ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

B.G. Konopelchenko

ON GENERALIZATION OF THE BACKLUND—CALOGERO TRANSFORMATIONS FOR INTEGRABLE EQUATIONS

PREPRINT 87-149

НОВОСИБИРСК

On Generalization of the Backlund — Calogero Transformations for Integrable Equations

B.G. Konopelchenko

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

It is shown that the consideration of dressing (gauge) transformations nonlocal on all spatial variables and spectral parameter allows one to extend the class of general Backlund — Calogero transformations for the Kadomtsev — Petviashvili equation.

A study of the recursion and group-theoretical properties of nonlinear equations integrable by the inverse spectral transform method (see e. g. [1—4]) is an important problem of the theory of nonlinear evolutions (see e. g. [2, 3, 5]). Recently an essential step has been done in the understanding of these properties for integrable equations in $1+2$ dimensions. Namely it was shown that the usual hierarchies of integrable equations in $1+2$ dimensions, their symmetries and Backlund — Calogero transformations are generated by a single bilocal recursion operator [6—13]. The bilocality on one of space variables (x or y) is an essential and common feature of these results. The bilocal approach is applicable to integrable equations in $1+1$ dimensions too [8, 10—12).

The purpose of the present letter is to demonstrate the possibility of constructing Backlund — Calogero transformations (BCTs) which are wider than those constructed earlier in [5—11]. These wider BCTs are related to the dressing (or gauge) transformations which are nonlocal on all spatial variables and spectral parameter. These generalized BCTs are calculated via a bilocal on all spatial variables and spectral parameter adjoint representation of given spectral problem. These generalized BCTs seems include also a t-, x-, y-dependent symmetries which were considered in [14—18, 7].

We will consider here a well known Kadomtsev — Petviashvili (KP) equation ($\sigma^2=\pm 1$)

$$U_t(x,y,t)=U_{xxx}+6UU_x+3\sigma^2\partial_x^{-1}U_{yy}. \qquad (1)$$

The equation (1) is integrable by the two-dimensional problem [1, 2]

$$L_{x,y}\Psi \stackrel{def}{=} (\sigma\partial_y + \partial_x^2 + U(x,y,t))\Psi = 0\,. \qquad (2)$$

A change $\Psi\to\hat{\Psi}$ given by $\Psi=\exp\left(i\lambda x+\frac{1}{\sigma}\lambda^2 y\right)\hat{\Psi}(x,y,\lambda)$ $(\lambda\in C)$ converts (2) into the spectral problem

$$(L_{x,y}+2i\lambda\partial_x)\hat{\Psi}(x,y,\lambda)=0\,. \qquad (3)$$

The spectral problem (3) has appeared in the framework of $\bar{\partial}$-approach to the KP equation (see e. g. [19]). This spectral problem is our starting point too.

Let us consider the completely nonlocal gauge (dressing) transformations

$$\hat{\Psi}(x,y,\lambda)\to\hat{\Psi}'(x',y',\lambda')=\int d\lambda\,dx\,dy\,G(x',x;y',y;\lambda',\lambda)\,\hat{\Psi}(x,y,\lambda) \qquad (4)$$

for the problem (3) and assume that:

$$(L'_{x',y'}+2i\lambda'\partial_{x'})\hat{\Psi}'(x',y',\lambda')\equiv(\sigma\partial_{y'}+\partial_{x'}^2+U'(x',y')+2i\lambda'\partial_{x'})\hat{\Psi}'=0. \qquad (5)$$

As a result G obeis the equation

$$\frac{i}{2}(L'_{x',y'}-L^+_{x,y})\,G(x',x;y',y;\lambda',\lambda)=(\lambda'\partial_{x'}+\lambda\partial_x)\,G \qquad (6)$$

where $L^+=-\sigma\partial_y+\partial_x^2+U(x,y)$ is the operator formally adjoint to L. Note that the bilocal quantity $\Phi(x',x;y',y;\lambda',\lambda)\stackrel{def}{=}\hat{\Psi}'(x',y',\lambda')\check{\Psi}(x,y,\lambda)$ where $(L^+_{x,y}-2i\lambda\partial_x)\check{\Psi}=0$ obeis equation (6). Equation (6) is the bilocal on x, y and λ adjoint representation of the problem (3).

An application of the completely local gauge transformations (i. e. $G=\delta(\lambda'-\lambda)\delta(y'-y)\delta(x'-x)\tilde{G}$) for the construction of the BCTs in 1+1 dimensions has been proposed in [20, 21]. The transformations (4) local on λ and bilocal either on x $(G=\delta(\lambda'-\lambda)\delta(y'-y)\tilde{G})$ or y $(G=\delta(\lambda'-\lambda)\delta(x'-x)\tilde{G})$ have been used in [7—9, 12]. Infinitesimal dressing transformations (4) nonlocal on λ and local on x and y $(G=\delta(x'-x)\delta(y'-y)\tilde{G})$ have been considered in [16].

Note that equation (6) is equivalent to the following

$$\frac{i}{2}(L'_{x',y'}-L^+_{x,y})\,G=(\lambda_+\partial_++\lambda_-\partial_-)\,G \qquad (7)$$

where $\lambda_\pm\stackrel{def}{=}\frac{1}{2}(\lambda'\pm\lambda)$ and $\partial_\pm\stackrel{def}{=}\partial_{x'}\pm\partial_x$.

A possibility of constructing different nonlinear transformations associated with the KP equation (1), in particular, the generalized BCTs is connected with a choice of different Ansatzs for G.

Here we will consider some simplest cases. Let us choose G as

$$G=\delta(\lambda'-\lambda)\sum_{n=0}^{N}\lambda_+^n\,\varphi_n(x',x;y',y,\lambda_+)$$

where $\varphi_0=\delta(x'-x)\delta(y'-y)\tilde{\varphi}_0$. Substituting such G into (7) one obtains

$$\Delta(\partial_+\Lambda_+\varphi_0)=0\,, \qquad (8a)$$

$$\partial_+\varphi_N=0\,,\qquad \Lambda_+\varphi_n=\varphi_{n-1}\qquad (n=1,\dots,N) \qquad (8b)$$

where Δ is a projection operator onto the diagonal $x'=x$, $y'=y$: $\Delta Q(x',x,y',y)\stackrel{def}{=}Q|_{x'=x,y'=y}$ and the operator Λ_+ is

$$\Lambda_+=\partial_+^{-1}\frac{i}{2}(L'_{x',y'}-L^+_{x,y})=$$

$$=\frac{i}{2}(\partial_{x'}+\partial_x)^{-1}(\sigma(\partial_{y'}+\partial_y)+\partial_{x'}^2-\partial_x^2+U'(x',y')-U(x,y))\,. \qquad (9)$$

The relations (8b) give $\varphi_N=f_N(x'-x,y',y)$ where f_N is an arbitrary function and $\varphi_0=\Lambda_+^N f_N$. Substituting this expression for φ_0 into (8a) we finally obtain

$$\Delta(\partial_+\Lambda_+^{N+1}f_N)=0\,. \qquad (10)$$

The consideration of the infinitesimal gauge transformations (4) $(\Psi'=\Psi+\delta\Psi,\ \delta\Psi=\varepsilon\Psi_t)$ with the same Ansatz for G gives the hierarchy of the integrable equations

$$U_t(x,y,t)=\Delta(\partial_+\Lambda_+^{N+1}\cdot 1) \qquad (11)$$

and their symmetry transformations

$$\delta U=\Delta(\partial_+\Lambda_+^{N+1}\cdot\tilde{f}_N)\,. \qquad (12)$$

In formulae (11) and (12) one must put $U'\equiv U(x',y',t)$ in the operator Λ_+.

Note that $\Delta=\Delta_x\Delta_y$ where Δ_x and Δ_y are the projection operations onto the diagonal $x'=x$ and $y'=y$ respectively: $\Delta_x Q(x',x;y',y)=Q|_{x'=x}$, $\Delta_y Q(x',x;y',y)=Q|_{y'=y}$. The operator Λ_+ contains the derivatives $\partial_{y'}$ and ∂_y only in the combination $\partial_{y'}+\partial_y$ and hence it admits a direct projection onto the diagonal $y'=y$. As a result the BCTs (10) and equations (11) can be rewritten in the form

$$\Delta_x(\partial_+ \Lambda_{x',x}^{N+1} f_N)=0 \tag{13}$$

and

$$U_t(x,y,t)=\Delta_x(\partial_+ \Lambda_{x',x}^{N+1}\cdot 1) \tag{14}$$

where $\Lambda_{x',x} \overset{def}{=} \Delta_y \Lambda_+$ is the operator bilocal on x:

$$\Lambda_{x',x}=(\partial_{x'}+\partial_x)^{-1}(\sigma\partial_y+\partial_{x'}^2-\partial_x^2+U'(x',y)-U(x,y)).$$

The operator Λ_+ does not admit a direct projection onto $x'=x$. But one can easily check that an action of the operator $\Delta_x\Lambda_+^2$ on the vector fields of the form $\Lambda_+^m\cdot 1$ is equivalent to the action of the bilocal on y operator

$$L_{y',y}=-\frac{1}{4}\{\partial_x^2+2\sigma(\partial_{y'}-\partial_y)+U'+U+\partial_x^{-1}(U'+U)\partial_x+$$
$$+\partial_x^{-1}(\sigma(\partial_{y'}+\partial_y)+U'-U)\,\partial_x^{-1}(\sigma(\partial_{y'}+\partial_y)+U'-U)\}$$

where $U'\equiv U'(x,y')$. Correspondingly the BCTs (10) and equations (11) can be reperesented in the forms $(N=2M+1)$

$$\Delta_y(\partial_x \Lambda_{y',y}^{M+1} f_M(y',y))=0 \tag{15}$$

and

$$U_t(x,y,t)=\Delta_y(\partial_x \Lambda_{y',y}^{M+1}\cdot 1). \tag{16}$$

The operator $\Lambda_{x',x}$ (up to the factor $i/2$), the general BCTs (13), the hierarchy (14) and their symmetries coincide with the bilocal on x recursion operator, the KP hierarchy and its symmetries constructed in [8, 12] by another approach. The operator $\Lambda_{y',y}$ bilocal on y, the BCTs (15), equations (16) and their symmetries coincide with those constructed in [6, 7, 9—11].

Now let us choose G in the form

$$G=\delta(\lambda'+\lambda)\sum_{m=0}^{M}\lambda_-^m\,\chi_m(x',x,y',y,\lambda_-).$$

Using this Ansatz for G we obtain from (7) the following BCTs

$$\Delta(\partial_-\Lambda_-^{M+1}\xi_M)=0 \tag{17}$$

where $\xi_M=\xi_M(x'+x,y',y)$ is an arbitrary function and

$$\Lambda_-\equiv\partial_-^{-1}\frac{i}{2}(L'_{x',y'}-L_{x,y}^+)=$$
$$=\frac{i}{2}(\partial_{x'}-\partial_x)^{-1}(\sigma(\partial_{y'}-\partial_y)+\partial_{x'}^2-\partial_x^2+U'(x',y')-U(x,y)). \tag{18}$$

The corresponding infinitesimal symmetry transformations are

$$\delta U=\Delta(\partial_-\Lambda^{M+1}\xi_M(x'+x,y',y)).$$

It is easy to see that $\partial_+\Lambda_+=\partial_-\Lambda_-=\frac{i}{2}(L'_{x',y'}-L_{x,y}^+)$. So in fact the operator $L'_{x',y'}-L_{x,y}^+$ plays a central role in our approach.

Emphasize also that

$$\Lambda_+(x',x,y',y)\,\Phi(\lambda,\lambda)=\lambda\,\Phi(x',x,y',y;\lambda,\lambda)$$

and

$$\Lambda_-(x',x,y',y)\,\Phi(\lambda,-\lambda)=\lambda\,\Phi(x',x,y',y;\lambda,-\lambda)$$

where $\Phi(x',x,y',y;\lambda',\lambda)\overset{def}{=}\hat{\Psi}'(x',y',\lambda')\check{\Psi}(x,y,\lambda)$.

Transformations and formulae (10) — (18) can be easily derived also for the Ansatzs $G=\delta(\lambda')\tilde{G}$ and $G=\delta(\lambda)\tilde{G}$. The corresponding results are given by (8) — (18) with an obvious change $\partial_+\to\partial_{x'}$, $\partial_-\to\partial_x$. In this case $f_N=f_N(x,y',y)$ and $\xi_M=\xi_M(x',y',y)$.

At last for the Ansatz

$$G=\delta(\lambda'^2-\lambda^2-4)\sum_{n=0}^{N}\lambda_+^n\,\varphi_n(x',x,y',y;\lambda_+)$$

the relation (7) gives

$$\partial_+\varphi_N=0,\qquad \frac{i}{2}(L'_{x',y'}-L_{x,y}^+)\,\varphi_0=\partial_-\varphi_1,\qquad \partial_-\varphi_0=0 \tag{19}$$

and

$$\frac{i}{2}(L'_{x',y'}-L_{x,y}^+)\,\varphi_n=\partial_+\varphi_{n-1}+\partial_-\varphi_{n+1}\qquad(n=1,\ldots,N-1).$$

As a result the generalized BCTs are of the form

$$\Delta(P_N(\Lambda_-,\partial_-^{-1}\partial_+)\,\varphi_0(x'+x,y',y))=0$$

where P_N is polynomial on Λ_- and $\partial_-^{-1}\partial_+$ the form of which is determined by the recurrent relation (19).

In a similar manner one can consider the general case $G=\delta(f(\lambda',\lambda))\,G(x',x,y',y,\lambda)$ where $f(\lambda',\lambda)$ is a some function. For example, at $f=\lambda'-\lambda^2$ and $G=\sum_{n=0}^{N}\lambda^n\varphi_n$ the generalized BCTs are given by the relation

$$\Delta(\partial_x P_N(\Lambda,\partial_x^{-1}\partial_{x'})\cdot 1)=0$$

where the polynomial P_N is determined by the recurrent relation $L\varphi_n=\partial_{x'}\varphi_{n-2}+\partial_x\varphi_{n-1}$ $(\varphi_N=\varphi_{N-1}=1)$ and the operator $L=\partial_x^{-1}\times\times\frac{i}{2}(L'_{x',y'}-L^+_{x,y})$.

In the one-dimensional limit $\partial_y\to 0$, $\partial_{y'}\to 0$ all these formulae give the corresponding generalized transformations and symmetries for the Korteweg—de Vries equations.

One can obtain the similar results for the matrix problem $(\partial_x+A\partial_y+P(x,y,t))\Psi=0$ and the problem $(\partial_x^2-\sigma^2\partial_y^2+ +\varphi(x,y)(\partial_x+\sigma\partial_y)+U(x,y))\Psi=0$ too.

I am thankful to F. Magri, A. Orlov and P. Santini for useful discussions.

REFERENCES

1. *Zakharov V.E., Manakov S.V., Novikov S.P., Pitaèvsky L.P.* Soliton Theory. — Moscow, Nauka, 1980.
2. *Ablowitz M.Y., Segur H.* Solitons and Inverse Scattering Transform. — SIAM, Philadelphia, 1981.
3. *Calogero F., Degasperis A.* Spectral Transform and Solitons, I. — North-Holland, Amsterdam, 1982.
4. *Newell A.C.* Solitons in Mathematics and Physics. — SIAM, Philadelphia, 1985.
5. *Konopelchenko B.G.* Nonlinear Integrable Equations. — Lect. Notes in Physics, v.270, Springer, Heidelberg, 1987.
6. *Fokas A.S., Santini P.M.* Stud. Appl. Math., 1986, v.75, p.179.
7. *Santini P.M., Fokas A.S.* Preprints INS 65, 67. Clarkson University, 1986.
8. *Santini P.M., Fokas A.S.* Preprint INS 79. Clarkson University, 1987.
9. *Boiti M., Leon Y.Y.P., Pempinelli F.* Preprint Montpellier PM/87-03, 1987.
10. *Konopelchenko B.G.* Phys. Lett., 1987, v.123A, p.451.
11. *Konopelchenko B.G.* Preprint INP 87-86. Novosibirsk, 1987.
12. *Boiti M., Leon Y.Y.P., Martina L., Pempinelli F.* Preprint Montpellier PM/87-25, 1987; Phys. Lett. A (to appear).
13. *Magri F., Morosi C., Tondo G.* Preprint Milano University, 1987.
14. *Chen H.H., Lee Y.C.* Physica, 1983 v.9D, p.439.

Chen H.H., Lee Y.C. Physica, 1987 v.26D, p.171.

15. *Schwarz F.* J. Phys. Japan, 1982, v.51, p.2387.
16. *Orlov A.Yu., Schulman E.I.* Preprint IA and E, N 217, 1984; N 277, 1983; Teor. Math. Fiz., 1985, v.64, p.323; Lett. Math. Phys., 1986, v.12, p.171.
17. *Fuchssteiner B.* Lect. Notes in Physics, 1985, v.216, p.305.
18. *David D., Kamran N., Winternitz P.* Phys. Rev. Lett., 1985, v.55, p.2111.
19. *Ablowitz M.Y., Fokas B.S.* Lect. Notes in Physics, 1982, v.189, p.137. *Ablowitz M.Y., Nachman A.I.* Physica, 1986, v.18D, p.223.
20. *Boiti M., Tu G.Z.* Nuovo Cim., 1982, v.B71, p.253.
21. *Levi D., Ragnisco O., Sym A.* Lett. Nuovo. Cim., 1982, v.33, p.401.

B.G. Konopelchenko

On Generalization of the Backlund — Calogero Transformations for Integrable Equations

Б.Г. Конопельченко

Обобщенные преобразования Бэклунд — Калоджеро для интегрируемых уравнений

Ответственный за выпуск С.Г.Попов

Работа поступила 22 октября 1987 г.
Подписано в печать 6.11 1987 г. МН 08440
Формат бумаги 60×90 1/16 Объем 0,8 печ.л., 0,7 уч.-изд.л.
Тираж 200 экз. Бесплатно. Заказ № 149

Набрано в автоматизированной системе на базе фотонаборного автомата ФА1000 и ЭВМ «Электроника» и отпечатано на ротапринте Института ядерной физики СО АН СССР,
Новосибирск, 630090, пр. академика Лаврентьева, 11.