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VANISHING OF THE CHIRAL ANOMALY FOR ANTISYMMETRIC TENSOR FIELD

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Vanishing of the Chiral Anomaly
for Antisymmetric Tensor Field

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ABSTRACT

We consider gauge antisymmetric tensor field (which is equivalent to a massless scalar field on-mass-shell). We demonstrate that the total chiral current which accounts for the chirality of the vector ghost fields does not possess anomaly. We also dwell on the relation between the number of zero modes of the antisymmetric tensor field and the anomaly in the chiral current of the vector field.

1. Chiral anomalies for bosonic fields have been discussed in a number of papers [1—7]. In particular, papers [1, 3, 4, 6, 7] address themselves to the problem of constructing chiral currents of bosonic fields and evaluating anomalous terms in their divergences. In this note we will consider chiral currents and their divergences in the case of antisymmetric tensor field $\varphi_{\mu\nu}$ and vector field A_μ interacting with external gravitational field.

The chiral current j_μ in the case of antisymmetric tensor $\varphi_{\mu\nu}$ was introduced first in Ref. [1] in the Feynman gauge and it looks as

$$j_\mu = -D_\lambda \tilde{\varphi}^{\lambda\nu}\varphi_{\mu\nu} - \tilde{\varphi}_{\mu\nu} D_\lambda \varphi^{\lambda\nu}. \qquad (1)$$

The anomalous divergence of this current is given by the following equation:

$$\langle D_\mu j^\mu \rangle = -\frac{1}{48\pi^2} R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}, \qquad (1a)$$

where D_μ is the covariant derivative, $\tilde{\varphi}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\alpha\beta}\varphi_{\alpha\beta}$, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor, $\tilde{R}^{\mu\nu\alpha\beta} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}$, and the brackets $\langle ... \rangle$ imply averaging over external gravitational field. Upon integrating over the Euclidean space the right-hand-side of eq. (1a) gives the difference in the number of the left and right zero modes of the antisymmetric tensor field in the external field considered.

The very existence of the anomaly (1) looks puzzling, however.

Indeed, the classical theory of the antisymmetric tensor field is known [8] (see also [9]) to be equivalent to that of a massless scalar field and the very notion of chirality is foreign to the scalar field. Moreover, one can seemingly evaluate the anomalous triangle graphs via the unitarity condition. The imaginary part is determined then by the Born graphs which are to be the same for classically equivalent theories. Basing on this kind of argument one would not expect any anomaly at all.

As for the chiral anomaly for the vector field it has been found most recently [4, 6, 7] and reads as

$$\langle D_\mu K^\mu \rangle = -\frac{1}{96\pi^2} R_{\mu\nu\alpha\beta} \tilde{R}^{\mu\nu\alpha\beta}, \tag{2}$$

where

$$K^\mu = -\frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta . \tag{2a}$$

What is unusual about this anomaly is that it is not related to zero modes of the vector field (moreover, the latter do not exist at all, see Ref. [10]). The integral over the right-hand-side of eq. (2) does count however number of zero modes of the antisymmetric tensor field, but not vector field.

Our observation is that there exist a close connection between the two anomalies discussed which resolves the both puzzles.

2. The relation between the anomalies arises in the most natural way if one pursues the unitarity argument mentioned above. Indeed, it is perfectly clear that chirality of the antisymmetric field refers only to the unphysical degrees of freedom which are introduced via the quantization procedure upon fixing the gauge of antisymmetric field $\varphi_{\mu\nu}$. As usual, one introduces at this point ghost fields which in the case considered include vector field η_μ. One can therefore introduce the corresponding chiral current $K_\mu(\eta, \bar{\eta})$,

$$K_\mu(\eta, \bar{\eta}) = -\frac{2}{\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} \bar{\eta}_\nu \partial_\alpha \eta_\beta , \tag{3}$$

which is normalized by the condition that the chiral charge is equal to $(+1)$ and (-1) for the left- and right-handed particles respectively. Moreover, the total chiral current in the theory of the antisymmetric tensor is the sum $j_\mu + K_\mu(\eta, \bar{\eta})$. One can readily verify then that the anomalies cancel each other so that the total chiral

current is divergenceless. Indeed, the anomaly in current K_μ for the non-Hermitian ghost fields is twice as that of the real vector field A_μ (see eq. (2)). Moreover, an extra minus sign arises there because of the wrong statistics of the ghost fields.

On the other hand, the absence of the anomaly in the total current could be inferred from the unitarity condition. Then it becomes much more natural that the anomaly (2) for the vector field is related to the number of the zero modes of the antisymmetric tensor which controls the coefficient in anomaly (1).

3. The argument above is clearly of heuristic nature. A more systematic consideration reveals at least two problems which call for further investigation. First, the current j_μ introduced above is not gauge invariant and the classical equations of motion are self-contradictory once the interaction with this current is accounted for even to first order. Second, any explicit evaluation of the anomalies assumes introduction of some ultraviolet or infrared cutoff. One possibility is to ascribe a nonvanishing mass to the fields. The advantage of such a regularization procedure is its simplicity and covariance. However, upon introducing a nonvanishing mass for the antisymmetric tensor field we change the number of degrees of freedom and it is far from being evident that the limit of a small but finite mass is the massless case indeed.

Our purpose is to learn to perform loop calculations for the current j_μ under the condition that the contribution of only the physical degrees of freedom in the intermediate state is kept. Usually, the explicit Lorentz invariance in gauge theories is ensured via the introduction of extra ghost fields while the unitarity is substituted for by the requirement of the BRST invariance. In the case considered this strategy fails because of the gauge noninvariance of the current j_μ. Nevertheless we shall be able to perform the loop calculations in a consistent way.

To this end let us start with a massive field $\Phi_{\mu\nu}$, the Lagrangian being of the form:

$$L = -\frac{1}{2} \partial_\lambda \tilde{\Phi}_{\lambda\nu} \partial_\beta \tilde{\Phi}_{\beta\nu} - \frac{1}{4} m^2 \Phi_{\mu\nu}^2 + h_\mu K_\mu(\Phi). \tag{4}$$

Here h_μ is some external field interacting with chiral current $K_\mu(\Phi)$:

$$K_\mu(\Phi) = -\partial_\lambda \tilde{\Phi}_{\lambda\nu} \Phi_{\mu\nu} . \tag{5}$$

For sake of simplicity we write down all the equations in Minkowski space keeping in mind that the generalization to the case of a nonvanishing external gravitational field is straightforward.

It is well known [8] that field $\Phi_{\mu\nu}$ is equivalent classically to the massive vector field b_μ described by the following Lagrangian:

$$L(b) = -\frac{1}{4} b_{\mu\nu}^2 + \frac{m^2}{2} b_\mu^2 + h_\mu K_\mu(b). \tag{6}$$

The correspondence between the fields is given by

$$b_\mu = \frac{1}{m} \partial_\lambda \tilde{\Phi}_{\lambda\mu}, \quad b_{\mu\nu} \equiv (\partial_\mu b_\nu - \partial_\nu b_\mu) = -m\tilde{\Phi}_{\mu\nu}. \tag{7}$$

In terms of field b_μ chiral current (5) looks as (see eq. (2a)):

$$K_\mu(b) = -b_\nu \tilde{b}_{\mu\nu} = -\varepsilon_{\mu\nu\alpha\beta} b_\nu \partial_\alpha b_\beta. \tag{8}$$

Classically the divergence of chiral current K_μ is equal to

$$\partial_\mu K_\mu = -\frac{1}{2} b_{\mu\nu} \tilde{b}_{\mu\nu} = \frac{m^2}{2} \Phi_{\mu\nu} \tilde{\Phi}_{\mu\nu}. \tag{9}$$

It is worth noting that the divergence of the chiral current, if expressed in terms of field $\Phi_{\mu\nu}$, is proportional to m^2. Thus, there exists an analogy to the case of the chiral current of a spinor field. It is clear, however, that the propagator of $\Phi_{\mu\nu}$ corresponding to Lagrangian (4) is singular in m^2 so that this vanishing of the divergence of the current in the limit of $m^2=0$ is only formal.

To trace these singularities it is convenient to abandon the Proca formalism for the vector field and use some other gauge like the Feynman one. To this end let us use the following substitution for the original field $\Phi_{\mu\nu}$:

$$\Phi_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{m} a_{\mu\nu},$$

$$a_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{10}$$

Since we have enlarged the number of the fields the equations of motion become invariant under the gauge transformations:

$$\delta\varphi_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu,$$

$$\delta a_\mu = -m\xi_\mu. \tag{11}$$

To fix the gauge we add the following term to the Lagrangian:

$$L_{gauge} = \frac{1}{2} \left[\partial_\lambda \varphi_{\lambda\mu} - m a_\mu - h_\lambda \left(\tilde{\varphi}_{\lambda\mu} - \frac{1}{m} \tilde{a}_{\lambda\mu}\right)\right]^2 + O(h_\mu^2). \tag{12}$$

The terms proportional to h_μ here are arranged in such a way as to produce its interaction with the chiral currents of the fields $\varphi_{\mu\nu}$ and a_μ (see eqs (1) and (2a)).

As usual along with L_{gauge} one has to add the Lagrangian of the ghost fields, L_{ghost}. The form of L_{ghost} is determined by the variation of L_{gauge} under gauge transformation (11). Explicitly,

$$L_{ghost} = -\frac{1}{2} \bar{\eta}_{\mu\nu} \eta_{\mu\nu} + m^2 \bar{\eta}_\mu \eta_\mu - 2h_\mu \eta_\nu \bar{\eta}_{\mu\nu}, \tag{13}$$

where η_μ and $\bar{\eta}_\mu$ are vector (Grassmanian) fields, $\eta_{\mu\nu}$ is equal to $\partial_\mu \eta_\nu - \partial_\nu \eta_\mu$ and $\tilde{\eta}_{\mu\nu}$ is given by $\tilde{\eta}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\gamma\delta} \eta_{\gamma\delta}$.

The total Lagrangian can be rewritten as

$$L + L_{gauge} + L_{ghost} = -\frac{1}{4} \varphi_{\mu\nu} \Box \varphi_{\mu\nu} - \frac{1}{4} m^2 \varphi_{\mu\nu}^2 - \frac{1}{4} a_{\mu\nu}^2 + \frac{m^2}{2} a_\mu^2 - \frac{1}{2} \bar{\eta}_{\mu\nu} \eta_{\mu\nu} + m^2 \bar{\eta}_\mu \eta_\mu + h_\mu \Big[-\partial_\lambda \tilde{\varphi}_{\lambda\nu} \varphi_{\mu\nu} - \partial_\lambda \varphi_{\lambda\nu} \tilde{\varphi}_{\mu\nu} - a_\nu \tilde{a}_{\mu\nu} - 2\eta_\nu \bar{\eta}_{\mu\nu} + \left(\frac{1}{m} \partial_\lambda \varphi_{\lambda\nu} \tilde{a}_{\mu\nu} - \frac{1}{m} \partial_\lambda \tilde{\varphi}_{\lambda\nu} a_{\mu\nu} + m a_\nu \tilde{\varphi}_{\mu\nu}\right)\Big]. \tag{14}$$

If one is interested in evaluating the loop graphs in the presence of external gravitational field to first order in h_μ the term in round brackets in the right-hand-side of eq. (14) can be dropped altogether. Indeed, we have fixed the gauge just in such a way that there is no Wick contraction between $\varphi_{\mu\nu}$ and a_μ. On the other hand, Lagrangian (14) is equivalent to Lagrangian (6) describing massive vector field. Thus we come to the following relation between the terms linear in h_μ:

$$\langle K_\mu(b) \rangle = \langle j_\mu(\varphi) + K_\mu(a) + K_\mu(\eta, \bar{\eta}) \rangle, \tag{15}$$

where $j_\mu(\varphi)$ is given by eq. (1) while $K_\mu(\eta, \bar{\eta})$ can be read off eq. (3). Moreover, since it is obvious that $\langle K_\mu(a) \rangle = \langle K_\mu(b) \rangle$ we come to the conclusion that

$$\langle j_\mu(\varphi) + K_\mu(\eta, \bar{\eta}) \rangle = 0, \tag{16}$$

i. e. the contributions of tensor field $\varphi_{\mu\nu}$ and that of the ghost fields cancel each other, as is expected.

Since

$$\langle K_\mu(\eta, \bar{\eta}) \rangle = -2 \langle K_\mu(a) \rangle$$

(see above), eq. (16) can be finally rewritten as

$$\langle j_\mu(\varphi) \rangle = 2 \langle K_\mu(a) \rangle . \qquad (17)$$

This is the relation we looked for.

One more test is provided by considering directly not the currents but the divergences. Namely, substituting expression (10) into eq. (9) and averaging over an external gravitational field we find

$$\langle \partial_\mu K_\mu(b) \rangle = \langle \frac{m^2}{2} \varphi_{\mu\nu}\tilde{\varphi}_{\mu\nu} + \frac{1}{2} a_{\mu\nu}\tilde{a}_{\mu\nu} + m\tilde{\varphi}_{\mu\nu}a_{\mu\nu} \rangle . \qquad (18)$$

Note that to consider the matrix elements of the current divergences one has to take proper care of the regulator fields (as far as matrix elements of the currents are concerned, it is mostly the problem of infrared regularization). Accounting for the regulator field $\varphi_{R\mu\nu}$ gives an extra term $(1/2)\langle M_R^2 \varphi_{R\mu\nu}\tilde{\varphi}_{R\mu\nu} \rangle$ where M_R is the regulator mass. It is just this term which results in the anomaly. As for the other terms in eq. (18) the piece $(1/2)\langle a_{\mu\nu}\tilde{a}_{\mu\nu} \rangle$ is identically equal to $-\langle \partial_\mu K^\mu \rangle$ while the last term, $\langle \tilde{\varphi}_{\mu\nu}a_{\mu\nu} \rangle$, can again be omitted. Thus, in the limit of vanishing mass m, $m \to 0$, we come again to the relation

$$\langle \partial_\mu j_\mu(\varphi) \rangle = 2 \langle \partial_\mu K^\mu(a) \rangle .$$

4. Hopefully, this somewhat lengthy though routine derivation does not conceal the simplicity of the result obtained. What we have proved is in fact the equivalence of Lagrangians (14) and (6) as far as loop calculations are concerned. On the other hand, that part Lagrangian (14) which contains field a_μ generates identically the same loops as propagation of massive field b_μ (see eq. (6)). This observation implies in turn that, upon subtracting the terms containing a_μ, Lagrangian (14) is equivalent to zero. Thus we come to a specific analogue of the BRST invariance. Relation (16) is just one particular manifestation of this triviality of the Lagrangian (14) (once a_μ is put to zero).

Once terms containing a_μ are subtracted, Lagrangian (14) allows for taking the limit of the vanishing mass $m=0$. To be more exact, one has to proceed first from the Proca formalism for the

ghost fields η_μ to the Feynman gauge using to this end the substitution similar to (10), $\eta_\mu \to \eta_\mu + \frac{1}{m}\partial_\mu \eta$. As a result each Proca field (3 degrees of freedom) is replaced by a Feynman field (4 degrees of freedom) plus a ghost scalar field of the opposite statistics.

Let us compare now Lagrangian (14) with $a_\mu = 0$ and that of the quantized massless tensor field. The quantization of the tensor field has been considered in a number of papers (see Refs [11—14]). In the Feynman gauge the full Lagrangian describes two ghost vector fields (also in the Feynman gauge) as well as three scalar fields of ordinary statistics. Thus, as compared to the massless limit of Lagrangian (14) there appears one extra scalar field. As a result once the interaction with the chiral current is introduced in the massless case just in the same way as specified by eq. (14) for the massive fields the equivalence of massless antisymmetric tensor and scalar fields appears apparent even off-mass-shell. The vanishing of the matrix element of the total chiral current, $\langle j_\mu(\varphi) + K_\mu(\eta, \bar{\eta}) \rangle = 0$, may be considered as manifestation of this general equivalence.

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