattering on the quasi-stable level of the energy $\varkappa_0^2$ and width $4\Gamma\varkappa_0$. Let us remind that eigenfunctions of operator $\delta^2S/\delta\varphi^2$ are the combinations of Bessel and Neuman functions like:

$$\cos\varphi J_1(\varkappa\rho)-\sin\varphi N_1(\varkappa\rho)\,, \qquad (3.22)$$

where $\rho=|z|$ or $\rho=|z-R|$. And near the resonance

$$\sin\varphi_\varkappa=\frac{\Gamma}{\sqrt{(\varkappa_0-\varkappa)^2+\Gamma^2}}\,,$$

$$\cos\varphi_\varkappa = \frac{\varkappa_0 - \varkappa}{\sqrt{(\varkappa_0-\varkappa)^2+\Gamma^2}} \,. \tag{3.23}$$

It's important that at small ρ the Neuman function goes to infinity $N_1(\varkappa\rho) = 2(\pi\varkappa\rho)^{-1}$, while the Bessel function goes to zero $J_1(\varkappa\rho) = \varkappa\rho/2$. Therefore one can say that the amplitude of probability for the exitation to penetrate inside pseudoparticle is proportional to $\sin\varphi$ (3.23).

Our purpose is to find such directions in the functional space where the small action variation corresponds to anomalously big variation of functional coordinate. In other words the small action variation is associated with macroscopic translation, rotation and scale transformation of pseudoparticles. Integrating along this directions one can't use the Gauss approximation — the anharmonic corrections are of greate importance because of slow convergence. Therefore the collective coordinates are necessary.

The obvious candidate of the directions which the collective coordinates are to be used along is the discrete spectrum eigenvector of $\delta^2 S/\delta\varphi^2$. In this case the corresponding eigenvalue ε is negative. This leads the Gauss approximation integral over the discrete spectrum eigenmodes of $\delta^2 S/\delta\varphi^2$ operator even diverge.

Another example of dangerous functional space directions are the operator $\delta^2 S/\delta\varphi^2$ eigenvectors close to the quasi-stable levels (3.21). Let us discuss the fluctuation of the type

$$\Psi = \int f(\varkappa)\, F_\varkappa \, d\varkappa \,. \tag{3.24}$$

The direct expression for $\Psi(z)$ is easy to found

$$\Psi = \left[\frac{p(\varkappa=\varkappa_a)}{z} + \frac{q(\varkappa=\varkappa_a)}{(z-R^*)}\right] \int f(\varkappa) \sin\varphi_\varkappa \, d\varkappa \,. \tag{3.25}$$

This expression is true near by to $z=0$ and $z=R$ while in all the other cases Ψ is small.

Adding of Ψ (3.25) to our initial configuration $\omega = \frac{A}{z} + \left(\frac{B}{z-R}\right)^*$ we just obtain the combination of rotations and scale transformations of the instanton and antiinstanton. It's easy to understand that the most «dangerous» direction in the functional space is the Ψ (3.24) vector with

$$f(\varkappa) \sim \sin\varphi_\varkappa = \frac{\Gamma}{\sqrt{(\varkappa_0-\varkappa)^2+\Gamma^2}} \,. \tag{3.26}$$

Formally the functional integration over the modes close to the quasi-stable level may be performed in the Gauss approximation. However, because the phase shift in the vicinity of quasi-stable level differs much from the trivial quantity $\varphi = n\pi$, the value of this Gauss integral should be anomalously big as compared to the vacuum integral. It's natural to interpret this big value of functional integral as a reflection of existence of the collective coordinate describing the pair of far separated pseudoparticles. In the Appendix C it is proved, that for the Gauss approximation integral near the instanton — antiinstanton pair to be equal to the product of integrals near the indivtdual pseudoparticles the functions $f_i(\varkappa)$ of the form (3.26) are to be used.

The formula (3.23) shows the value of phase shift in the vicinity of resonance only. So, one can substitute $f_i(\varkappa)$ of the form (3.26) into (3.24) and (3.18) if only all the integrals over $\varkappa$ converge at $|\varkappa-\varkappa_0| \sim \Gamma$. Unfortunately this is not so.

We saw that in (3.24) and (3.18) the Bessel function appears in the only combination

$$\int f(\varkappa) \cos\varphi_\varkappa J_1(\varkappa\rho)\, d\varkappa \sim \int \frac{\Gamma(\varkappa_0-\varkappa)}{(\varkappa_0-\varkappa)^2+\Gamma^2} J_1(\varkappa\rho)\, d\varkappa \,. \tag{3.27}$$

Of course, at $\rho \gg \varkappa_0^{-1}$ this integral converge well because of the J_1 function oscillations. But we are most interested in $\rho \ll \varkappa_0^{-1}$, where (3.27) converges very badly. The formula (3.26) define the $f_i(\varkappa)$ only close to resonance. To improve the convergence of the integrals one have to redifine the integrals so that at $|\varkappa-\varkappa_0| > \Delta$ ($\Gamma \ll \Delta \ll \varkappa_0$) function $f_i(\varkappa)$ should go to zero very fast. Without additional physical information we can say nothing about the character of function $f_i(\varkappa)$ decrease. However, we should see, that to calculate the first term in the instanton — antiinstanton potential the inequality $\Gamma \ll \Delta \ll \varkappa_0$ is enough.

With two complex equations (B.5) two values λ_i and two functions $f_i(\varkappa)$ normalization constants are easy to found. After that from (3.18) one sees:

$$\omega = \frac{1}{z}\left\{A - |z|^2 \frac{\ln|z/R|}{\ln|R/A|}\left(\frac{B}{R^2}\right)^* + \sum_{n=1}^{\infty} a_n (\varkappa_0|z|)^{2n}\right\} +$$

$$+\frac{1}{(z-R)^*}\left\{B^*-|z-R|^2\frac{\ln|(z-R)/R|}{\ln|R/B|}\frac{A}{R^2}+\sum_{n=1}^{\infty}b_n(\varkappa_0|z-R|)^{2n}\right\}, \quad (3.28)$$

where a_n, $b_n \ll L_A$, L_B. As we don't know how the integrals of (3.27) type were defined, we can say nothing about the values a_n, b_n. The series (3.28) are good convergent at $|z|$, $|z-R| \ll |R|\sqrt{L}$ (where L is a big logarithm either $\ln|R/A|$ or $\ln|R/B|$). Contribution to the action by $|z|$, $|z-R| \geqslant |R|\sqrt{L}$ is only $\sim(|R|^2L)^{-1}$.

The corrections to action appearing due to pseudoparticles interaction may be divided in two classes. First appear if we choose the instanton—antiinstanton configuration as a direct sum of the fields $\omega=\frac{A}{z}+\left(\frac{B}{z-R}\right)^*$ (the sum ansatz)

$$S_{\text{sum}}=8\int\frac{|\frac{A}{z^2}|^2+|\frac{B}{(z-R)^2}|^2}{\left(1+|\frac{A}{z}+\left(\frac{B}{z-R}\right)^*|^2\right)^2}dx\,dy=16\pi+32\pi\,\mathrm{Re}\left(\frac{AB}{R^2}\right)+O\left(\frac{1}{|R|^4}\right). \quad (3.29)$$

This instanton—antiinstanton potential was found in [7].

Another corrections appear because the configuration (3.28) differs from the sum ansatz. With this corrections the interaction of far separated pseudoparticles becomes twice more than (3.29)

$$S=16\pi+64\pi\,\mathrm{Re}\left(\frac{AB}{R^2}\right)+O\left(\frac{1}{|R|^2L}\right). \quad (3.30)$$

It is quite unexpected that in the approximation we use the first two terms in ω (3.18) $\sim 1/z$ and $\sim 1/(z-R)^*$ cancels

$$\omega=-\sum\frac{\lambda_i F_i}{\varkappa_i^2}+\sum\int\frac{f_j(\varkappa)F_\varkappa}{\varkappa^2}d\varkappa. \quad (3.31)$$

At $\varkappa_0|z| \gg 1$ functions F_i in (3.31) decrease like $\exp(-\varkappa_0|z|)$. The continuous spectrum functions at $\varkappa|z| \gg 1$ decrease very slowly $F_\varkappa \sim 1/\sqrt{z}\exp(i\varkappa|z|)$. However, after integration over $\varkappa$ in (3.31) the first term in the $F_\varkappa$ asymptotic cancels and so $\omega \sim |z|^{-3/2}$ at $\varkappa_0|z| \gg 1$. We see that at large distances the field of instanton—antiinstanton configuration decrease much faster then the individual pseudoparticles fields.

We have calculated only the interaction induced correction to the

pseudoparticle pair classic action $\Delta S=64\pi\,\mathrm{Re}\left(\frac{AB}{R^2}\right)$. Further corrections to this action may be of the order $\sim[\ln|R/A|]^{-1}$, $[\ln|R/B|]^{-1}$. Unfortunately, so far as we do not understand how to define $f_j(\varkappa)$ (3.26) at $|\varkappa-\varkappa_0| \gg \Gamma$ we can say nothing about the value of this corrections. We can also say nothing about the difference of Gauss approximation integral near the instanton-antiinstanton pair from the product of individual pseudoparticle fields (see Appendix C).

4. CONCLUSION

We have found the long range behaviour of the instanton—antiinstanton interaction in the nonlinear O(3)-sigma model and discovered also that the field of pseudoparticle pair at large distance decreases much faster than the sum of instanton and antiinstanton fields.

We can't calculate any further corrections to the instanton—antiinstanton interaction because of principle difficulties. In single pseudoparticle case the collective variables are associated with localized eigenvectors of the operator $\delta^2S/\delta\varphi^2$ which have even zero eigenvalues. In the pair of pseudoparticles case some of «dangerous» directions, which the collective coordinates are necessary along, become not single eigenvectors but narrow packets of $\delta^2S/\delta\varphi^2$ eigenfunctions close to a quasi-stable level. To find the shape of this packet the phase shift $\varphi(\varkappa)$ behaviour near a resonance is to be considered. When we calculate the first term only in the pseudoparticle interaction the information from the simple Breit—Wigner formulas is enough. Collective variables are chosen so that the Gauss integral over other variables equals to the product of single pseudoparticle integrals. The further corrections can not be found before we formulate the better criterion for extraction of correspondent to the collective motion part of function $\varphi(\varkappa)$.

The next, much more important application of this paper method should be the calculation of an instanton—antiinstanton potential in the Yang—Mills theory. It is to be noted that Balitsky—Yung method [12, 13] has recently been applied to this problem [18].

The usefulness of the collective coordinate method to describe the instanton—antiinstanton pair (at least for large distances) is evident. It's interesting to understand whether there exists any other

class of configurations which have no such a well established asymptotic as pseudoparticle pair, but to calculate the functional integral around them the collective variables are necessary? Formula (2.3) gives us the natural way to look for such configurations. In Appendix D the attempt to redefine the meron solutions of nonlinear O(3)-sigma model by means of equations (2.3) is examined.

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Appendix A

Let us discuss the instanton and antiinstanton placed at the points R_I, R_A inside a big circular volume of a radius $D(|R_I|, |R_A| \ll D)$. We are to formulate the boundary conditions which the solution of eq. (3.8) should satisfy at $|z|=0$. The most popular conditions are either $F(|z|=D)=0$ or $\left(\frac{\partial}{\partial|z|}F(z)\right)_{|z|=D}=0$. It occurs, however, that the most suitable to us boundary conditions are mixed:

$$\begin{cases} [F(z)-F(-z)]_{|z|=D}=0 \\ \left[\frac{\partial}{\partial|z|}(F(z)+F(-z))\right]_{|z|=D}=0 \end{cases} \tag{A.1}$$

Really, at $|z| \gg |R|$ one may write $|z-R| = |z| - |z|\,\mathrm{Re}\,(R/z)$ and so at large z (3.16) transforms into

$$F \sim p\,e^{-i\alpha}[\cos\Delta_I \cos(\varkappa|z|+\varphi_I) + \sin\Delta_I \sin(\varkappa|z|+\varphi_I)] + \\ + q\,e^{i\alpha}[\cos\Delta_A \cos(\varkappa|z|+\varphi_A) + \sin\Delta_A \sin(\varkappa|z|+\varphi_A)]\,, \tag{A.2}$$

where $z=|z|e^{i\alpha}$, $\Delta_I = \varkappa|z|\,\mathrm{Re}\,(R_I/z)$, $\Delta_A = \varkappa|z|\,\mathrm{Re}\,(R_A/z)$. It's important that for transformation $z \to -z$ the function $\cos\Delta$ is even and $\sin\Delta$ — odd. So, both terms in the first (or in the second) addentum in (A.2) satisfy the condition (A.1) at the same value of $\varkappa$. However, the whole function F (A.2) is consistent with (A.1) only if $\varphi_I = \varphi_A$.

In principle, one may solve eq. (3.8) for other boundary conditions as well. The only inequality $|R_I|$, $|R_A| \ll D$ is necessary. It can be shown that the final result for instanton — antiinstanton interaction don't change, but the solution become much more complicated.

Appendix B

Expression (3.18) is correct at $|z| \gg |A|$ and $|z-R| \gg |B|$ only. On the other hand at $|z| \sim |A|$, for example, $\omega = A/z + C + \delta\omega$ where $\delta\omega \sim |R|^{-2}$, and $C \sim |R|^{-1}$ constant which doesn't change the instantonic character of ω. We are to find $\delta\omega$ from the exact eq. (3.7) at $|z| \sim |A|$ so to satisfy the (3.18) at $|z| \gg |A|$.

At small z (3.18) takes the form:

$$\omega = \frac{A}{z} + C - z^* \sum \frac{\lambda_i p_i}{2}\left[\ln\left(\frac{\gamma\varkappa_i|z|}{2}\right) - \frac{1}{2}\right] - \\ - z^* \sum \int \frac{f_i p_\varkappa}{2}\left\{\left[\ln\left(\frac{\gamma\varkappa|z|}{2}\right) - \frac{1}{2}\right]\sin\varphi - \frac{\pi}{2}\cos\varphi\right\} d\varkappa - z^*\left(\frac{b}{R^2}\right)^*, \tag{B.1}$$

where C is a constant of no interest which may be easily found from (3.15), (3.16), (3.18), $\ln(\gamma) = 0.577...$ is the Eurler's constant. The explicit formula for b is also easy to be found. In the following we shall use only that $b=B$ at $\varkappa \ll |R|^{-1}$. The wellknown properties of cylinder functions were used in (B.1) (see e. g. [19]).

On the other hand, let's search for a solution of (3.7) at small z like $\omega = \frac{A+V(|z|^2)}{z} + \frac{\mathrm{const}}{R^*}$. In the first order over V

$$tV'' + \frac{2\alpha}{t+\alpha}V' = -F_0\,, \tag{B.2}$$

where $\alpha = |A|^2$, $t = |z|^2$, $F_0 = \sum \frac{\lambda_i p_i}{4} + \sum \int \frac{f_i p_\varkappa}{4} \sin\varphi_\varkappa\, d\varkappa$. Solution for this equation is

$$V = -F_0 \int_0^{|z|^2} \left[\left(\frac{t+\alpha}{t}\right)^2 \ln\left(\frac{t+\alpha}{\alpha}\right) - \frac{t+\alpha}{t}\right] dt \tag{B.3}$$

and for large t

$$V = -F_0\left[t\ln\left(\frac{t}{\alpha}\right) - 2t\right]. \tag{B.4}$$

Comparing (B.4) and (B.1), (B.2) it's easy to write the equation for values λ_i and for $f_i(\varkappa)$ normalization constants. It occurs that except for big parameter $|R/A|^2$ (or $|R/B|^2$) there appears a big value $\ln|R/A|$ (or $\ln|R/B|$). In the leading order over $[\ln|R/A|]^{-1}$ (or $[\ln|R/B|]^{-1}$) we get:

$$\begin{cases} L_A\left[\sum \frac{\lambda_i p_i}{2}+\sum\int \frac{f_j p_\varkappa}{2}\sin\varphi_\varkappa\, d\varkappa\right]=\left(\frac{B}{R^2}\right)^*, \\ L_B\left[\sum \frac{\lambda_i q_i}{2}+\sum\int \frac{f_j q_\varkappa}{2}\sin\varphi_\varkappa\, d\varkappa\right]=\frac{A}{R^2}, \end{cases} \tag{B.5}$$

where $L_A=\ln|R/A|$, $L_B=\ln|R/B|$. The second equation (B.5) is found at $|z-R|\sim|B|$.

Now let investigate the operator $\delta^2S/\delta\varphi^2$ eigenvectors. We are to understand what restrictions to F (3.15), (3.16) may be found at $|z|\sim|A|$ and $|z-R|\sim|B|$. Let's discuss first the bound states. Keeping in (3.15) at $|z|\ll|R|$ only the terms with angular behaviour like $\sim 1/z$ (or $\sim z^*$) and neglecting the terms ~ 1 as compared to big $\ln(\varkappa|z|)$ one gets:

$$F=\frac{p}{z}+pz^*\frac{\varkappa^2}{2}\ln(\varkappa|z|)-q\left(\frac{z}{R^2}\right)^*. \tag{B.6}$$

Now we are to get the solution of eq. (3.8) equal p/z at $|z|\ll|A|$ and turning into (B.6) at $|A|\ll|z|\ll|R|$. Let the solution of (3.8) at $|z|\sim|A|$ be of the form:

$$F=\frac{p+\psi(|z|^2)}{z}+\ldots \tag{B.7}$$

where ... are small corrections with angular behaviour different from $1/z$. Substituting ω (B.3) and F (B.7) into the (3.8) one gets the differential equation for ψ. Its solution at $t\gg|A|^2$ is

$$\psi=t\left[\ln\left(\frac{t}{\alpha}\right)-2\right]\frac{\varkappa^2 p}{4}+t\left(\frac{p}{A}\right)^* F_0. \tag{B.8}$$

From (B.7) and (B.6) we see:

$$\begin{cases} \frac{\varkappa^2}{2}L_A p=-\frac{q}{R^{*2}}, \\ \frac{\varkappa^2}{2}L_B q=-\frac{p}{R^2}, \end{cases} \tag{B.9}$$

where $L_A=\ln|R/A|$, $L_B=\ln|R/B|$. The second equation (B.9) was found at $|z-R|\sim|B|$.

For the continuous spectrum repeating all upper reasons almost

without change we see:

$$\begin{cases} \frac{\varkappa^2}{2}\left[L_A+\frac{\pi}{2}\operatorname{ctg}\varphi\right]p=\frac{q}{R^{*2}}, \\ \frac{\varkappa^2}{2}\left[L_B+\frac{\pi}{2}\operatorname{ctg}\varphi\right]q=\frac{p}{R^2}. \end{cases} \tag{B.10}$$

Appendix C

The Gauss Integral Calculation Near Instanton—Antiinstanton Configuration

As we have yet seen, the most interesting information about the pseudoparticle interaction comes from the long wave fluctuations. So let's discuss only the small momentum excitations $\varkappa\ll|R|^{-1}$.

The problem of Gauss integral calculation is known to be reduced to the problem of phase shifts calculation for $\delta^2S/\delta\varphi^2$ operator eigenfunctions. We are most interested in the massless theories where the $\delta^2S/\delta\varphi^2$ eigenvalue spectrum begins at $\omega^2=0$. The Gauss integral over the states with small ω converges badly. So the Gauss approximation can be only applied if the long wave contribution to all physical quantities is small. In other words at a big wavelength the phase shift must go to zero.

It's easy to show that in the field of one pseudoparticle the phase shift comes to zero not like the power of $\varkappa$ but much slowly for only fluctuations $F\sim 1/z$ at $z\to 0$ (the instanton case) or $F\sim 1/(z-R)^*$ at $z\to R$ (the antiinstanton case). For example, in the field of single instanton ($\omega=A/z$)

$$\operatorname{ctg}\varphi=\frac{2}{\pi}\left[\ln\left(\frac{\gamma\varkappa|A|}{2}\right)+\frac{1}{2}\right]\approx\frac{2}{\pi}\ln(\varkappa|A|). \tag{C.1}$$

On the other hand, in the case of pseudoparticle pair at small $\varkappa$ only excitations like $1/z$ in the field of instanton and $1/(z-R)^*$ in the field of antiinstanton mix strongly. So to calculate the long wave contribution to the Gauss integral we are to discuss $\delta^2S/\delta\varphi^2$ eigenfunctions of the type (3.16) only.

Let's put for definition $|A|>|B|$. Graph of the $\varphi(\varkappa)$ function (3.20) consisting of two curves is shown on Fig. 1 (curves I and II). One curve (I) begins at $\varphi=\pi$, $\varkappa=0$. Another (II) begins at $\varphi=0$, $\varkappa=0$ but at $\varkappa\approx\varkappa_0$ the phase jumps quickly from $\varphi\approx 0$ to

$\varphi \approx \pi$. At $|R|^{-1} \ll \varkappa \ll |A|^{-1}, |B|^{-1}$ we have $\operatorname{ctg}\varphi = \frac{2}{\pi} L_B$ (I) or $\operatorname{ctg}\varphi = \frac{2}{\pi} L_A$ (II). At $\varkappa \gg |R|^{-1}$ the excitations of the type I correspond to the excitations near the antiinstanton and know nothing about the instanton, and excitations of the type II are the near instanton excitations.

Let's remind that if $L_A \gg 1$, $L_B \gg 1$ all the eq. (3.8) solutions are double degenerate. The function $\varphi(\varkappa)$ consists not of two, but four in pair coinciding curves.

Besides the continuous spectrum functions, as we saw in section 3, the eq. (3.8) has two discrete spectrum solutions. Let's consider these solutions as a part of curve I of function $\varphi(\varkappa)$.

It's easy to find $\varkappa$ which makes the (3.16) to satisfy the boundary conditions:

$$\varkappa_N = \frac{1}{D}\left\{\left(N + \frac{5}{4}\right)\pi - \varphi(\varkappa)\right\}. \tag{C.2}$$

D is the volume radius.

Let's discuss the contribution to the functional integral from the type I excitations. Dividing this contribution by that to the integral near vacuum ($\varphi = 0$) one gets:

$$\frac{\pi}{D}\frac{\prod_{N=0}^{M}\left(N+\frac{5}{4}\right)}{\prod_{N=1}^{M}\left(N+\frac{5}{4}-\frac{\varphi}{\pi}\right)} = \varkappa_{\max}\exp\left[\frac{1}{\pi}\int_0^{\varkappa_{\max}}(\varphi-\pi)\frac{d\varkappa}{\varkappa}\right]. \tag{C.3}$$

The number of multipliers in (C.3) numerator is one more than in denominator because in the case of pseudoparticle pair one variable in the functional integral corresponds to the discrete spectrum level and we introduce the collective coordinate to integrate over it.

We know that at $\varkappa \gg |B|^{-1}$ the phase shift goes to zero. So in (C.3) the dependence on ultraviolet cut-off cancels.

At $|R|^{-1} \ll \varkappa \ll |B|^{-1}$ the phase shift in curve we discuss is:

$$\varphi = \pi\left(1 - \frac{1}{2L_B}\right) = \pi\left(1 + \frac{1}{2\ln(\varkappa|B|)}\right). \tag{C.4}$$

At $\varkappa \ll |R|^{-1}$ the φ goes to π quite quickly. Then contribution to the Gauss integral of the Fig. 1 curve I excitations is

$$\frac{\text{const}}{\sqrt{\ln|R/B|}}. \tag{C.5}$$

Here the unknown constant depends only on the individual antiinstanton properties and doesn't influence the pseudoparticle interaction.

We have yet said in section 3 that to calculate the functional integral over the excitations of Fig. 1 curve II the Gauss approxi-

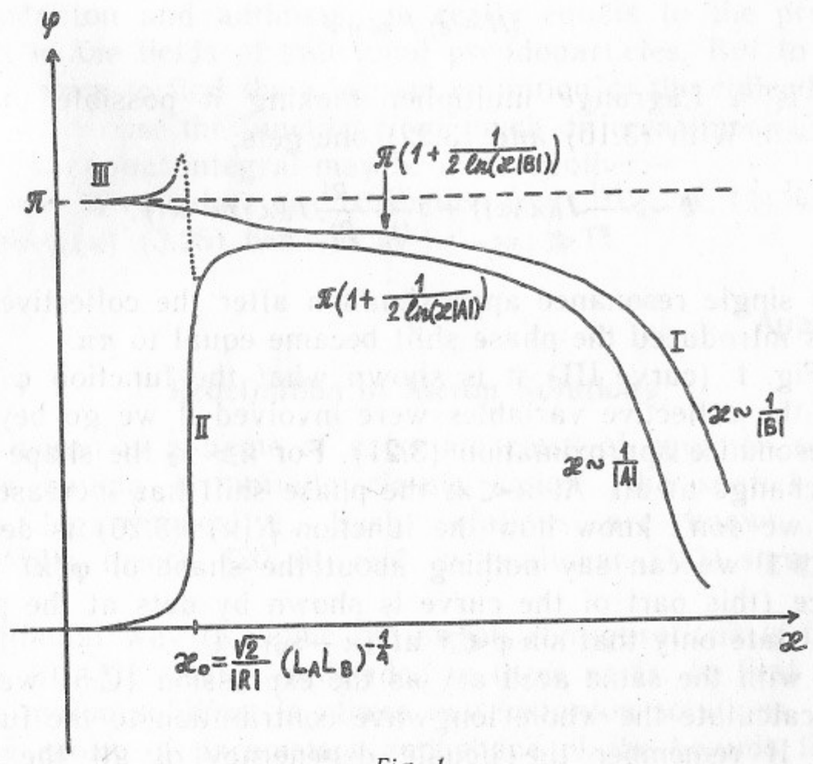

Fig. 1.

mation is inapplicable. Formally calculated Gauss integral (divided on corresponding vacuum integral) in this case is

$$\frac{\text{const}}{\varkappa_0\sqrt{\ln|R/A|}}. \tag{C.6}$$

Because of a resonance behaviour of the function $\varphi(\varkappa)$ there appears a big multiplier $\varkappa_0^{-1} \sim |R|$ in the integral. But we know that in massless theories the Gauss approximation may be applied only if the long-wave contribution cancels in all physical quantities.

To introduce the collective coordinate we multiply the functional

integral (2.1) by the Faddeev—Popov unit containing the delta function $\delta(\langle\omega-\omega_0|\Psi\rangle)$ where Ψ is defined by (3.24), (3.26)

$$\Psi=\int\frac{\Gamma}{\sqrt{(\varkappa_0-\varkappa)^2+\Gamma^2}}\,F_\varkappa\,d\varkappa. \tag{C.7}$$

If rewrite (3.8) shortly in the form $HF=\varepsilon F$, then to calculate the Gauss integral we are to solve

$$(H-\varepsilon)\,F=\eta\Psi. \tag{C.8}$$

Here η is a Lagrange multiplier making it possible to satisfy $\langle F|\Psi\rangle=0$. With (3.16) and (3.22) one gets:

$$F\sim p\frac{|z|}{z}J_1(\varkappa|z|)+q\frac{|z-R|}{(z-R)^*}J_1(\varkappa|z-R|). \tag{C.9}$$

I. e in a single resonance approximation after the collective coordinate was introduced the phase shift became equal to $n\pi$.

On Fig. 1 (curve III) it is shown what the function $\varphi(\varkappa)$ may be after the collective variables were involved if we go beyond the single resonance approximation (3.21). For $\varkappa>\varkappa_0$ the shape of $\varphi(\varkappa)$ doesn't change at all. At $\varkappa<\varkappa_0$ the phase shift has increased on π. Because we don't know how the function $f_j(\varkappa)$ (3.26) is defined at $|\varkappa-\varkappa_0|\gg\Gamma$ we can say nothing about the shape of $\varphi(\varkappa)$ close to resonance (this part of the curve is shown by dots at the picture). One can state only that $\sin\varphi\ll 1$ at $|\varkappa-\varkappa_0|\sim\Gamma$.

Now with the same accuracy as the expression (C.5) was found we can calculate the whole longwave contribution to the functional integral. If remember the double degeneracy of all the $\delta^2S/\delta\varphi^2$ eigenvalues one gets:

$$\frac{\text{const}}{\ln\left|\frac{R}{A}\right|\ln\left|\frac{R}{B}\right|}. \tag{C.10}$$

The unknown constant determines the single pseudoparticles properties only and doesn't influence their interaction.

We have said nothing about the value of the multiplier appearing in the functional integral (2.7)

$$\prod\delta(\langle\omega-\omega_0|\Psi_i\rangle)\prod\delta(\langle\omega-\omega_0|F_j\rangle)\det\left|\left|\frac{\partial\langle\omega-\omega_0|\Psi_l,F_l\rangle}{\partial r_k}\right|\right|\prod dr_s. \tag{C.11}$$

It's easy to show that if neglect all the corrections $\sim L_A^{-1}$, L_B^{-1} the (C.11) gives:

$$\text{const}\ln\left|\frac{R}{A}\right|\ln\left|\frac{R}{B}\right|\prod dr_s. \tag{C.12}$$

Here r_s is the set of collective variables describing positions, orientations and sizes of pseudoparticles.

So, we have shown, that Gauss integral in the field of far separated instanton and antiinstanton really equals to the product of integrals in the fields of individual pseudoparticles. But to get this result we were to find the adequate definition of the collective coordinates. If one use the function $f_j(\varkappa)$ much different from (3.26) the value of functional integral may be quite another.

In order to calculate any correction to (C.10), (C.12) we are to know how $f_j(\varkappa)$ (3.26) behaves at $|\varkappa-\varkappa_0|\gg\Gamma$.

Appendix D

Redefinition of Meron Solutions

The meron is a name of a certain class of equation of motion solutions having an infinite classic action (the spatial integral diverges logarithmically). Such solutions are known both in Yang—Mills theory [20, 6] and in nonlinear O(3)-sigma model [8].

Traditional way to define the meron configurations is the following [6, 8]. All volume is devided in three parts. In first part one uses the meron solution, in others two instanton solutions. With the adequate choice of parameters and shape of the bounds this solutions may be maid equal at the bound (with break of derivatives). As we should see, at least in the nonlinear O(3)-sigma model such a definition doesn't allow to determine even the number of meron collective coordinates.

Let's discuss in details the nonlinear O(3)-sigma model merons. Every function like [8]:

$$\omega=e^{i\tau}\frac{u(z)}{u^*} \tag{D.1}$$

is, in particular, the solution of classic equation of motion (eq. (3.7) with zero r.h.s.). Here u—an arbitrary analytic function,

τ — a real parameter. The simplest localized solution of this type is

$$\omega=\sqrt{\frac{(z-A)}{(z-A)^*}\frac{(z-B)^*}{(z-B)}}\,. \tag{D.2}$$

It's easy to understand the nature of the solutions like (D.1). Substituting (D.1) in (3.4) one sees that nonlinear O(3)-sigma model meron is a nonlinear O(2)-sigma model whirl immersed into O(3).

It occures that the eq. (3.8) solutions describing the operator $\delta^2 S/\delta\varphi^2$ eigenvectors in meron (D.1) field with $\varepsilon=0$ may be found in explicit form. In particular, for (D.2) meron the eq. (3.8) with $\varepsilon=0$ has a solution:

$$F\sim\sqrt{\frac{(z-A)}{(z-A)^*}\frac{(z-B)^*}{(z-B)}}\,\sin\left[\sqrt{2}\ln\left|\frac{z-A}{z-B}\right|+\varphi\right] \tag{D.3}$$

i. e. while approaching the points A and B function F changes the sign an infinite number of times. In quantum mechanics function like (D.3) appears if the particle falls on the point [21]. The appearance of every new zero means the birth of a new bound state.

To define the finite action meron configuration it seems very natural to use (2.3)

$$\frac{\delta S}{\delta\varphi}=\sum_1^N \lambda_i F_i \tag{D.4}$$

instead of $\delta S/\delta\varphi=0$. Here F_i — a set of bound state eigenvectors of $\delta^2 S/\delta\varphi^2$. Such a definition means that we search for configuration minimizing the action in all the directions except for that which the Gauss integral diverge along.

Let's discuss ω like (D.2). If keep in the eq. (D.4) r.h.s. only functions F localized at $|z-A|\sim\rho_A$ and $|z-B|\sim\rho_B$ we should find the solution quite smooth near $z=A$ and $z=B$, and equal to (D.2) at $|z-A|\gg\rho_A$ and $|z-B|\gg\rho_B$. The explicit form of the (D.4) solution may be found only numerically. Nevertheless if ρ_A, ρ_B are small enough the classic action of this configuration may be found:

$$S=2\pi\ln\left(\frac{|B-A|^2}{\rho_A\rho_B}\right)+O(1)\,. \tag{D.5}$$

We can also find the number of collective variables necessary for meron description

$$N\sim\ln\left(\frac{|B-A|^2}{\rho_A\rho_B}\right). \tag{D.6}$$

Generally accepted [6, 8] is the point of view, that meron solutions begin to make a significant contribution to the physic values only at such a big coupling constant that logarithmic growth of the action (D.5) is canceled by further growth of the coupling constant g.

We get accustomed to that any collective variable leads to appearance of a big multiplier in the functional integral (the whole volume, the distance between pseudoparticles etc.). Now, of course, we can say nothing about what the integration over meron collective variables will lead to. Nevertheless the fact that with growth of the action (D.5) the number of collective variables (D.6) grows simultaneously gives us a hope that meron configurations should play an important role at small coupling constants also.

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