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DYNAMICS AND STATISTICS OF UNSTABLE QUANTUM STATES

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Dynamics and Statistics of Unstable Quantum States

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ABSTRACT

Statistical theory of spectra formulated in terms of random matrices is extended to unstable states. The energies and widths of these states are treated as real and imaginary parts of complex eigenvalues for an effective non-hermitian Hamiltonian. Eigenvalue statistics are investigated under simple assumptions. If the coupling through common decay channels is weak we obtain Wigner distribution for level spacings and Porter—Thomas one for widths with the only exception of spacings less than widths where level repulsion fades out. Meanwhile in the complex energy plane the repulsion of eigenvalues is quadratic in accordance with T-noninvariant character of decaying systems. In the opposite case of strong coupling with continuum k short-lived states are formed (k is the number of open decay channels). These states accumulate almost the whole total width the rest of states becoming long-lived. Such a perestroika corresponds to separation of direct processes (a nuclear analogue of Dicke coherent superradiance). At small channel number, Ericson fluctuations of cross sections turn out to be suppressed.

The one-channel case is considered in detail. The joint distribution of energies and widths is obtained. Average cross sections and density of unstable states are calculated.

1. INTRODUCTION

Statistical methods are now getting deeply insight into physics of real quantum systems. It is caused by structure uncertaincies of systems under consideration (for instance, disordered solids or atoms in random external fields) as well as by complexity of wave functions. The latter can take place not only in many-body systems but, at some conditions, even for small number of degrees of freedom. Statistical properties of a quantum system turned out to be profoundly interrelated with dynamical stability of phase space trajectories of a corresponding classical system.

One can schematically subdivide existing statistical approaches [1] into global and local ones. Theories of the first kind deal with wide energy regions. The main statistical quantity here is the level density $\rho_I(E)$ for given energy E and other exact integrals of motion I. It results in the coarse grained description using usually thermodynamical concepts. Actually, one should take an ensemble average over typical eigenstates of the given Hamiltonian. The Bohr idea of compound nucleus [2] is an excellent example.

In the local-type approach [3] a sequence of closely spaced levels is considered and the main subjects of interest are correlations and fluctuations of quantities relating to specific states. As for regular dependence of state properties on excitation energy, it is of minor importance here. This field can be naturally referred to as statistical spectroscopy [4]. Its extensive development was initiated by pioneering works of Wigner [5] and Dyson [6] although empiri-

cal studies of local regularities have been started long before [7]. Usually one considers statistics of energy levels and matrix elements of simple operators [8]. The progress in this area has provided us with understanding of difference between regular and stochastic quantum systems as connected with quasi-periodic or ergodic behaviour of corresponding classical systems [9—11].

The random matrix theory [3, 6, 12] proved to be an adequate tool for investigating local statistics of quantum states. This way implies higher degree of abstractness in comparison with Gibbs thermodynamics and allows to reveal the most profound properties determined by general features of dynamics (hermiticity, unitarity, T-invariance) rather than by specific details. Stochasticity of motion gives rise to the hypothesis of equivalence of basises in the space of states. It means that the distribution function of matrix elements of a Hamiltonian is invariant with respect to orthogonal transformations. Thus, one obtains the Gaussian orthogonal ensemble (GOE) of random real symmetrical matrices which gives a good description of stochastic motion in actual systems (for example, isolated neutron resonances in nuclei) as well as in model calculations. Lifting the T-invariance restriction one comes to the Gaussian unitary ensemble (GUE) of complex hermitian matrices. As it well known GOE and GUE predict respectively linear and quadratic level repulsion at small spacings. The repulsion reflects possibility of level crossing only on a manifold of zero measure in the space of random matrix elements.

Actually, excited states of physical systems have a finite lifetime. An unstable state is created by an external field and decays afterwards via open channels. Strictly speaking, above-mentioned standard Gaussian ensembles are applicable to discrete stationary levels only. The influence of coupling with continuum on the properties of internal states has not been sufficiently understood up to now. In fact, such effects have being considered only in some numerical calculations (see e.g. [13]) of nuclear reactions. On the physical grounds, one should expect that even in the case of separated resonances ($\Gamma/D \ll 1$) the level statistics will be modified due to instability. In particular, one can guess that the level repulsion at small distances $\lesssim \Gamma$ should be washed out. The remote classical analog for this phenomenon is the possibility of intersection of «corridors» originating from noncrossing phase space trajectories of conservative motion as a result of dissipation. In the opposite case of overlapped resonances ($\Gamma/D \gtrsim 1$) the whole pattern is set by the openness

of a system, namely by external mixing via continuum [14]. This situation is new for statistical spectroscopy.

Since unstable states emerge as intermediate ones in various reactions, the complete statistical description calls for extra hypotheses concerning reaction amplitudes. Assuming that the decay of the internal state n into the channel a performs an analysis of a complicated wave function and extracts the modulus squared $|A_n^a|^2$ of its component matched to this channel, one can as well obtain the distribution of decay probabilities within the scope of the Wigner—Dyson theory. In such a way, the Porter—Thomas distribution [15] for neutron widths of isolated resonances arises in good agreement with experiment [1]. In contrast to this approach, we use below independent statistical assumptions on internal states and decay amplitudes. For separated resonances nevertheless, we come to the same Porter—Thomas width distribution.

For years statistical reaction theory and statistical spectroscopy were being advanced practically independently. Only recently the progress was achieved in the direction of combined description of reaction amplitudes and cross sections together with levels of an open quantum system [16—18]. Having in mind primarily investigation of unstable intermediate states we will pursue essentially the same aim of combined description using methods generalizing those of traditional statistical spectroscopy. Such an approach leads to the problem of spectral properties of nonhermitian random matrices. Ensembles of those matrices provide generalization of Gaussian ensembles of hermitian Hamiltonians to unstable systems.

We proceed from the general theory of resonance nuclear reactions [19]. In good approximation reaction amplitudes can be represented as sums of pole terms in the complex energy plane. Such poles $\mathscr{E}_n = E_n - \frac{i}{2}\Gamma_n$ being the eigenvalues of the nonhermitian effective Hamiltonian $\mathscr{H}$ correspond to unstable intermediate states with energies E_n and widths Γ_n. We adopt simple statistical hypotheses concerning matrix elements of hermitian and anti-hermitian parts of $\mathscr{H}$. If coupling of internal states with continuum is weak (small external mixing) and resonances do not overlap, $\Gamma/D \ll 1$, our assumptions give the same results as the Wigner—Dyson theory. The only significant difference is disappearance of level repulsion at small distances $|E_m - E_n| < \max\{\Gamma_m, \Gamma_n\}$.

In the opposite limiting case of strong external mixing, the results are governed by the algebraic structure of the anti-hermitian

part of $\mathscr{H}$. This structure follows, in fact, from the unitarity condition. Therefore its consequences are of a general nature. Due to this structure, sharp redistribution («collectivization») of widths occurs at $\Gamma/D \gtrsim 1$. As a result, k rapidly decaying states are formed (k is a number of open channels) reflecting segregation of fast «direct» processes. The rest of intermediate states are long-lived and have small excitation cross sections. This discrimination of widths is consistent with the picture [20] of the two-step course of nuclear reactions in the region of overlapping resonances. The similar phenomenon—formation of a short-lived coherently decaying state in a system with almost degenerate levels—is known in quantum optics as Dicke superradiance. Such a state of N identical two-level atoms coupled via the common radiation field has a width $\Gamma = N\gamma$ where γ stands for the width of the excited level of a single atom.

In section 2 we discuss the phenomenological nonhermitian effective Hamiltonian $\mathscr{H}$ and its connection to the scattering matrix $\hat{S}$ arising from the general reaction theory. In particular, it is shown that the number of nonzero eigenvalues of the anti-hermitian part of $\mathscr{H}$ can not exceed the number k of open channels. Two different forms of the secular equation for complex energies $\mathscr{E}_n$ are considered in section 3. In the case of strong external mixing, the width accumulation at a single state is demonstrated for one-channel scattering. We introduce the phenomenological R-matrix and compare different representations of the scattering matrix. Sections 4 and 5 are devoted to discussion of our statistical hypotheses concerning the distributions of matrix elements of hermitian and anti-hermitian parts of $\mathscr{H}$. The distributions of corresponding eigenvalues are analyzed. In sections 4 and 6 the ensemble averages of R- and S-matrices are found as well as the averaged cross sections of scattering and absorption processes. The joint distribution function for energies E_n and widths Γ_n is obtained in section 7 for the one-channel case. In particular, quadratic repulsion of complex energies is revealed. Due to the unitarity condition, residues in the poles of an amplitude for the one-channel scattering can be expressed in terms of complex energies $\mathscr{E}_n$ only. Therefore the statistics of residues follow from the level statistics in this case. Finally, the mean density of unstable states in the complex energy plane is found in section 8. Both cases of weak and strong external mixing are explicitly examined.

2. EFFECTIVE HAMILTONIAN AND SCATTERING MATRIX

Suppose a complex quantum system (for instance, a highly excited atomic nucleus) near an excitation energy E is considered. Let the given energy region contain N resonance states studied with reactions $a \to b$ ($a, b = 1, 2, \ldots, k$ are open channels). In accordance with [19], the scattering matrix

$$S^{ba}(E) = \delta^{ba} - iT^{ba}(E) \tag{2.1}$$

can be expressed in terms of amplitudes A_n^a connecting an internal basis state n ($n = 1, 2, \ldots, N$) with a channel state a as follows:

$$T^{ba}(E) = \sum_{m,n=1}^{N} A_m^{b*} \left(\frac{1}{E - \mathscr{H}} \right)_{mn} A_n^a . \tag{2.2}$$

Here the $N \times N$ matrix in the internal space

$$\mathscr{H} = H - \frac{i}{2} W \tag{2.3}$$

plays the role of the effective Hamiltonian; matrices H and W both are hermitian. For simplicity, we do not take into account pure potential scattering. The corresponding contribution can be introduced without difficulties.

The anti-hermitian part W of (2.3) has the form

$$W_{mn} = \sum_{a=1}^{k} A_m^a A_n^{a*} \tag{2.4}$$

where the sum goes over k open channels. Eq. (2.4) assures unitarity of S-matrix (2.1) [14]. Generally speaking, amplitudes A_n^a and matrix elements H_{mn} are energy dependent. However, this dependence is rather smooth. Since the level density is high in the energy region we are interested in, the regular energy variations of Hamiltonian matrix elements within this region are negligible (one should assume that reaction thresholds are sufficiently far from this interval). Hence, one can restrict oneself to considering only explicite pole energy dependence in (2.2). Just this allows to use the matrix $\mathscr{H}$ as an effective Hamiltonian for studying local dynamical and statistical properties of an unstable system.

Eq. (2.2) corresponds to the resonance diagram

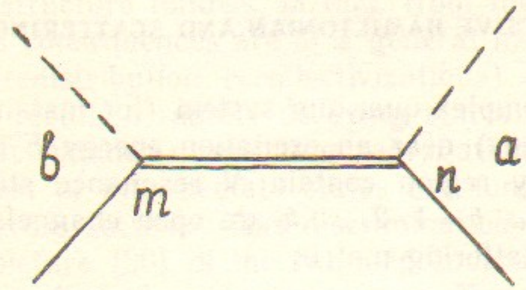


Fig. 1.

where the internal doubled line depicts the exact Green function G_{mn} of an intermediate system. Even in the basis of eigenstates of the hermitian part H, the matrix G is not diagonal because of coupling of these states through common decay channels. This coupling can be represented by the mass operator

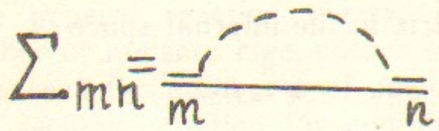


Fig. 2.

The δ-functional contributions to Σ corresponding to real decays with energy conservation give rise to the anti-hermitian part (2.4). On the other hand, virtual decays described by the principal value integrals renormalize the hermitian part of $\mathscr{H}$. The approximation dropping off-diagonal matrix elements of Σ is used sometimes in the nuclear reaction theory as well as in the theory of resonance light scattering by quantum systems. Meanwhile if the coupling via continuum is strong enough such an approximation violates unitarity.

For convenience, we introduce operator notations in the channel space marking them with caret. Using rectangular $N\times k$ matrices $\mathbf{A}=\{A_n^a\}$ we write down eqs. (2.2) and (2.4) in the compact form

$$\hat{T}(E)=\mathbf{A}^+\frac{1}{E-\mathscr{H}}\mathbf{A}, \qquad W=\mathbf{A}\mathbf{A}^+. \tag{2.5}$$

Separability of W causes the important consequence: for $k<N$ the matrix W cannot have more than k nonzero eigenvalues. Indeed, columns of the matrix $\mathbf{A}$ are N-dimensional vectors in the inner space. Provided that $k<N$ these vectors span the k-dimensional subspace of the state space. One can choose remaining $N-k$ basis

vectors to be orthogonal to the columns of the matrix $\mathbf{A}$. Then it is clear that the rank of the matrix W is equal to k. It can be shown readily that the nonzero eigenvalues of W coincide with those of the $k\times k$ matrix

$$\hat{X}=\mathbf{A}^+\mathbf{A} \tag{2.6}$$

in the channel space. Its elements $X^{ba}=\sum_n A_n^{b*}A_n^a$ are scalar products of k N-dimensional vectors $\mathbf{A}^a$.

For definiteness, we will consider below the T-invariant theory. As it is known, then the choice of channel states is feasible making the S-matrix symmetric. Without loss of generality, the inner basis also can be chosen for the T-invariant case to make the effective Hamiltonian $\mathscr{H}$ symmetric, $\mathscr{H}^T=\mathscr{H}$ (the superscript T marks transposed matrices). As a result, the amplitudes A_n^a become real, $\mathbf{A}^+=\mathbf{A}^T$.

N eigenvectors $\psi^{(n)}$ $(n=1, 2, \ldots, N)$ of the nonhermitian Hamiltonian $\mathscr{H}$ correspond to exponentially decaying states and the complex eigenvalues

$$\mathscr{E}_n=E_n-\frac{i}{2}\Gamma_n \tag{2.7}$$

give their energies E_n and widths Γ_n. With notation Ψ for the $N\times N$ matrix composed of columns $\psi^{(n)}$, one can write the diagonalizing transformation of $\mathscr{H}$ as

$$\mathscr{H}=\Psi\mathscr{H}_d\Psi^{-1}, \tag{2.8}$$

where $(\mathscr{H}_d)_{mn}=\delta_{mn}\mathscr{E}_n$. Symmetry of $\mathscr{H}$ leads to

$$\Psi^T\Psi=1=\Psi\Psi^T. \tag{2.9}$$

Here the first equality expresses the orthonormalization condition whereas the second one is the completeness condition. At the same time $\Psi^+\Psi\neq 1$ since the Hamiltonian $\mathscr{H}$ is nonhermitian. Thus eigenstates $\psi^{(n)}$ are nonorthogonal because of existence of common decay channels. Multiplying the equation $\mathscr{H}\Psi=\Psi\mathscr{H}_d$ of the eigenvalue problem by Ψ^+ from the left and the conjugate equation by Ψ from the right we get

$$\Psi^+\Psi\mathscr{H}_d-\mathscr{H}_d^+\Psi^+\Psi=\Psi^+(\mathscr{H}-\mathscr{H}^+)\Psi=-i\Psi^+W\Psi=-i\,\tilde{\mathbf{A}}^*\tilde{\mathbf{A}}^T. \tag{2.10}$$

Unlike to $\mathbf{A}$, the tilded amplitudes

$$\tilde{\mathbf{A}}=\Psi^T\mathbf{A}, \quad \tilde{A}^a_m=\sum_n A^a_n \psi^{(m)}_n \tag{2.11}$$

are complex, $\tilde{A}^a_m$ being the decay amplitude of an unstable eigenstate m into a channel a. Eq. (2.10) is nothing but the matrix form of the Bell—Steinberger relation [22]. In components, it reads

$$\langle \psi^{(m)}|\psi^{(n)}\rangle = i\Big(\sum_a \tilde{A}^{a*}_m \tilde{A}^a_n\Big)(\mathscr{E}^*_m-\mathscr{E}_n)^{-1}. \tag{2.12}$$

After diagonalization of $\mathscr{H}$, the scattering matrix (2.2) takes a form of a sum of resonance terms

$$T^{ba}(E)=\Big(\tilde{\mathbf{A}}^T\frac{1}{E-\mathscr{H}_d}\tilde{\mathbf{A}}\Big)^{ba}=\sum_n \frac{\tilde{A}^b_n\tilde{A}^a_n}{E-\mathscr{E}_n}. \tag{2.13}$$

Thus, in adopted approximation, reaction amplitudes are meromorphic functions in the complex energy plane having poles in the points of eigenvalues of $\mathscr{H}$. Residues in the poles are complex. It is essential for interference of close resonances [23, 24].

3. SECULAR EQUATION AND R-MATRIX. «COLLECTIVIZATION» OF WIDTHS

Complex energies (2.7) of intermediate states are roots of the secular equation

$$\mathrm{Det}\,(\mathscr{E}-\mathscr{H})=0. \tag{3.1}$$

The structure (2.4) of the operator W guarantees nonnegativity of widths Γ_n.

Let us introduce the resolvents $G^0(\mathscr{E})$ and $G(\mathscr{E})$ of the matrices H and $\mathscr{H}$ respectively,

$$G^0(\mathscr{E})=(\mathscr{E}-H)^{-1}, \quad G(\mathscr{E})=(\mathscr{E}-\mathscr{H})^{-1}, \quad \mathscr{E}=E-\frac{i}{2}\Gamma, \tag{3.2}$$

In what follows we will need also the $k\times k$ matrix in the channel space

$$\hat{R}(\mathscr{E})=\mathbf{A}^TG^0(\mathscr{E})\,\mathbf{A}, \tag{3.3}$$

similar to the Wigner R-matrix of the nuclear reaction theory. Matrix elements of $\hat{R}(\mathscr{E})$ have poles in the real points $\mathscr{E}=\varepsilon_n$ of eigenvalues of the hermitian part H. Eq. (3.1) can be transformed as follows

$$\mathrm{Det}\,(\mathscr{E}-\mathscr{H})=\mathrm{Det}\,(\mathscr{E}-H)\ \mathrm{Det}\Big(1+\frac{i}{2}G^0(\mathscr{E})W\Big)=$$

$$=\mathrm{Det}\,(\mathscr{E}-H)\exp\Big\{\mathrm{Tr}\log\Big(1+\frac{i}{2}G^0(\mathscr{E})\,\mathbf{A}\mathbf{A}^T\Big)\Big\}. \tag{3.4}$$

It is easy to verify that one can permute cyclically the matrix $\mathbf{A}^T$ under log. As a result one reduces the secular equation (3.1) to

$$\det\Big[1+\frac{i}{2}\hat{R}(\mathscr{E})\Big]=0 \tag{3.5}$$

where det stands for the determinant in the k-dimensional channel space. In the practically interesting case $k<N$, the new equation (3.5) is simpler than the original one (3.1).

Using separability of W we get the Dyson-type relation connecting resolvents (3.2) for stable and unstable systems

$$G(\mathscr{E})=G^0(\mathscr{E})-\frac{i}{2}G^0(\mathscr{E})\,\mathbf{A}\Big[1+\frac{i}{2}\hat{R}(\mathscr{E})\Big]^{-1}\mathbf{A}^TG^0(\mathscr{E}). \tag{3.6}$$

Insertion of eq. (3.6) into eqs. (2.5) and (2.1) gives

$$\hat{T}(E)=\mathbf{A}^TG(E)\,\mathbf{A}=\frac{\hat{R}(E)}{1+\frac{i}{2}\hat{R}(E)}, \tag{3.7}$$

$$\hat{S}(E)=\frac{1-\frac{i}{2}\hat{R}(E)}{1+\frac{i}{2}\hat{R}(E)}. \tag{3.8}$$

In addition, the useful formula for the trace of the Green function

$$\mathrm{Tr}\,G(\mathscr{E})=\mathrm{Tr}\,G^0(\mathscr{E})+\frac{i}{2}K(\mathscr{E}), \tag{3.9}$$

follows from (3.6) where the function $K(\mathscr{E})$ is given by the trace (tr) in the channel space

$$K(\mathscr{E})=\mathrm{tr}\left\{\frac{d\hat{R}(\mathscr{E})}{d\mathscr{E}}\,\frac{1}{1+\frac{i}{2}\hat{R}(\mathscr{E})}\right\}. \tag{3.10}$$

As it should be, complex poles in eqs. (3.6) — (3.10) coincide with roots of eq. (3.5). The contributions of real poles $\mathscr{E}=\varepsilon_n$ from two terms of eq. (3.9) cancel out.

In the simplest one-channel case the R-matrix (3.3) turns out to be just a meromorphic function with positive residues (Wigner R-function) and the secular equation (3.5) takes the form

$$1+\frac{i}{2}R(\mathscr{E})=1+\frac{i}{2}\,\mathrm{Tr}\,[G^0(\mathscr{E})\,W]=0. \tag{3.11}$$

If $\mathscr{E}_n=E_n-\frac{i}{2}\Gamma_n$ is a root of eq. (3.11) then $1-\frac{i}{2}R(\mathscr{E}_n^*)=0$ i. e. the one-channel S-matrix (3.8) vanishes at $\mathscr{E}=\mathscr{E}_n^*$. Taking into account the asymptotic condition $\lim\limits_{E\to\infty}S(E)=1$ we obtain the factorized representation

$$S(E)=\prod_n\frac{E-\mathscr{E}_n^*}{E-\mathscr{E}_n}=\prod_n\left(1-i\frac{\Gamma_n}{E-E_n+\frac{i}{2}\Gamma_n}\right), \tag{3.12}$$

used repeatedly in literature (see for example [26, 27]). This representation is equivalent to the sum (2.13) over poles provided that complex parameters $\mathscr{E}_n$ are roots of the secular equation (3.11). However, the expression (3.12) remains unitary for arbitrary $\mathscr{E}_n$. If one takes, as in [26, 27], energies E_n and widths Γ_n as unspecified parameters then the two representations of the one-channel S-matrix cease to be equivalent. The noted fact often slips away. Meanwhile the parametrization (3.12) with arbitrary $\mathscr{E}_n$ may become incompatible with the algebraic structure of W stressed in section 2.

Such a nonequivalence is especially important in the case of overlapped resonances when the anti-hermitian part W dominates. In the basis of eigenvectors of H one has

$$R(\mathscr{E})=\sum_n\frac{\gamma_n}{\mathscr{E}-\varepsilon_n},\qquad \gamma_n\equiv A_n^2. \tag{3.13}$$

Accordingly, the secular equation (3.11) can be written down as a pair of coupled real equations

$$\frac{1}{4}\Gamma\sum_n\frac{\gamma_n}{(E-\varepsilon_n)^2+\frac{1}{4}\Gamma^2}=1,\qquad \frac{1}{4}\Gamma\sum_n\frac{\gamma_n\varepsilon_n}{(E-\varepsilon_n)^2+\frac{1}{4}\Gamma^2}=E. \tag{3.14}$$

In the limiting case of the completely degenerate operator $H=\varepsilon\cdot 1$, one obtains immediately that there exists the unique eigenstate of the effective Hamiltonian $\mathscr{H}$ with the nonzero width

$$\Gamma=\sum_n\gamma_n=\mathrm{Tr}\,W\equiv w \tag{3.15}$$

accumulated all individual widths γ_n. The rest $N-1$ states remain stable. Similar situation ruling out the arbitrary choice of widths Γ_n always occurs if external mixing is strong. Indeed, it follows from (3.14) that if the summarized width w exceeds the energy range ΔE where levels ε_n are located, then the broad level with the width $\Gamma_1\sim w$ emerges whereas remaining widths prove to be small. In the limit $w/\Delta E\gg 1$ and $N\gg 1$, assuming the distribution of ε_n to be homogeneous and symmetric around $\varepsilon=0$, one can obtain easily for the broad level $E_1=\frac{1}{w}\sum_n\gamma_n\varepsilon_n\approx 0$, $\Gamma_1=w\left[1-O\left(\left(\frac{\Delta E}{w}\right)^2\right)\right]$. As for narrow levels ($n=2, 3,\ldots, N$), their energies E_n occur intermittently beween neighbouring ε_n and their widths $\Gamma_n\sim\left(\frac{\Delta E}{w}\right)^2\frac{w}{N}$ complement Γ_1 to the total value w.

For strong external mixing the more convenient form of the secular equation can be obtained with the help of the orthogonal transformation diagonalizing the anti-hermitian part W. Since in the one-channel case the matrix W has $N-1$ degenerate zero eigenvalues, one can diagonalize in addition the $(N-1)\times(N-1)$ submatrix of the hermitian part H with the extra orthogonal transformation of the $N-1$-dimensional subspace. As a result, eq. (3.11) takes the form

$$\mathscr{E}-h-\tilde{R}(\mathscr{E})+\frac{i}{2}w=0, \tag{3.16}$$

where notations

$$\tilde{R}(\mathscr{E})=\sum_{\nu=2}^{N}\frac{h_\nu^2}{\mathscr{E}-\varepsilon_\nu},\qquad h=H_{11},\qquad h_\nu=H_{1\nu}=H_{\nu 1} \tag{3.17}$$

are introduced ε_ν being eigenvalues of the $(N-1)$-dimensional submatrix of H. At $h_\nu=0$ only one level is unstable having a width w. Remaining levels ε_ν get the widths through admixture of the unstable one. This mixing is described by the function $\tilde{R}$ (3.17) analogous to the function R (3.13) describing mixing with continuum states.

In the basis of the preceeding paragraph, the scattering amplitude is

$$T(E)=\mathrm{Tr}[\,G(E)\,W]=\frac{w}{E-h-\tilde{R}(E)+\frac{i}{2}w}. \tag{3.18}$$

Since $\tilde{R}(E)\to\infty$ at $E\to\varepsilon_\nu$ the amplitude (3.18) and the cross section have $N-1$ zeroes coinciding with ε_ν. Peaks of the cross section are disposed at energies satisfying the equation

$$E-h-\tilde{R}(E)=0\,, \tag{3.19}$$

which is nothing but the secular equation for eigenvalues of the hermitian part H. The R-function (3.13) has roots of eq. (3.19) as its poles[1]. Therefore, independently of the power of external mixing, the one-channel cross section has (under neglecting the potential contribution) N peaks alternating with $N-1$ zeroes[2]. However, only in the case of weak coupling via continuum, the cross section peaks lie, accordingly to (3.14) and (2.13), in the points $E_n=\varepsilon_n$ of energies of unstable intermediate states and have near those points the Breit—Wigner shape with widths $\Gamma_n=\gamma_n$. In the opposite case of strong coupling, $E_\nu\approx\varepsilon_\nu$ ($\nu=2, 3, \ldots, N$; see eq. (7.36) below) so that now the cross section minima rather than its maxima fall at energies of unstable states. The contributions of the broad and narrow resonances happen to be opposite in phase and cancel out. Outside the interval ΔE occupied by the narrow resonances, the amplitude (3.18) has the Breit—Wigner shape with the width w.

One can conclude that the variables h, ε_ν, h_ν, and w are more

[1] We recall that the energy shifts occuring in the general theory of nuclear reactions are incorporated into H.

[2] Such a behaviour of the cross section is typical for the one-channel case only. If, apart from the elastic channel, some inelastic channels are opened the elastic cross section has no zeros. This is the case also for inelastic cross sections at $k\geqslant 3$ [28]. The cross section peaks do not coincide, in the general case, with R-matrix poles. These results follow from the many-channel analogue of eq. (3.18).

adequate for parametrizing energy dependence of the one-channel S-matrix than energies E_n and widths Γ_n of intermediate unstable states. The latter are suitable for analysis of time dependence of the process. These two ways of treatment are complementary. We would like to note though that the pole parametrization (2.13) is universal for any channel number whereas the simple properties of the expression (3.18) have no analogues in the many-channel case.

Thus, the strong coupling with continuum drives to sharp redistribution of widths creating a short-lived state. Similarly, if there exist k open channels with comparable decay probabilities, k broad states are formed absorbing practically the whole summarized width. This phenomenon observed in realistic calculations of specific nuclear reactions [13] implies predominance of direct processes over relaxation to a compound nucleus. Quite abrupt «phase transition» from the uniform pattern of separated resonances to development of broad coherently decaying states occurs at $N\gg 1$ near the value $w\sim\Delta E$ i. e. $\langle\gamma\rangle\equiv\frac{w}{N}\sim D=\frac{\Delta E}{N}$.

Using the term «direct process» we have in mind actually any fast process independently of its mechanism. The duration $\hbar/w$ of this process is essentially shorter not only than the recurrence time $\hbar/D$ but also than the time interval $\hbar/\Delta E$ of fragmentation of the entrance state into more complicated configurations. In standard notations of the nuclear reaction theory, it means that $\Gamma^\uparrow\sim w\gg\Gamma^\downarrow\sim\Delta E$. On the other hand, we have shown that the lifetime of long-lived states is $\tau\sim\frac{w}{\Delta E}\frac{\hbar}{D}\gg\frac{\hbar}{D}$. This time exceeds the delay time obtained in [29] by the factor $w/\Delta E$. The discrepance is due to the difference of input assumptions. We have started from the S-matrix representation (2.1), (2.2) based on the general reaction theory. In the one-channel case we have obtained from here eq. (3.18) as a consequence. Hence, the only parameter characterizing instability of intermediate states is $\mathrm{Tr}\,W=w$. Unlike to this, the explicitly unitary form (3.12) of the S-matrix with arbitrary widths was used in [29]. As it have been stressed earlier, the two representations are not equivalent. The large lifetime τ of compound states as compared with the recurrence time $\hbar/D$ implies that the system reaches thermal equilibrium.

Elastic scattering through the single-particle resonance coupled with the background of stable many-particle levels [30] can be con-

sidered as a particular example. The single-particle resonance is formed by the optical shell-model potential. The scattering amplitude found in Ref. [30] coincides with (3.18) w being the width of the single-particle resonance. Generally, short-lived states are generated by the anti-hermitian part W of the effective Hamiltonian. This part forms superpositions of internal states matched to channel wave functions.

As it has been mentioned in Introduction, width collectivization induced by strong coupling via common decay channels («self-organization» observed in numerical calculations [13]) can be considered as a nuclear analogue of Dicke coherent superradiance [21]. Similar effects were found in numerical simulation [31] of a dissipative spin system.

4. STATISTICS OF THE HERMITIAN PART OF THE HAMILTONIAN. AVERAGING OVER GOE

In preceding sections we have discussed dynamical properties of a phenomenological Hamiltonian $\mathscr{H}$ of an unstable system. Now we proceed treating $\mathscr{H}$ as a member of an ensemble of random matrices. We assume the simplest statistical postulates consistent with general physical principles.

We consider the hermitian H and anti-hermitian W parts of $\mathscr{H}$ to be statistically independent. We suppose that a real symmetric $N\times N$ matrix H belongs to the GOE, i. e. matrix elements H_{mn} are not correlated and their distribution function is invariant with respect to orthogonal transformations. Such assumptions are known [1] to imply the Gaussian distribution for matrix elements of H,

$$\mathscr{P}(H)=2^{\frac{N(N-1)}{4}}\left(\frac{N}{\pi a^2}\right)^{\frac{N(N+1)}{4}}\exp\left(-\frac{N}{a^2}\operatorname{Tr}H^2\right). \tag{4.1}$$

Here the centre of the energy scale is placed at the origin. Eq. (4.1) leads to the joint distribution function of N eigenvalues $\varepsilon_1, \ldots, \varepsilon_N$

$$\mathscr{P}(\varepsilon_1,\ldots,\varepsilon_N)=C_N\prod_{m<n}|\varepsilon_m-\varepsilon_n|\exp\left(-\frac{N}{a^2}\sum_n\varepsilon_n^2\right). \tag{4.2}$$

The normalization constant in (4.2) is equal [32] to

$$C_N=2^{\frac{N(N-1)}{4}}\left(\frac{N}{a^2}\right)^{\frac{N(N+1)}{4}}\frac{1}{N!}\left[\prod_{n=1}\Gamma\left(\frac{n}{2}\right)\right]^{-1}. \tag{4.3}$$

As in [18] we have introduced the scale factor N in the exponent of eq. (4.1). As a result, in the limit $N\to\infty$ the energy levels ε_n are located within the N-independent interval $(-a, a)$ the mean spacing being equal to $D=2a/N$.

The essential feature of the GOE is correlation of levels. The probability for two levels to coincide tends to zero in proportion to their spacing (so-called linear level repulsion). For $N=2$ the Wigner level spacing distribution ($s=|E_1-E_2|/D$)

$$\mathscr{P}_W(s)=\frac{\pi}{2}s\exp\left(-\frac{\pi}{4}s^2\right) \tag{4.4}$$

follows from (4.2). It is evident that level crossing is possible only if the matrix element H_{12} and the difference $H_{11}-H_{22}$ both are equal to zero, i. e. on a manifold of zero measure in the random parameter space. Eq. (4.4) is nothing but the Rayleigh distribution for lengths of random two-dimensional vectors having uncorrelated normally distributed components. This formula is sufficiently accurate for the spacing of adjacent levels at any N [1] when exact results could be achieved only numerically. If T-invariance is not supposed and $\operatorname{Re}H_{12}$ and $\operatorname{Im}H_{12}$ are independent, the level repulsion becomes quadratic [6],

$$\mathscr{P}(s)\sim s^2, \quad s\to 0. \tag{4.5}$$

In what was stated above we have had in mind level sets with fixed values of exact constants of motion. On the contrary, if many spectra corresponding to different values of those constants are superimposed [8], level correlations are washed away converting the level spacing distribution into the Poisson one

$$\mathscr{P}_P(s)=e^{-s} \tag{4.6}$$

so that level repulsion disappears. Finally, adding to a regular Hamiltonian a randomizing perturbation, which destructs certain integrals of motion, one gets the domain of spacings less than the typical magnitude of perturbation where levels with given values of survived integrals repulse each other [33].

Due to orthogonal invariance, wave functions $\varphi^{(n)}$ of stationary states may be represented in the GOE by N-component unit vectors $\{\varphi_m^{(n)}\}$ spreaded isotropically over the hypersphere $\sum_m (\varphi_m^{(n)})^2 = 1$. In the large N limit, it gives the Gaussian distribution of components $\varphi_m^{(n)}$ with zero mean values and variances $\langle (\varphi_m^{(n)})^2 \rangle = 1/N$. Postulating that amplitudes A_n^a have the same statistical properties as components $\varphi_m^{(n)}$ one easily obtains the χ_k-square distribution of decay widths where the number of degrees of freedom k is equal to the channel number (Porter-Thomas distribution for $k=1$). This procedure, however, has no serious justification so we prefer to derive the width distribution with the aid of independent statistical hypotheses concerning the anti-hermitian part W (see the next section).

Computation of mean values of various quantities over the GOE is simplified in the limit of large N. In particular, in order to average the Green function $G^0(\mathscr{E})$ (3.2), one can expand it into a formal series about H and average each item with the use of the pair contraction formula (see (4.1))

$$\langle H_{mm'} H_{nn'} \rangle = \frac{a^2}{4N} (\delta_{mn} \delta_{m'n'} + \delta_{mn'} \delta_{m'n}) . \tag{4.7}$$

At $N \to \infty$ nonintersecting contractions are only surviving [16] since there a trace in the internal space being proportional to N compensates the factor N^{-1} in (4.7). The summation of those terms gives[3)]

$$\langle G^0(\mathscr{E}) \rangle = \frac{1}{\mathscr{E} - \frac{a^2}{4N} \operatorname{Tr} \langle G^0(\mathscr{E}) \rangle} . \tag{4.8}$$

From eq. (4.8) we get [34]

$$g^0(\mathscr{E}) \equiv \frac{1}{N} \langle \operatorname{Tr} G^0(\mathscr{E}) \rangle = \frac{2}{a^2} \left[\mathscr{E} - \sqrt{\mathscr{E}^2 - a^2} \right] \tag{4.9}$$

where the sign is determined by the asymptotic condition $g^0(\mathscr{E}) = 1/\mathscr{E}$ at $\mathscr{E} \to \infty$).

Using expression (4.9) we obtain the mean level density for a stable system

[3)] This method is applicable also to Hamiltonians containing a regular part along with a random one.

$$\rho^0(E) = \langle \sum_n \delta(E - \varepsilon_n) \rangle = \frac{1}{\pi} \operatorname{Im} \langle \operatorname{Tr} G^0(E - i0) \rangle =$$

$$= N \frac{2}{\pi a^2} \sqrt{a^2 - E^2}\ \theta(a^2 - E^2) \tag{4.10}$$

(the semicircle Wigner rule [5]).

The analogous GOE-averaging the R-matrix (3.3) leads in the limit $N \to \infty$ to

$$\langle \hat{R}(\mathscr{E}) \rangle_{\text{GOE}} = \langle G^0(\mathscr{E}) \rangle \hat{X} \to g^0(\mathscr{E})\, \hat{X} \tag{4.11}$$

where the matrix $\hat{X}$ has been defined by eq. (2.6). In particular, in the one-channel case

$$\langle R(\mathscr{E}) \rangle_{\text{GOE}} = g^0(\mathscr{E})\, w . \tag{4.12}$$

In the similar manner averaging the function $\tilde{R}(\mathscr{E})$ (3.17), we find

$$\langle \tilde{R}(\mathscr{E}) \rangle = \sum_v \langle h_v^2 \rangle \langle \frac{1}{\mathscr{E} - \varepsilon_v} \rangle \to \frac{a^2}{4} g^0(\mathscr{E}) . \tag{4.13}$$

For GOE-averaging the S-matrix connected with the R-matrix by the nonlinear relation (3.8), we note that at $N \to \infty$ contractions (4.7) of random matrix elements belonging to different R-matrices reduce the number of independent traces leading to contributions of higher order in $1/N$. If the channel number $k \ll N$, then the smallness of $1/N$ can not be recompenced by a trace in the channel space arising when contracting two matrix elements of H from neighbouring R-matrices. Therefore, up to terms of order k/N,

$$\langle \hat{S}(E) \rangle_{\text{GOE}} = \frac{1 - \frac{i}{2} g^0(E) \hat{X}}{1 + \frac{i}{2} g^0(E) \hat{X}} . \tag{4.14}$$

As one can see the GOE-averaging has smoothed contributions of long-lived compound states so that the result depends only on the matrix $\hat{X}$ rather than on the original matrix W. In other words, the average S-matrix (4.14) describes fast «direct» processes.

5. STATISTICS OF THE ANTI-HERMITIAN PART OF THE HAMILTONIAN

In the T-invariant theory, the anti-hermitian part W of the effective Hamiltonian is defined by specifying Nk real amplitudes. Let us regard them as independent random Gaussian variables assuming that[4)]

$$\langle A_n^a \rangle = 0, \quad \langle A_n^a A_m^b \rangle = \frac{1}{N} \eta^a \delta^{ab} \delta_{mn}. \tag{5.1}$$

Like in (4.7), we have introduced here the scale factor $1/N$ so that the average trace

$$\langle w \rangle = \langle \mathrm{Tr}\, W \rangle = \sum_a \eta^a \tag{5.2}$$

is independent of N serving as an invariant characteristic parameter of decay channels under study. As eq. (4.1), the Gaussian distribution of amplitudes A_n^a follows practically uniquely from statistical independence and orthogonal invariance in the internal space.

For simplicity, we suppose open channels to be equiprobable ($\eta^a = \eta$, $\langle w \rangle = k\eta$). Then additional orthogonal invariance in the channel space arises resulting in the joint amplitude distribution function

$$\mathscr{P}(A_1^1, \ldots, A_N^k) = \left(\frac{N}{2\pi\eta}\right)^{Nk/2} \exp\left(-\frac{N}{2\eta} w\right) \tag{5.3}$$

depending on $w = \mathrm{Tr}\, W = \sum_{a,n} (A_n^a)^2$ only.

The distribution (5.3) allows us to derive statistical properties of matrix elements W_{mn} (2.4). Diagonal ones $W_{nn} = \sum_a (A_n^a)^2 \equiv \gamma_n$ are positive, statistically independent and distributed by the χ_k-square rule,

$$\mathscr{P}_k(\gamma_n) = \frac{1}{\Gamma\left(\frac{k}{2}\right)} \left(\frac{N}{2\eta}\right)^{k/2} \gamma_n^{k/2-1} \exp\left(-\frac{N}{2\eta}\gamma_n\right). \tag{5.4}$$

[4)] In the paper by Weidenmüller [18], it is supposed that the amplitude randomness takes place in the eigenbasis of the hermitian part H only. However, it is difficult to adjust this assumption to orthogonal invariance of the GOE of hermitian parts.

This derivation needs no identification of decay amplitudes with components of wave functions (section 4). With the channel number increasing, the expression (5.4) tends to the Gaussian distribution

$$\mathscr{P}_k(\gamma_n) = \frac{N}{2\eta\sqrt{\pi k}} \exp\left\{-\frac{N^2}{4k\eta^2}\left(\gamma_n - \frac{k}{N}\eta\right)^2\right\} \tag{5.5}$$

with the mean value and the variance

$$\langle \gamma_n \rangle = \frac{k}{N}\eta, \quad \langle \gamma_n^2 \rangle - \langle \gamma_n \rangle^2 = \frac{2k}{N^2}\eta^2 = \frac{2}{k}\langle \gamma_n \rangle^2 \tag{5.6}$$

respectively. These equations were used frequently [35] in order to extract the effective number of open channels from experimental data. Such a procedure can be meaningful only if widths are small and resonances are not overlapping. The matrix W is then a weak perturbation and the γ_n's indeed can be treated as resonance widths.

The trace w of the matrix W is a sum of independent random quantities γ_n and its distribution function

$$\mathscr{P}_k(w) = \frac{1}{\Gamma\left(\frac{1}{2}Nk\right)} \frac{N}{2\eta} \left(\frac{N}{2\eta} w\right)^{Nk/2-1} \exp\left(-\frac{N}{2\eta} w\right) \tag{5.7}$$

at $Nk \gg 1$ turns into Gaussian one

$$\mathscr{P}_k(w) = \frac{1}{2\eta}\sqrt{\frac{N}{\pi k}} \exp\left\{-\frac{N}{4k\eta^2}(w - k\eta)^2\right\} \tag{5.8}$$

with the mean value $\langle w \rangle = k\eta$ and the small variance

$$\langle w^2 \rangle - \langle w \rangle^2 = \frac{2k}{N}\eta^2 = \frac{2}{Nk}\langle w \rangle^2. \tag{5.9}$$

As for off-diagonal elements W_{mn}, $m \neq n$, their distribution function depends on $|W_{mn}|$ only and can be expressed in terms of the MacDonald function $K_\mu(z)$,

$$\mathscr{P}_k(W_{mn}) = \frac{1}{\Gamma\left(\frac{k}{2}\right)} \frac{N}{\sqrt{\pi}\eta} \left(\frac{N}{2\eta}|W_{mn}|\right)^{(k-1)/2} K_{(k-1)/2}\left(\frac{N}{\eta}|W_{mn}|\right), \tag{5.10}$$

$$\langle W_{mn} \rangle = 0, \quad \langle W_{mn}^2 \rangle = \frac{k}{N^2}\eta^2 = \frac{1}{N^2 k}\langle w \rangle^2. \tag{5.11}$$

The analogous distribution function of matrix elements for a transition between complicated states has been considered in [8].

Unlike original amplitudes A_n^a, elements of a certain row or column of W are not statistically independent. Variables independent of γ_n are the angles θ_{mn} between N k-dimensional vectors $\mathbf{A}_n$ with components A_n^a. Using such a representation we have

$$W_{mn}=(\mathbf{A}_m\cdot\mathbf{A}_n)=\sqrt{\gamma_m\gamma_n}\ \cos\theta_{mn}\,. \tag{5.12}$$

The distribution of any angle θ_{mn} is given by the formula $(k\geqslant 2)$

$$\mathscr{P}_k(\theta)=\frac{1}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k-1}{2}\right)}\,\sin^{k-2}\theta\,, \tag{5.13}$$

ensuing from the expression for a solid angle in a k-dimensional space. Since a rotation in the channel space does not change a scalar product (5.12) the total number of random parameters specifying the matrix W is equal to $Nk-\frac{1}{2}k(k-1)$ for $k\leqslant N$. For $k>N$ one should add the number $\frac{1}{2}(k-N)(k-N-1)$ of rotation generators for the complementary $(k-N)$-dimensional space getting as a result $\frac{1}{2}N(N+1)$ independently of k.

The distribution of k nontrivial eigenvalues of the matrix W at $k<N$ is of particular interest. As it has been mentioned in section 2 they coincide with k eigenvalues γ^a of the matrix $\hat{X}=\mathbf{A}^T\mathbf{A}$ (2.6) in the channel space.

The distribution sought for can be obtained by the stochastic equation method [36, 37]. A stochastic process is constructed to describe evolution of the amplitude matrix $\mathbf{A}$ in fictitious time τ,

$$d\mathbf{A}(\tau)=-\mathbf{A}(\tau)\,d\tau+d\mathbf{a}(\tau), \tag{5.14}$$

where the Gaussian random force is defined by

$$\overline{da_n^a}=0,\qquad \overline{da_n^a\,da_m^b}=\frac{2}{N}\,\eta^a\,\delta^{ab}\,\delta_{mn}\,d\tau\,. \tag{5.15}$$

Independently of initial conditions, the distribution function of amplitudes A_n^a tends to that of eq. (5.3) at $\tau\to\infty$. Evolution of eigenvalues $\gamma^a(\tau)$ is determined by (5.14) as well. Diagonalizing

the matrix $\hat{X}(\tau+d\tau)$ with the help of perturbation theory with respect to the parameter $(d\tau)^{1/2}$ one can get drift and diffusion coefficients and then write down the Focker—Planck equation for the distribution function of eigenvalues. Its steady solution is (recall that all $\gamma^a>0$)

$$\mathscr{P}(\gamma^1,\ldots,\gamma^k)=C_{kN}\Big(\prod_{a<b}|\gamma^a-\gamma^b|\Big)\Big(\prod_c\gamma^c\Big)^{\frac{N-k-1}{2}}\exp\Big(-\frac{N}{2\eta}\sum_a\gamma^a\Big) \tag{5.16}$$

where $\sum_a\gamma^a=\operatorname{tr}\hat{X}=\operatorname{Tr}W=w$. The normalization constant can be calculated by methods developed in [32] and proves to be equal

$$C_{kN}=\Big(\frac{N}{2\eta}\Big)^{Nk/2}\pi^{k/2}\Big[\Gamma(k+1)\prod_a\Gamma\Big(\frac{a}{2}\Big)\Gamma\Big(\frac{N-a-1}{2}\Big)\Big]^{-1} \tag{5.17}$$

It can be seen from (5.16) that $\langle\gamma^a\rangle=\eta$.

When applied to the hermitian part H the stochastic equation method leads naturally to eq. (4.2). The distinction between eqs. (4.2) and (5.16) is caused by the specific form (2.6) of the matrix $\hat{X}$: its matrix elements are finite sums of products of normally distributed random variables. In the limit $N\to\infty$ this difference fades and the expression (5.16) turns into

$$\mathscr{P}(\gamma^1,\ldots,\gamma^k)=\Big[2^{k/2}\,\Gamma(k+1)\prod_a\Gamma\Big(\frac{a}{2}\Big)\Big]^{-1}\Big(\frac{N}{\eta^2}\Big)^{\frac{k(k+1)}{4}}\times$$

$$\times\prod_{a<b}|\gamma^a-\gamma^b|\exp\Big\{-\frac{N}{2\eta^2}\sum_c(\gamma^c-\eta)^2\Big\} \tag{5.18}$$

similar to (4.2). Note that formulae (5.16) and (5.18) manifest linear repulsion of close eigenvalues γ^a.

6. AVERAGE S-MATRIX AND CROSS SECTIONS

Let us return to averaging over the ensemble of effective Hamiltonians started in section 4. In accordance with (5.1), the mean value $\langle X^{ab}\rangle=\eta^a\delta^{ab}$ is diagonal and independent of N. Therefore the average R-matrix (4.11) is diagonal as well,

$$\langle R^{ab}(\mathscr{E})\rangle=g^0(\mathscr{E})\,\eta^a\delta^{ab}\,. \tag{6.1}$$

When averaging the expression (4.14) one should not contract amplitudes belonging to different matrices $\hat{X}$ provided that the channel number $k \ll N$. Such a contraction would break a trace in the internal space contributing a term of relative order k/N. Hence, within this accuracy we obtain the entirely averaged S-matrix

$$\langle S^{ba}(E)\rangle = \frac{1-\frac{i}{2}\eta^a g^0(E)}{1+\frac{i}{2}\eta^a g^0(E)}\delta^{ab} \equiv \langle S^a(E)\rangle \delta^{ab}, \tag{6.2}$$

which is diagonal and coincides with the result of Ref. [18] achieved in a different way.

In a customary manner we adopt that any ensemble average is equivalent to a corresponding energy average. In the nuclear reaction theory, the energy-averaged S-matrix is identified with the S-matrix of the optical model [20]. Eq. (6.2) allows us to find the average total cross section $\sigma_t(E)$ as well as the shape elastic cross section $\sigma_{se}(E)$. The latter describes a fast elastic process avoiding formation of an equilibrium compound nucleus. At energies $|E|>a$ outside the domain of compound resonances the two cross sections coincide being equal to (in units of $\frac{\pi}{k^2}(2J+1)$ where J is the angular momentum and k is the relative initial wave number)

$$\sigma_t(E)=\sigma_{se}(E)=\frac{\eta^2}{\left[\frac{1}{2}(E+\sqrt{E^2-a^2})\right]^2+\frac{1}{4}\eta^2}=\frac{\varkappa^2}{\left[\frac{1}{2}\left(\frac{E}{a}+\sqrt{\frac{E^2}{a^2}-1}\right)\right]^2+\frac{1}{4}\varkappa^2}. \tag{6.3}$$

The parameter $\varkappa=\eta/a$ introduced in (6.3) is a measure of external mixing and resonance overlapping. In the far regions $|E|\gg a$, the function (6.3) looks like wings of a Breit—Wigner resonance with the broad width η (we omit the channel superscript for simplicity).

In the compound resonance region $|E|<a$, the projectile can be captured following by formation of an intermediate compound system. As a result, the total cross section

$$\sigma_t(E)=2(1-\mathrm{Re}\langle S(E)\rangle)=\frac{\varkappa^2+\varkappa\sqrt{1-\frac{E^2}{a^2}}}{\frac{1}{2}\varkappa\sqrt{1-\frac{E^2}{a^2}}+\frac{1}{4}(\varkappa^2+1)} \tag{6.4}$$

and the shape elastic one

$$\sigma_{se}(E)=|\langle S(E)\rangle-1|^2=\frac{\varkappa^2}{\frac{1}{2}\varkappa\sqrt{1-\frac{E^2}{a^2}}+\frac{1}{4}(\varkappa^2+1)} \tag{6.5}$$

do not coincide, their difference

$$\sigma_{abs}(E)=1-|\langle S(E)\rangle|^2=\frac{\varkappa\sqrt{1-\frac{E^2}{a^2}}}{\frac{1}{2}\varkappa\sqrt{1-\frac{E^2}{a^2}}+\frac{1}{4}(\varkappa^2+1)} \tag{6.6}$$

being the cross section of absorption due to compound nucleus formation. In accordance with the qualitative analysis of Ref. [20], the absorption cross section (6.6) decreases when the degree of resonance overlap increases.

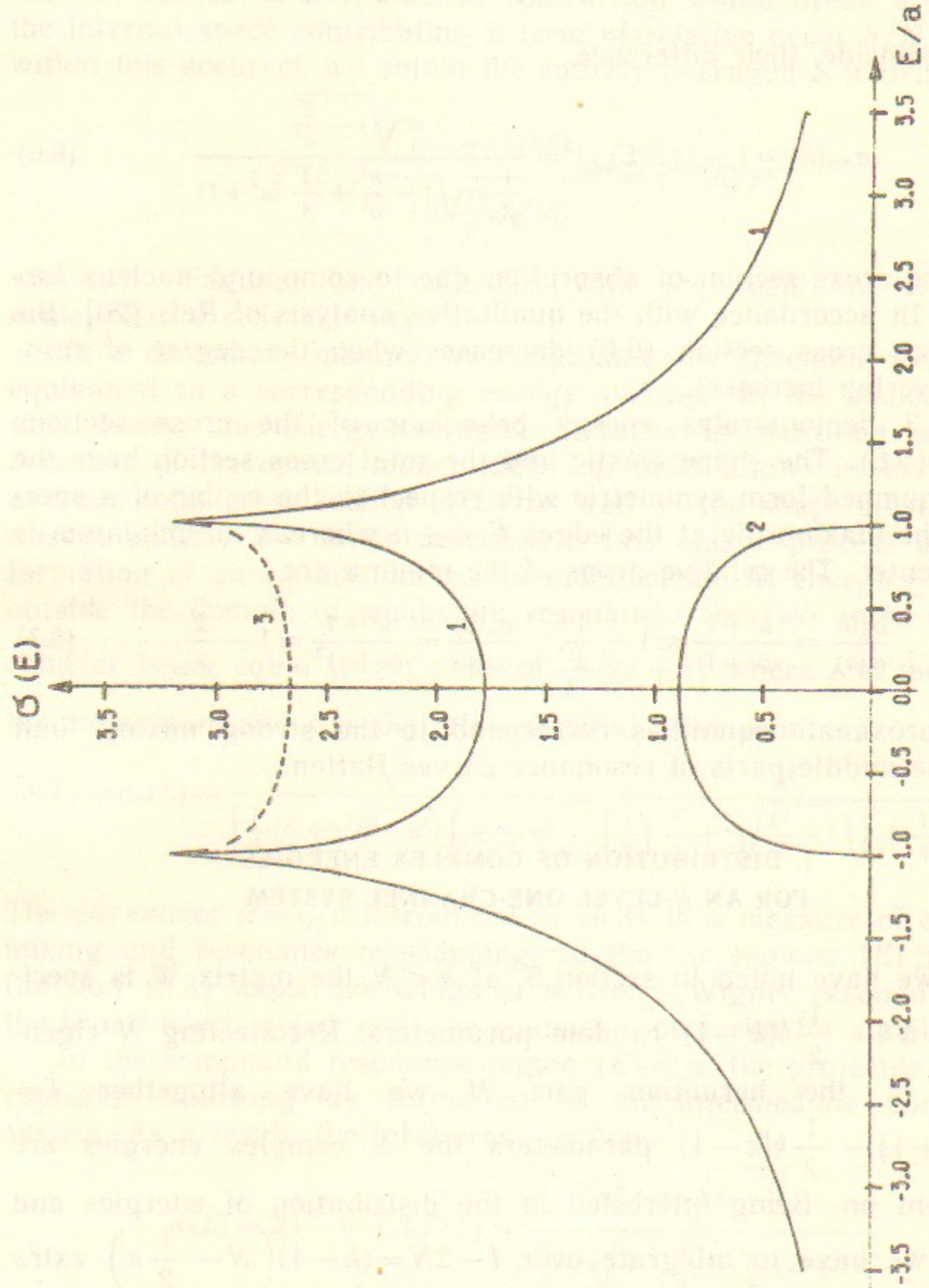
Fig. 3 demonstrates energy behaviour of the cross sections (6.4) — (6.6). The shape elastic and the total cross section have the double-humped form symmetric with respect to the center of a spectrum. The maxima lie at the edges $E=\pm a$ whereas the minimum is in the center. The relative drops of the minima are

$$\frac{\sigma_t(0)}{\sigma_t(a)}=\frac{\varkappa^2+1}{\varkappa(\varkappa+1)}\approx 1-\frac{1}{\varkappa}, \quad \frac{\sigma_{se}(0)}{\sigma_{se}(a)}=\frac{\varkappa^2+1}{(\varkappa+1)^2}\approx 1-\frac{2}{\varkappa}. \tag{6.7}$$

The approximate equalities correspond to the strong mixing limit when the middle parts of resonance curves flatten.

7. DISTRIBUTION OF COMPLEX ENERGIES FOR AN N-LEVEL ONE-CHANNEL SYSTEM

As we have noted in section 5, at $k<N$ the matrix W is specified by $Nk-\frac{1}{2}k(k-1)$ random parameters. Recollecting N eigenvalues of the hermitian part H we have altogether $l=N(k+1)-\frac{1}{2}k(k-1)$ parameters the N complex energies are dependent on. Being interested in the distribution of energies and widths we have to integrate over $l-2N=(k-1)\left(N-\frac{1}{2}k\right)$ extra variables. In the exceptional case of $k=1$ the number l of original variables is just equal to the number $2N$ of parameters $\{E_n, \Gamma_n\}$ so



Puc. 3. Energy behaviour of average cross sections:
solid lines *1* and *2* correspond to the shape elastic cross section and to the absorption cross section respectively, the dashed line *3* corresponds to the total cross section.

that the necessity of integration drops out and the problem becomes reduced to variable transformation and calculation of the Jacobian. This one-channel case will be considered below in detail.

We use the secular equation (3.11) with the R-matrix in the form (3.3) symmetric with respect to levels and perform the transformation $\{\varepsilon_n, \gamma_n\} \to \{E_n, \Gamma_n\}$. Let us define characteristic polynomials of $N \times N$ matrices H, $\mathscr{H} = H - \frac{i}{2} W$ and $\mathscr{H}^+ = H + \frac{i}{2} W$:

$$Q_0(\lambda) \equiv \mathrm{Det}\,(\lambda - H) = \prod_n (\lambda - \varepsilon_n)\,, \tag{7.1}$$

$$Q(\lambda) \equiv \mathrm{Det}\,(\lambda - \mathscr{H}) = \prod_n (\lambda - \mathscr{E}_n)\,, \tag{7.2}$$

$$\bar{Q}(\lambda) \equiv \mathrm{Det}\,(\lambda - \mathscr{H}^+) = \prod_n (\lambda - \mathscr{E}_n^*)\,.$$

One can easily verify that due to eq. (3.13)

$$Q(\lambda) = Q_0(\lambda) - \frac{i}{2} \sum_m \gamma_m \frac{\partial Q_0(\lambda)}{\partial \varepsilon_m}\,. \tag{7.3}$$

The coefficients $\Lambda_n^{(N)}$ of polynomials (7.1) and (7.2) defined according to

$$Q_0(\lambda) = \sum_{n=0}^{N} (-1)^n \Lambda_n^{(N)}(\varepsilon)\, \lambda^{N-n}\,,$$

$$Q(\lambda) = \sum_{n=0}^{N} (-1)^n \Lambda_n^{(N)}(\mathscr{E})\, \lambda^{N-n}, \qquad \bar{Q}(\lambda) = \sum_{n=0}^{N} (-1)^n \Lambda_n^{(N)}(\mathscr{E}^*)\, \lambda^{N-n}, \tag{7.4}$$

are symmetric over their N variables and linear in each of them:

$$\Lambda_0^{(N)} = 1, \quad \Lambda_1^{(N)}(\varepsilon) = \sum_n \varepsilon_n, \quad \Lambda_2^{(N)}(\varepsilon) = \sum_{m<n} \varepsilon_m \varepsilon_n, \ldots, \Lambda_N^{(N)}(\varepsilon) = \prod_n \varepsilon_n\,. \tag{7.5}$$

Derivatives of coefficients (7.5)

$$M_{nm}^{(N)}(\varepsilon) = \frac{\partial \Lambda_n^{(N)}(\varepsilon_1, \ldots, \varepsilon_N)}{\partial \varepsilon_m}, \qquad n, m = 1, 2, \ldots, N, \tag{7.6}$$

coincide with analogous coefficients of characteristic polynomials of the $(N-1)$-dimensional matrix having the same eigenvalues with ε_m omitted. Therefore

$$M_{nm}^{(N)}(\varepsilon) = \Lambda_{n-1}^{(N-1)}(\varepsilon_1, \ldots, \varepsilon_{m-1}, \varepsilon_{m+1}, \ldots, \varepsilon_N) \equiv \Lambda_{n-1;m}^{(N-1)}(\varepsilon)\,. \tag{7.7}$$

In the same way

$$\frac{\partial^2 \Lambda_n^{(N)}(\varepsilon)}{\partial \varepsilon_m \partial \varepsilon_l} = \frac{\partial \Lambda_{n-1;m}^{(N-1)}(\varepsilon)}{\partial \varepsilon_l} = \Lambda_{n-2;ml}^{(N-2)}(\varepsilon) \tag{7.8}$$

and so on. The linearity property implies equivalence of eq. (7.8) to

$$M_{nl}^{(N)}(\varepsilon) - M_{nm}^{(N)}(\varepsilon) = (\varepsilon_m - \varepsilon_l)\, \Lambda_{n-2;ml}^{(N-2)}(\varepsilon)\,. \tag{7.9}$$

We can calculate the determinant of the $N \times N$ matrix $M^{(N)}(\varepsilon)$ with elements $M_{nm}^{(N)}(\varepsilon)$ as follows. Subtracting the first column of this determinant from others we find with the help of (7.9)

$$\operatorname{Det} M^{(N)}(\varepsilon) \equiv \Delta^{(N)}(\varepsilon_1, ..., \varepsilon_N) = \prod_{n=2}^{N} (\varepsilon_1 - \varepsilon_n)\, \Delta^{(N-1)}(\varepsilon_2, ..., \varepsilon_N)\,. \tag{7.10}$$

Continuing in the same manner we obtain

$$\Delta^{(N)}(\varepsilon) = \prod_{m<n} (\varepsilon_m - \varepsilon_n)\,. \tag{7.11}$$

The connection (7.3) between polynomial expansions (7.4) for $Q(\lambda)$ and $Q_0(\lambda)$ leads to

$$\Lambda_n^{(N)}(\mathscr{E}) = \Lambda_n^{(N)}(\varepsilon) - \frac{i}{2} \sum_m M_{nm}^{(N)}(\varepsilon)\, \gamma_m\,, \qquad n = 1, 2, ..., N\,. \tag{7.12}$$

This relation together with the conjugate one gives explicitly the transformation $\{\varepsilon_n, \gamma_n\} \to \{\mathscr{E}_n, \mathscr{E}_n^*\}$. To derive the Jacobian of this transformation we need derivatives

$$\sigma_{mn} = \frac{\partial \varepsilon_m}{\partial \mathscr{E}_n}, \qquad \tau_{mn} = \frac{\partial \gamma_m}{\partial \mathscr{E}_n}. \tag{7.13}$$

Differentiating eqs. (7.6), (7.12) and conjugate ones with respect to $\mathscr{E}$ we find (in the matrix notation)

$$M(\varepsilon)\sigma = \frac{1}{2} M(\mathscr{E}), \tag{7.14}$$

$$M(\varepsilon)\tau = iM(\mathscr{E}) - L(\varepsilon, \gamma)\sigma; \quad L_{nm}(\varepsilon, \gamma) \equiv \sum_l \frac{\partial M_{nl}^{(N)}(\varepsilon)}{\partial \varepsilon_m}\, \gamma_l\,. \tag{7.15}$$

Since the matrices $M(\varepsilon)$ and $L(\varepsilon, \gamma)$ are real it follows from (7.14) and (7.15) that

$$\tau = 2i\sigma - M^{-1}(\varepsilon)\, L(\varepsilon, \gamma)\sigma, \qquad \tau^* = -2i\sigma^* - M^{-1}(\varepsilon)\, L(\varepsilon, \gamma)\sigma^*\,. \tag{7.16}$$

The Jacobian of the transformation $\{\varepsilon, \gamma\} \to \{\mathscr{E}, \mathscr{E}^*\}$ is equal to

$$\Delta_J = \begin{vmatrix} \sigma & \sigma^* \\ \tau & \tau^* \end{vmatrix}. \tag{7.17}$$

Because of the standard determinant property, the second items in eqs. (7.16) do not contribute so that

$$\Delta_J = (-4i)^N\, |\operatorname{Det} \sigma|^2\,. \tag{7.18}$$

On the other hand, from eq. (7.14)

$$\operatorname{Det} \sigma = \frac{1}{2^N} \frac{\Delta^{(N)}(\mathscr{E})}{\Delta^{(N)}(\varepsilon)}. \tag{7.19}$$

Finally, we obtain

$$\Delta_J = (-i)^N \left| \frac{\Delta^{(N)}(\mathscr{E}_1, ..., \mathscr{E}_N)}{\Delta^{(N)}(\varepsilon_1, ..., \varepsilon_N)} \right|^2. \tag{7.20}$$

To complete the transformation of the distribution function we have to express the combination $F \equiv [\Delta^{(N)}(\varepsilon)]^2 \prod_n \gamma_n$ in terms of new variables. Substituting $\mathscr{E} = \varepsilon_n$ in eq. (7.3) and taking into account that $Q_0(\varepsilon_n) = 0$ and $\left(\frac{\partial Q_0(\lambda)}{\partial \varepsilon_m}\right)_{\lambda = \varepsilon_n} = -\delta_{mn} \prod_{l \neq n} (\varepsilon_n - \varepsilon_l)$ we get

$$\gamma_n = -2i \frac{Q(\varepsilon_n)}{\prod_{l \neq n} (\varepsilon_n - \varepsilon_l)}. \tag{7.21}$$

With the help of eqs. (7.1), (7.2) and (7.11) we come to

$$F = (-1)^{\frac{N(N-1)}{2}} (-2i)^N \prod_n Q(\varepsilon_n) = (-1)^{\frac{N(N-1)}{2}} (-2i)^N \prod_n (-1)^N Q_0(\mathscr{E}_n). \tag{7.22}$$

Since $Q(\mathscr{E}_n) = 0$ it follows from (7.12) along with the conjugate equation and eqs. (7.5) that

$$Q_0(\mathscr{E}_n) = \frac{1}{2} \bar{Q}(\mathscr{E}_n) = \frac{1}{2} \prod_m (\mathscr{E}_n - \mathscr{E}_m^*)\,. \tag{7.23}$$

Therefore the expression (7.22) turns out to be reduced to

$$F=(-1)^{\frac{N(N-1)}{2}}(-i)^N\prod_{m,n}(\mathscr{E}_n-\mathscr{E}_m^*)=\prod_n|\mathscr{E}_n-\mathscr{E}_n^*|\prod_{m<n}|\mathscr{E}_m-\mathscr{E}_n^*|^2. \quad (7.24)$$

The distribution function of original variables $\{\varepsilon_n, \gamma_n\}$ at $k=1$ can be written down, according to eqs. (4.2) and (5.16), as

$$\mathscr{P}(\varepsilon_1,\ldots,\varepsilon_N;\gamma_1,\ldots,\gamma_N)=C_NC_{1N}|\Delta^{(N)}(\varepsilon_1,\ldots,\varepsilon_N)|\times$$
$$\times\prod_n\frac{1}{\sqrt{\gamma_n}}\exp\left[-N\left(\frac{1}{a^2}\sum_n\varepsilon_n^2+\frac{1}{2\eta}\sum_n\gamma_n\right)\right]. \quad (7.25)$$

To transform the exponent in eq. (7.25) note that

$$\mathrm{Tr}\,\mathscr{H}^2=\sum_n\mathscr{E}_n^2=\mathrm{Tr}\left(H^2-iHW-\frac{1}{4}W^2\right)=\sum_n(\varepsilon_n^2-i\varepsilon_n\gamma_n)-\frac{1}{4}\mathrm{Tr}\,W^2. \quad (7.26)$$

In the one-channel case the matrix W has only one nonzero eigenvalue so that

$$\mathrm{Tr}\,W^2=(\mathrm{Tr}\,W)^2=\left(\sum_n\Gamma_n\right)^2 \quad (7.27)$$

and the real part of eq. (7.26) gives

$$\sum_n\varepsilon_n^2=\sum_nE_n^2-\frac{1}{4}\left[\sum_n\Gamma_n^2-\left(\sum_n\Gamma_n\right)^2\right]=\sum_nE_n^2+\frac{1}{2}\sum_{m<n}\Gamma_m\Gamma_n. \quad (7.28)$$

Gathering all the results we gain the desired distribution function for energies and widths of intermediate unstable states:

$$\mathscr{P}(E_1,\ldots,E_N;\Gamma_1,\ldots,\Gamma_N)=C_NC_{1N}\prod_{m<n}\frac{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m-\Gamma_n)^2}{\sqrt{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m+\Gamma_n)^2}}\times$$
$$\times\prod_n\frac{1}{\sqrt{\Gamma_n}}\exp\left\{-N\left[\frac{1}{a^2}\sum_nE_n^2+\frac{1}{2a^2}\sum_{m<n}\Gamma_m\Gamma_n+\frac{1}{2\eta}\sum_n\Gamma_n\right]\right\}. \quad (7.29)$$

As one can see from the preexponent factor, energies and widths of unstable states are correlated. According to the qualitative arguments mentioned in Introduction, unlike to (4.2), the probability for two energies E_m and E_n to coincide does not vanish provided that $\Gamma_m\neq\Gamma_n$. At the same time, the complex energies $\mathscr{E}_m$ and $\mathscr{E}_n$ repulse each other the coincidence probability being in proportion to the

square of distance $r_{mn}=\sqrt{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m-\Gamma_n)^2}$ in the complex plane of $\mathscr{E}=E-\frac{i}{2}\Gamma$. This reflects T-noninvariant nature of a decaying system.

Features of distribution (7.29) at N fixed change drastically depending on the value of the parameter $\varkappa=\eta/a$. The mean level spacing is $D\sim a/N$. In the weak overlap regime widths are typically $\Gamma\sim\eta/N$ (see eq. (5.6)) and $\Gamma/D\sim\varkappa\ll1$. Then the bilinear in Γ terms in the exponent (7.29) are negligible so that for energy spacings $s_{mn}=\frac{|E_m-E_n|}{D}\gg\varkappa$ the distribution function factorizes in the product of independent distributions of energies (see eq. (4.2)) and widths (see eq. (5.4)). At these distances, energy repulsion is linear. Only at very small spacings $s_{mn}\lesssim\varkappa$ correlations differ from those in a stable system resulting in disappearance of repulsion.

In the opposite limit of $\varkappa\gg1$, widths are accumulated by the exceptional eigenvalue having a large width of order η as it was shown earlier (section 3). In line with such predictions, eq. (7.28) demonstrates that the probability density (7.29) is appreciable in the only case $\left(\sum_n\Gamma_n\right)^2\approx\sum_n\Gamma_n^2$, i. e. if only one of widths, say Γ_1, is large, $\Gamma_1\approx\eta$, the others $\Gamma_{n\geqslant2}$ being small. Since then $|E_1-E_n|\lesssim a\ll\eta$, we have

$$\prod_{n=2}^N\frac{(E_1-E_n)^2+\frac{1}{4}(\Gamma_1-\Gamma_n)^2}{\sqrt{(E_1-E_n)^2+\frac{1}{4}(\Gamma_1+\Gamma_n)^2}}\approx\left(\frac{\Gamma_1}{2}\right)^{N-1}, \quad (7.30)$$

$$\left(\sum_n\Gamma_n\right)^2-\sum_n\Gamma_n^2=2\Gamma_1\sum_{n=2}^N\Gamma_n+\left[\left(\sum_{n=2}^N\Gamma_n\right)^2-\sum_{n=2}^N\Gamma_n^2\right]. \quad (7.31)$$

After integrating over E_1 and Γ_1 one obtains the distribution function for long-lived resonances ($m, n=2, 3, \ldots, N$):

$$\mathscr{P}(E_2,\ldots,E_N;\Gamma_2,\ldots,\Gamma_n)=C_NC_{1N}2^{1-N}\sqrt{\frac{\pi a^2}{N}}\left(\frac{2\eta}{N}\right)^{N-1/2}\frac{\Gamma\left(N-\frac{1}{2}\right)}{\left(1+\frac{\varkappa^2}{\eta}\sum_n\Gamma_n\right)^{N-1/2}}\times$$

$$\times \prod_{m<n} \frac{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m-\Gamma_n)^2}{\sqrt{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m+\Gamma_n)^2}} \prod_n \frac{1}{\sqrt{\Gamma_n}} \exp\Big[-N\Big(\frac{1}{a^2}\sum_n E_n^2+$$

$$\frac{1}{2a^2}\sum_{m<n}\Gamma_m\Gamma_n+\frac{1}{2\eta}\sum_n\Gamma_n\Big)\Big]. \qquad (7.32)$$

To estimate values of typical small widths it is convenient to introduce amplitudes $\beta_n=\sqrt{\Gamma_n}$ $(n\geqslant 2)$ assuming that they run over the whole real axis. Then

$$\prod_n \frac{d\Gamma_n}{\sqrt{\Gamma_n}} \to \prod_n d\beta_n = \tilde{w}^{(N-3)/2} d\tilde{w}\, dO_{N-1}, \qquad (7.33)$$

where $\tilde{w}=\sum_{n=2}^{N}\Gamma_n$ and dO_{N-1} stands for an infinitesimal solid angle of a $(N-1)$-dimensional space with β_n as its coordinates. It is easy to show that the probability density (7.32) has a maximum at

$$\tilde{w}=\frac{\eta}{\varkappa^2}\ll\Gamma_1\approx\eta, \qquad (7.34)$$

whereas the angle distribution in the space of β_n is nearly isotropic. Hence, the individual small widths are of order $\Gamma_{n\geqslant 2}\sim\tilde{w}/(N-1)$ i. e., at large N, $\Gamma_{n\geqslant 2}\sim\eta/\varkappa^2 N$ and the corresponding overlap parameter is $\Gamma/D\sim\frac{\eta}{\varkappa^2 N}\frac{N}{a}=\frac{1}{\varkappa}\ll 1$. As a consequence, long-lived states fall again in the small overlap regime and, at spacings $s_{mn}\gtrsim 1/\varkappa$, the energy distribution coincides with that of eq. (4.2).

These results are evidently of dynamical rather than statistical origin. Therefore they can be derived from the secular equation (section 3) straightforwardly.

The version (3.16) obtained by diagonalization of the matrix W is appropriate for the case $\varkappa\gg 1$. Since, due to eq. (4.7), $\langle h_\nu^2\rangle=\frac{a^2}{4N}=\frac{\eta^2}{4N\varkappa^2}$ is small in the limit under study, we can use the perturbational formulae:

$$\mathscr{E}_1=h-\frac{i}{2}w+\sum_{\nu=2}^{N}\frac{h_\nu^2}{h-\varepsilon_\nu-\frac{i}{2}w}=h-\frac{i}{2}w+\tilde{R}\Big(h-\frac{i}{2}w\Big), \qquad (7.35)$$

$$\mathscr{E}_\nu=\varepsilon_\nu+\frac{h_\nu^2}{\varepsilon_\nu-h+\frac{i}{2}w}, \quad \nu\geqslant 2. \qquad (7.36)$$

Neglecting $(h-\varepsilon_\nu)^2\ll a^2$ as compared with $w^2\approx\eta^2$ we find from eqs. (7.35) and (7.36) the large width

$$\Gamma_1=w-w\sum_{\nu=2}^{N}\frac{h_\nu^2}{(h-\varepsilon_\nu)^2+\frac{1}{4}w^2}\approx\Big(1-\frac{1}{\varkappa^2}\Big)w \qquad (7.37)$$

as well as small ones

$$\Gamma_\nu=\Gamma_1\frac{h_\nu^2}{(h-\varepsilon_\nu)^2+\frac{1}{4}\Gamma_1^2}\approx\frac{w}{N\varkappa^2}. \qquad (7.38)$$

The similar picture has been discussed in Ref. [27] basing on quite different arguments. In contrast to that paper, we are not in need of any assumptions concerning the preexistence of narrow resonances on the background of a broad one. The separation of width scales happens automatically by virtue of structure of the matrix W which is dictated ultimately by the unitarity principle.

It should be stressed that in the one-channel case the width rearrangement leaves no room for manifestating Ericson fluctuations [38]. Indeed, at $N\gg 1$ conditions necessary for the strong overlap of resonances cannot be implemented. To reveal such fluctuations, there should be $k\gg 1$ of open channels when k short-lived states with comparable widths and weakly correlated decay amplitudes arise. To investigate rigorously cross section fluctuations in frames of our approach, one should derive distribution of decay amplitudes (2.11). We hope to return to this problem elsewhere.

8. AVERAGE DENSITY OF UNSTABLE STATES

An important quantity for an N-level unstable system is the mean density of complex energies $\mathscr{E}_n = E_n - \frac{i}{2}\Gamma_n$ (compare (4.10)),

$$\rho(E,\Gamma) = \langle \frac{1}{2}\sum_n \delta^{(2)}(\mathscr{E}-\mathscr{E}_n)\rangle = \langle \sum_n \delta(E-E_n)\,\delta(\Gamma-\Gamma_n)\rangle \tag{8.1}$$

normalized according to

$$\int_{-\infty}^{\infty} dE \int_0^{\infty} d\Gamma\, \rho(E,\Gamma) = N. \tag{8.2}$$

The straightforward calculation of $\rho(E,\Gamma)$ by multidimensional integrating the joint distribution function (7.29) is extremely difficult. It is more convenient, analogously to eq. (4.10), to start with evaluating the average trace of the Green function

$$g(E,\Gamma) = \frac{1}{N}\langle \operatorname{Tr} G(\mathscr{E})\rangle = \frac{1}{N}\langle \sum_n \frac{1}{\mathscr{E}-\mathscr{E}_n}\rangle . \tag{8.3}$$

We use an electrostatic analogy to establish the connection between functions ρ and g in the complex plane $\mathscr{E} = E - \frac{i}{2}\Gamma$. Let us interpret the function (8.1) as a density of two-dimensional charge distribution across the (E,Γ)-plane. Then the function $Ng(E,\Gamma)$ acquires the meaning of the two-dimensional electric field created by this charge distribution. In our notations, Maxwell equations take the form

$$N\left[\frac{\partial g(E,\Gamma)}{\partial E} - 2i\frac{\partial g(E,\Gamma)}{\partial \Gamma}\right] = 4\pi\,\rho(E,\Gamma). \tag{8.4}$$

Since all $\Gamma_n \geqslant 0$ it is clear that $\rho(E,\Gamma)=0$ in the half-plane $\Gamma<0$. Eq. (8.4) reduces then to the Cauchy—Riemann condition expressing analyticity of g in this half-plane. On the contrary, $\rho \neq 0$ in points where $g(E,\Gamma)$ loses its analyticity.

To calculate $g(E,\Gamma)$ we use eq. (3.9). At $\Gamma>0$ the only surviving contribution to eq. (8.4) stems from the function $K(\mathscr{E})$ which can be presented in the form

$$K(\mathscr{E}) = -\theta\left[\operatorname{Im}\left(1+\frac{i}{2}R(\mathscr{E})\right)\right]K_+(\mathscr{E}) - \theta\left[-\operatorname{Im}\left(1+\frac{i}{2}R(\mathscr{E})\right)\right]K_-(\mathscr{E}). \tag{8.5}$$

Here the new functions

$$K_\pm(\mathscr{E}) = \frac{d}{d\mathscr{E}}\int_0^{\infty}\frac{d\lambda}{\lambda}\left\{\exp\left[\pm i\lambda\left(1+\frac{i}{2}R(\mathscr{E})\right)\right]-1\right\} \tag{8.6}$$

are introduced each of them being analytical in its own part of the complex plane cut by the corresponding step function θ. In these parts the functions (8.6) both satisfy the Cauchy-Riemann condition

$$\frac{\partial}{\partial E}K_\pm(\mathscr{E}) - 2i\frac{\partial}{\partial\Gamma}K_\pm(\mathscr{E}) = 0. \tag{8.7}$$

Since $R(\mathscr{E})$ is analytical everywhere with exception of the real axis we get at $\Gamma>0$

$$\left(\frac{\partial}{\partial E} - 2i\frac{\partial}{\partial\Gamma}\right)\operatorname{Im}\left(1+\frac{i}{2}R(\mathscr{E})\right) = \frac{1}{2}\frac{\partial R(\mathscr{E}^*)}{\partial\mathscr{E}^*}. \tag{8.8}$$

Eq. (8.5), (8.7) and (8.8) give

$$\left(\frac{\partial}{\partial E} - 2i\frac{\partial}{\partial\Gamma}\right)K(\mathscr{E}) = \frac{\pi}{2}\left|\frac{\partial R(\mathscr{E})}{\partial\mathscr{E}}\right|^2\delta^{(2)}\left(1+\frac{i}{2}R(\mathscr{E})\right) \tag{8.9}$$

and, finally,

$$\rho(E,\Gamma) = \frac{1}{8}\left\langle\left|\frac{\partial R(\mathscr{E})}{\partial\mathscr{E}}\right|^2\delta^{(2)}\left(1+\frac{i}{2}R(\mathscr{E})\right)\right\rangle. \tag{8.10}$$

This result is valid independently of particular statistic assumptions. With the aid of the integral representation of the two-dimensional δ-function we can carry out explicitly averaging of ρ (8.10) over the distribution function (5.4) of residues γ_n of the R-function:

$$\rho(E,\Gamma) = -\frac{1}{2\pi^2}\frac{\partial^2}{\partial\mathscr{E}\,\partial\mathscr{E}^*}\int\frac{d^2\lambda}{|\lambda|^2}\exp(-2i\operatorname{Im}\lambda)\times$$

$$\times\left(\left\langle\exp\left\{-\frac{1}{2}\operatorname{Tr}\log\left(1-i\frac{\eta}{N}\left[\lambda^* G^0(\mathscr{E})+\lambda G^0(\mathscr{E}^*)\right]\right)\right\}\right\rangle_\varepsilon - 1\right). \tag{8.11}$$

GOE-averaging over the distribution function (4.2) of energies ε_n is still to be done. This task is not trivial in general case. To get explicit formulae we consider the large N limit where GOE expressions (4.9) and (4.10) are valid.

At $\varkappa \ll 1$ the anti-hermitian part W is a small perturbation so that in the lowest order $\mathscr{E}_n = \varepsilon_n - \frac{i}{2}\gamma_n$. The density $\rho(E, \Gamma)$ is then factorized,

$$\rho(E, \Gamma) = \rho^0(E)\, \mathscr{P}_1(\Gamma)\,, \tag{8.12}$$

where $\rho^0(E)$ is the semicircular distribution (4.10) and $\mathscr{P}_1(\Gamma)$ stands for the Porter—Thomas distribution (5.4) with the mean value $\langle \Gamma \rangle = \eta/N$.

In the opposite case of $\varkappa \gg 1$, it is convenient to use the alternative version (3.16) of the secular equation. Eigenvalues of $\mathscr{H}$ can be found by perturbation theory (7.35) — (7.36). Separating in eq. (8.1) the short-lived ($n=1$) state from long-lived ones ($n=\nu=2, 3, ..., N$) we have

$$\rho(E, \Gamma) = \frac{1}{2}\langle \delta^{(2)}(\mathscr{E} - \mathscr{E}_1)\rangle + \frac{1}{2}\sum_{\nu}\langle \delta^{(2)}(\mathscr{E} - \mathscr{E}_\nu)\rangle. \tag{8.13}$$

Averaging over h_ν and ε_ν in the first term (denote it by $\rho_1(E, \Gamma)$) can be carried out at $N \to \infty$ like in (4.12) resulting in

$$\rho_1(E, \Gamma) = \frac{1}{2}\langle \delta^{(2)}\Big[\mathscr{E} - h + \frac{i}{2}w - \frac{a^2}{4}g^0\Big(h - \frac{i}{2}w\Big)\Big]\rangle_{h,w}. \tag{8.14}$$

Quantities h and w are normally distributed with mean values $\langle h \rangle = 0$ and $\langle w \rangle = \eta$ and variances $\langle h^2 \rangle = \eta^2/2\varkappa^2 N$ and $\langle (w-\eta)^2 \rangle = 2\eta^2/N$ so that we can expand the argument of the δ-function in eq. (8.14) over h/w. In the real part, the corrections cancel out up to the second order inclusively whereas in the imaginary part the result coincides with that of eq. (7.37). Hence, we obtain in this approximation

$$\rho_1(E, \Gamma) = \Big\langle \delta(E-h)\, \delta\Big(\Gamma - w + \frac{w}{\varkappa^2}\Big)\Big\rangle_{h,w}, \tag{8.15}$$

Final averaging over h and w can be performed trivially giving

$$\rho_1(E, \Gamma) = \sqrt{\frac{N}{\pi a^2}}\exp\Big(-\frac{N}{a^2}E^2\Big)\frac{\sqrt{N/\pi}}{2\eta(1-\varkappa^{-2})} \times$$

$$\times \exp\Big\{-\frac{N}{4}\frac{[\Gamma - \eta(1-\varkappa^{-2})]^2}{\eta^2(1-\varkappa^{-2})^2}\Big\}. \tag{8.16}$$

Within our accuracy, the energy and the width of the short-lived state are not correlated. This resonance is located near the center of the region of levels coupled with a given channel. The whole energy region is covered by the resonance width. As for fluctuations of E and Γ, they are relatively small, $\sim 1/\sqrt{N}$.

The second term of eq. (8.13), $\rho_2(E, \Gamma)$, contains $N-1$ identical contributions. Each of them is equal to

$$\frac{1}{2}\langle \delta^{(2)}(\mathscr{E} - \mathscr{E}_\nu)\rangle = \Big\langle \frac{w\bar{f}}{\Gamma(w+\Gamma)}\delta(\varepsilon_\nu - \bar{\varepsilon})\, \delta(h_\nu^2 - \bar{f})\Big\rangle, \tag{8.17}$$

with notations

$$\bar{\varepsilon} = \frac{wE + h\Gamma}{w+\Gamma}, \quad \bar{f} = w\Gamma\Big[\frac{1}{4} + \Big(\frac{E-h}{w+\Gamma}\Big)^2\Big]. \tag{8.18}$$

Now it is obvious that (see eq. (4.11))

$$\langle \delta(\varepsilon_\nu - \bar{\varepsilon})\rangle_{\varepsilon_\nu} = \frac{1}{N-1}\rho^0(\bar{\varepsilon}) \underset{N\to\infty}{\longrightarrow} \frac{2}{\pi a^2}\sqrt{a^2 - \bar{\varepsilon}^2}\;\theta(a^2 - \bar{\varepsilon}^2)\,, \tag{8.19}$$

$$\langle \delta(h_\nu^2 - \bar{f})\rangle_{h_\nu} = \sqrt{\frac{2N}{\pi a^2 \bar{f}}}\exp\Big(-\frac{2N}{a^2}\bar{f}\Big). \tag{8.20}$$

Remaining averaging over h and w can be done by the saddle-point method. The result is

$$\rho_2(E, \Gamma) = (N-1)\frac{2}{\pi a^2}\sqrt{a^2 - E^2}\;\theta(a^2 - E^2)\sqrt{\frac{N\varkappa^2}{2\pi\eta\Gamma}}\exp\Big(-\frac{N\varkappa^2}{2\eta}\Gamma\Big). \tag{8.21}$$

We keep here unity side by side with N to guarantee proper normalization. Thus, for $N-1$ long-lived states the semicircle GOE rule (4.11) along with the Porter—Thomas distribution are recovered. However, the latter has a small average width (7.38), $\langle \Gamma \rangle = \eta/\varkappa^2 N$. As we have emphasized earlier overlap of long-lived resonances is small, $\langle \Gamma \rangle / D \sim 1/\varkappa$.

9. CONCLUSION

In this paper we have attempted to extend standard statistical spectroscopy of discrete levels to unstable states. Decaying systems are described by ensembles of random nonhermitian Hamiltonians represented by N-dimensional matrices. Complex eigenvalues of these matrices determine energies E_n and widths Γ_n of states with a

finite lifetime. Under simple statistical assumptions corresponding to completely stochastical motion, the eigenvalue distribution in the complex plane (E, Γ) is obtained. The one-channel case is discussed in detail.

Instability of states changes significantly energy statistics removing level repulsion at distances smaller than widths. As for complex energies $\mathscr{E}_n = E_n - \frac{i}{2}\Gamma_n$ they repulse quadratically with respect to the distance in the complex plane[5]. At increasing coupling with continuum dynamical restructuring of a system takes place. When matrix elements of the anti-hermitian part of the effective Hamiltonian become comparable with spacings of eigenvalues of the hermitian part, a transition to a new regime occurs. In the new regime the picture is determined by the algebraic structure of the anti-hermitian part dictated by unitarity rather than by particular statistical hypotheses. As a result, decay probabilities turn out to be redistributed in such a way to distinguish short-lived states (their number is equal to that of open decay channels) saturating approximately the whole summarized width of levels. Remaining long-lived states are quite isolated and their statistics are essentially the same as those of stable states. Such a picture implies coexistence of direct and compound processes resulting in a gross structure of cross section. Similar structure with several short-lived resonance states has been observed in numerical simulations of nuclear reactions [13]. In this work formation of such a structure along with the state overlap increasing was referred to as «selforganization». As one can see its mechanism is of quite general nature being analogous to Dicke superradiance in quantum optics.

In the one-channel case there is no room for manifestation of Ericson fluctuations. In principle, they might be observed if the channel number is large enough to generate a number of interfering resonances. This problem is under study.

The main results of this paper have been published in [40].

[5] This is equivalent to cubic repulsion if one uses polar coordinate in the (E, Γ)-plane. Such a rule is confirmed by numerical calculations [39] for a kicked nonlinear top with dissipation.

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Dynamics and Statistics of Unstable Quantum States

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