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B.G. Konopelchenko

ON OPERATOR REPRESENTATION OF INTEGRABLE EQUATIONS

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On Operator Representation of Integrable Equations

B.G. Konopelchenko

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

An algebraic operator $[L_i, L_k] = \sum_{l=1}^{d} \gamma_{ikl} L_l$ of the nonlinear equations integrable by the inverse spectral transform method is discussed.

A central point of the inverse spectral transform method (see, e. g. [1—4]) is a correspondence between a nonlinear differential equation and a certain set of auxiliary linear problems

$$L_i \Psi = 0\,, \quad i = 1, \ldots, d\,, \tag{1}$$

where L_i are linear (usually, differential) operators. Namely, a given nonlinear equation is a condition of the compatibility of the linear system (1) [1—4]. Algebraic (operator) forms of this compatibility condition are the well known Lax pair [5], the commutativity condition $[L_1, L_2] \equiv L_1 L_2 - L_2 L_1 = 0$ [6], Manakov's triad $[L_1, L_2] = BL_1$ [7] or Zakharov's algebraic system [8]. A rather general method of construction of the compatible multidimensional linear problems has been proposed recently in [9].

In the present paper we discuss the operator form of the compatibility condition for the linear system (1). A rather generic form of these compatibility conditions is

$$[L_i, L_k] = \sum_{l=1}^{d} \gamma_{ikl} L_l\,, \quad (i, k = 1, \ldots, d)\,, \tag{2}$$

where γ_{ikl} are some operators and $[L_i, L_k] \equiv L_i L_k - L_k L_i$. The relations (2) are equivalent to nonlinear equations for the coefficients of the operators L_i. In the generic case the number d of the auxiliary linear problems (1) and number of the independent variables $x_1, \ldots, x_d$ in the corresponding integrable equation coincide. We present here different examples of the nonlinear equations representable in the form (2).

We firstly consider the case of two independent variables x and y $(d=2)$. In this case the system (2) is

$$[L_1, L_2]=\gamma_1 L_1+\gamma_2 L_2. \tag{3}$$

For $\gamma_1=\gamma_2=0$ it is the commutativity condition $[L_1, L_2]=0$, for $\gamma_2=0$ (or $\gamma_1=0$) it is a Manakov's triad representation. The example of the linear equation which has the generic representation (3) is the equation

$$\Delta\varphi_y(x, y)-4\alpha\varphi_{xxy}+6\alpha^2(\varphi\cdot\Delta^{-1}(\varphi^2)_{xy})_x+6\alpha\beta\varphi_y\cdot\Delta^{-1}(\varphi^2)_{yy}=0, \tag{4}$$

where $\varphi(x, y)$ is a scalar function, $\Delta\equiv\alpha\partial_x^2-\beta\partial_y^2$, $\partial_x\equiv\frac{\partial}{\partial x}$, $\partial_y\equiv\frac{\partial}{\partial y}$, $\varphi_x\equiv\frac{\partial\varphi}{\partial x}$, $\varphi_y\equiv\frac{\partial\varphi}{\partial y}$ and α, β are arbitrary constants. Equation (4) is equivalent to the relation

$$[L_1, L_2]=-2\alpha\varphi_x L_1-\varphi_y L_2, \tag{5}$$

where

$$L_1=\partial_x\partial_y+\frac{1}{2}\varphi\partial_y-\frac{1}{4}\varphi_y-\frac{3}{4}\alpha\Delta^{-1}(\varphi^2)_{xy},$$

$$L_2=\alpha\partial_x^2+\beta\partial_y^2-\alpha\varphi\partial_x+\frac{\alpha}{2}\varphi_x+\frac{3}{4}\alpha\varphi^2-\frac{3}{2}\alpha\Delta^{-1}(\varphi^2)_{xx}. \tag{6}$$

The nonlocal equation (4) can be obviously rewritten as a local system by introducing the function $\Psi(x, y)$ such that $\Delta\Psi=(\varphi^2)_y$.

At $\beta=0$ equation (4) is reduced to a very simple equation

$$\varphi_{xy}-2\varphi\partial_x^{-1}(\varphi^2)_y=0. \tag{7}$$

An example of the system which has the Manakov's triad representation is the system

$$\Delta q(x, y)-\alpha(q_x)^2+\beta(q_y)^2-2\alpha\partial_y^{-1}U_x+2\beta\partial_x^{-1}U_y=0,$$

$$\Delta U(x, y)+2\alpha(Uq_x)_x-2\beta(Uq_y)_y=0. \tag{8}$$

For this system

$$L_1=\partial_x\partial_y+q_x\partial_y+U,$$

$$L_2=\alpha\partial_x^2+\beta\partial_y^2+2\beta q_y\partial_y+2\alpha\partial_y^{-1}U_x \tag{9}$$

and $\gamma_1=2\Delta q$ and $\gamma_2=0$. The system (8) is a stationary version $(q_t=U_t=0)$ of the system constructed in [10, eq. (5)]. A stationary limit of the known three-dimensional (t, x, y) integrable equations with the Manakov's triad representation (see e. g. [10—13]) gives the other examples of the two-dimensional equations which possess the representation (3) with $\gamma_2=0$.

It is interesting to note that at $\alpha=0$ the system (8) with $\varphi=q_x$ is reduced to equation (7). For $\beta=0$ it is reduced to (7) with the change $x\leftrightarrow y$.

Equation (7) is also the stationary version of equation (7.29) from Ref. [14] with the additional reduction $U^*=U\equiv\varphi$.

Emphasize now that for given integrable equation which possesses the representation (3) all the operators L_1 L_2, γ_1, γ_2 are not determined uniquelly but up to the transformations

$$L_i\to L_i'=\sum_{k=1}^{2}Q_{ik}L_k, \quad C_i\to C_i'=\sum_{k=1}^{2}C_k\tilde{Q}_{ki}, \tag{10}$$

where $C_1\equiv\gamma_1+L_2$, $C_2\equiv\gamma_2-L_1$ and Q_{ik} and $\tilde{Q}_{ki}$ are arbitrary differential operators which obey the constraint $\sum_{k=1}^{2}\tilde{Q}_{ik}Q_{ki}=\delta_{li}$. In the particular case $Q_{ii}=\tilde{Q}_{ii}\equiv1$, $Q_{12}=\tilde{Q}_{12}=0$ and $Q_{21}=-\tilde{Q}_{21}\equiv Q$ the transformation (10) is reduced to the transformation $L_1\to L_1'=L_1$, $L_2\to L_2'=L_2+QL_1$, $\gamma_1\to\gamma_1'=\gamma_1+[L_1, Q]-\gamma_2 Q$, $\gamma_2\to\gamma_2'=\gamma_2$. At $\gamma_2=0$ (Manakov's triad representation) these transformations coincide with those firstly considered in [14]. One can use the uncertainty (10) to find (similar to [10]) equivalent matrix triad or even commutativity ($[L_1', L_2']=0$) representations for general case (3). Note that the noncommutative representation of the two-dimensional equations different from (3) has been considered in [15].

In the generic three-dimensional case (x_1, x_2, x_3) one has three auxiliary linear problems (1) $(d=3)$ [9]. The simplest example of such system is the three-dimensional integrable chiral field type model which is described by the equation [9]

$$\frac{\partial U_{ik}^i}{\partial x^l}-U_{ik}^l U_{il}^i+U_{lk}^l U_{li}^i+U_{lk}^k U_{ik}^i=0, \tag{11}$$

where $U_{ik}^k=-\frac{\partial g_k}{\partial x^i}g_k^{-1}+g_k\frac{A_i}{\Lambda_i-\Lambda_k}g_k^{-1}$, $[A_i, A_k]=0$, $\Lambda_i(i=1, 2, 3)$ are constant and $g_k(x_1, x_2, x_3)$ $(k=1, 2, 3)$ are $N\times N$ matrices. The summation over repeated indices in (11) and further formulae is absent. In this case it is more convenient to use the double—indices notations. The corresponding linear operators are [9]

$$L_{ik}=\partial_{x_i}\partial_{x_k}+U^i_{ik}\partial_{x_i}+U^k_{ik}\partial_{x_k}, \quad (i,\ k=1,2,3) \qquad (12)$$

and the equation (2) is of the form

$$[L_{ik},L_{nk}]=\alpha_{ink}L_{in}+\beta_{ink}L_{ik}+\gamma_{ink}L_{nk}, \quad (i,\ k,\ n=1,2,3,\ i\neq k\neq n,\ i\neq n), \qquad (13)$$

where

$$\alpha_{ink}=\frac{\partial U^n_{nk}}{\partial x^k}-\frac{\partial U^i_{ik}}{\partial x^k}-[U^n_{nk},U^i_{ik}],$$

$$\beta_{ink}=\frac{\partial U^k_{nk}}{\partial x^i}-\frac{\partial U^i_{ik}}{\partial x^n}-[U^k_{nk},U^i_{ik}],$$

$$\gamma_{ink}=\frac{\partial U^n_{nk}}{\partial x^i}-\frac{\partial U^k_{ik}}{\partial x^k}-[U^n_{nk},U^k_{ik}].$$

In the two-dimensional limit (e. g. $\frac{\partial U^i_{ik}}{\partial x^3}=0$) the system (11) is reduced to the well known two-dimensional chiral field equations $(g_{x_2}g^{-1})_{x_1}+(g_{x_1}g^{-1})_{x_2}=0$ $(g\equiv g_3)$ where $U^3_{23}=-g_{x_2}g^{-1}$, $U^3_{13}=g_{x_1}g^{-1}$ and $U^1_{13}=-U^2_{23}\equiv 1$, $U^1_{12}=-\frac{1}{2}g_{x_2}g^{-1}$, $U^2_{12}=-\frac{1}{2}g_{x_1}g^{-1}$. The operators L_{ik} are reduced to the following three operators $(\partial_{x_3}\to\Lambda)$:

$$L_{12}\to\tilde{L}_3=\partial_{x_1}\partial_{x_2}-\frac{1}{2}g_{x_2}g^{-1}\partial_{x_1}-\frac{1}{2}g_{x_1}g^{-1}\partial_{x_2},$$

$$L_{13}\to\tilde{L}_1=(\Lambda+1)\partial_{x_1}-\Lambda g_{x_1}g^{-1}, \qquad (14)$$

$$L_{23}\to\tilde{L}_2=(\Lambda-1)\partial_{x_2}-\Lambda g_{x_2}g^{-1}.$$

The relations (13) are reduced to the following

$$[\tilde{L}_1,\tilde{L}_2]=0, \qquad (15a)$$

$$[\tilde{L}_3,\tilde{L}_1]=\frac{1}{2}(g_{x_2}g^{-1})_{x_1}\tilde{L}_1-\frac{1}{2}(g_{x_1}g^{-1})_{x_1}\tilde{L}_2,$$

$$[\tilde{L}_3,\tilde{L}_2]=-\frac{1}{2}(g_{x_2}g^{-1})_{x_2}\tilde{L}_1+\frac{1}{2}(g_{x_1}g^{-1})_{x_2}\tilde{L}_2, \qquad (15b)$$

The operators $\tilde{L}_1$, $\tilde{L}_2$ and the commutativity representation (15a) of the principal chiral equations are well known [16]. The system of

relations (15) characterises the two-dimensional principal chiral field equations as the stationary version of the corresponding three-dimensional equations (11). Note that the relations (15) have a structure which is distinguished from that discussed in [15].

In the degenerated (non generic) cases the three-dimensional integrable equations have two auxiliary spectral problems [8, 9]. In such situation the corresponding general compatibility condition is $[L_1,L_2]=\gamma_1L_1+\gamma_2L_2$. In particular, Zakharov's algebraic system, described in [8] for the operators $L_i=A_i(\partial_{x_3})\partial_{x_i}-B_i(\partial_{x_3})$ $(i=1,2)$ where A_i, B_i are the differential over x_3 operators, is equivalent to the operator equation $[L_1,L_2]=\gamma_1L_1+\gamma_2L_2$ with $\gamma_1=D_1(\partial_{x_3})\partial_{x_2}-C_1(\partial_{x_3})$, $\gamma_2=D_2(\partial_{x_3})\partial_{x_1}-C_2$ where D_i, C_i are differential over x_3 operators.

Emphasize that in the generic case one should solve simultaneously all the linear problems $L_i\Psi=0$ $(i=1,\dots,d)$ together, in contrast to the usual situation [1—4]. For example, for equation (4) one must compatibly solve the two two-dimensional linear problems $L_1\Psi=0$, $L_2\Psi=0$ where the operators L_1 and L_2 are given by (9).

Note also that the weak commutativity condition $[L_1,L_2]\Psi=0$ which has been discussed in [13] is equivalent to the Manakov's operator triad representation.

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B.G. Konopelchenko

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Б.Г. Конопельченко

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