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V.V Sokolov, V.G. Zelevinsky

ON STATISTICAL THEORY OF OVERLAPPING RESONANCES

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On Statistical Theory of Overlapping Resonances

V.V Sokolov, V.G. Zelevinsky

Institute of Nuclear Physics
630090, Novosibirsk, USSR

ABSTRACT

The statistics of unstable N-level quantum systems are considered. The distribution function for complex energies and the density of overlapped resonances are obtained. It is shown that the instability removes the level repulsion at the energy distance less than the width meanwhile the complex energies repulse quadratically. In the case of the strong overlapping k «collective» states (k is the number of channels) are formed absorbing the total width.

1. The statistical approach is only possible one for description of high-lying levels of complex quantum systems. Statistical spectroscopy dated back to Wigner and Dyson (see [1] and references therein) grows more and more important for various fields of quantum physics [2]. It is found now [3] that the statistics of quantum spectra are closely connected with the local instability of trajectories in the phase space of corresponding classical systems [4].

Actually, the excited states have a finite lifetime. Their properties are studied by means of reactions. In the statistical analysis, reaction amplitudes are assumed to be random variables. The Porter — Thomas distribution for neutron widths of separated resonances and Ericson fluctuations of cross sections in the case of overlapping resonances are typical examples of corresponding statistical regularities. In Refs [5] the statistical approach to nuclear reactions has got further development.

One can construct the general phenomenological description of resonance reactions in terms of the nonhermitian Hamiltonian $\hat{\mathcal{H}}=\hat{H}-\frac{i}{2}\hat{\Gamma}$. Eigenvalues $\mathcal{E}_n=E_n-\frac{i}{2}\Gamma_n$ of an operator $\hat{\mathcal{H}}$ give energies E_n and widths Γ_n of unstable states. The amplitude T^{ba} of reaction $a\rightarrow b$ can be expressed with the use of matrix elements $\mathcal{H}_{mn}$ and decay amplitudes A_n^a as

$$T^{ba}(E)=\sum_{m,n} A_m^{b*}\left(\frac{1}{E-\hat{\mathcal{H}}}\right)_{mn} A_n^a . \qquad (1)$$

Here the superscripts a, $b=1, 2, \ldots, k$ label the reaction channels whereas subscripts m, $n=1, 2, \ldots, N$ run over the basis states of the decaying system. Due to the unitarity condition, the matrix elements Γ_{mn} of the antihermitian part $\hat{\Gamma}$ are related to the amplitudes A_n^a:

$$\Gamma_{mn}=\sum_a A_m^a A_n^{a*}. \tag{2}$$

Such an approach makes it possible to interconnect the statistical spectroscopy as it was formulated in pioneering papers [1] with the statistics of reaction amplitudes. Important results along this line were obtained by Weidenmüller with coworkers [7—9] considering the statistical properties of the amplitudes of processes going through unstable intermediate states.

As distinct from Refs [7—9] we will study the statistical distribution of resonances as revealed by spectra of random nonhermitian Hamiltonians. In this respect our treatment is more close to the traditional one.

2. Let us formulate our statistical hypotheses. As usually, we assume the hermitian part $\hat{H}$ of the total effective Hamiltonian $\hat{\mathscr{H}}$ to belong to the Gaussian Orthogonal Ensemble (GOE) of random matrices $N\times N$. Physically, N is determined by the number of states decaying into k common channels. As it well known [1] the distribution function for the eigenvalues ε_n of such matrices is given by

$$\mathscr{P}(\varepsilon_1, \ldots, \varepsilon_N)=C_N \prod_{m<n} |\varepsilon_m-\varepsilon_n| \exp\left(-\frac{N}{4a^2}\sum_n \varepsilon_n^2\right). \tag{3}$$

Similar to Ref. [8] we introduce the factor N in the exponent in order to locate for $N\gg 1$ the eigenvalues in the fixed interval $\sim a$. Then the mean distance D between adjacent eigenvalues is of order a/N. The distribution (3) predicts the linear level repulsion at small distances.

3. Our main goal is to investigate the influence of the coupling with continuum on the statistical properties of spectrum. It is physically obvious that the finite level width should wash away the strong correlations at small energy distances.

Concerning the statistical properties of the antihermitian part $\hat{\Gamma}$, we formulate them in terms of the random channel amplitudes A_n^a. As in Refs [8, 9] we assume them to be real. It implies that the total Hamiltonian matrix $\mathscr{H}_{mn}$ is symmetric. It is the natural generali-

zation of GOE of symmetric matrices used for T-invariant stable systems.

Let the decay amplitudes A_n^a into various channels be statistically independent Gaussian random variables so that

$$\langle A_m^a A_n^b \rangle=\frac{1}{N}\delta^{ab}\delta_{mn}\gamma^a. \tag{4}$$

Here again we have introduced the scale factor $1/N$ to ensure the independence of $\mathrm{Tr}\,\hat{\Gamma}$ of N. For equivalent channels ($\gamma^a=\gamma=\mathrm{const}$) one obtains from (4) the chi-square distribution of diagonal matrix elements $\Gamma_{nn}\equiv\gamma_n>0$ (the Porter—Thomas distribution in the one-channel case $k=1$)

$$\mathscr{P}(\gamma_n)=\frac{1}{\left(\frac{k}{2}-1\right)!}\,\frac{N}{2\gamma}\left(\frac{N\gamma_n}{2\gamma}\right)^{k/2-1}\exp\left(-N\frac{\gamma_n}{2\gamma}\right). \tag{5}$$

In particular, this formula directly describes the width distribution of isolated resonances.

It should be noted that, independently of statistical hypotheses, the algebraic structure (2) of the matrix $\hat{\Gamma}$ leads to the important conclusion concerning its eigenvalues: if the channel number k is less than the level number N there exist in general only k nonzero eigenvalues of this matrix.

4. Having the statistical ensemble of matrices $\hat{\mathscr{H}}$ completely determined we will find out the distribution of complex energies $\mathscr{E}_n$. The ensemble under study is invariant with respect to orthogonal transformations. A general nonhermitian matrix $\hat{\mathscr{H}}$ can't be diagonalized by a transformation of such kind. Nevertheless, one can diagonalize with such a transformation either hermitian $\hat{H}$ or antihermitian $\hat{\Gamma}$ part of the total Hamiltonian $\hat{\mathscr{H}}$. Correspondingly, one has two versions of secular equation for $\mathscr{E}_n$.

We write down these equations for the one-channel case:

$$1+\frac{i}{2}\sum_{n=1}^{N}\frac{\gamma_n}{\mathscr{E}-\varepsilon_n}\equiv 1+\frac{i}{2}R(\mathscr{E})=0, \tag{6a}$$

$$\mathscr{E}-\sum_{\nu=2}^{N}\frac{h_\nu^2}{\mathscr{E}-\varepsilon_\nu}-h+\frac{i}{2}w\equiv\tilde{R}(\mathscr{E})-h+\frac{i}{2}w=0. \tag{6b}$$

In the second case we have used the degeneracy of $N-1$ zero eigenvalues of $\hat{\Gamma}$ to diagonalize, in addition, the $(N-1)\times(N-1)$ submatrix of the hermitian part, ε_ν $(\nu=2,3,\dots,N)$ being the eigenvalues of this submatrix. In (6b) $\Gamma_{11}=\mathrm{Tr}\,\hat{\Gamma}\equiv w$ is the sole nonzero matrix element of $\hat{\Gamma}$, $h=H_{11}$ and $h_\nu=H_{1\nu}=H_{\nu 1}$ are the elements of the first row (column) of $\hat{H}$ after additional diagonalization.

To obtain the distribution function for energies and widths of unstable states we have to convert from variables $\{\varepsilon_n,\gamma_n\}$ (or $\{h,w,h_\nu,\varepsilon_\nu\}$ in the second case) to the physical ones $\{E_n,\Gamma_n\}$. One can calculate the Jacobian of the transformation with the help of the secular equation (6a) (or (6b)). The result is

$$\mathscr{P}(E_1,\dots,E_N;\Gamma_1,\dots,\Gamma_N)=C_N\prod_{m<n}\frac{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m-\Gamma_n)^2}{\sqrt{(E_m-E_n)^2+\frac{1}{4}(\Gamma_m+\Gamma_n)^2}}\prod_n\frac{1}{\sqrt{\Gamma}}\times$$

$$\times\exp\left(-\frac{N}{4a^2}\sum_n E_n^2-\frac{N}{4a^2}\sum_{m<n}\Gamma_m\Gamma_n-\frac{N}{2\gamma}\sum_n\Gamma_n\right). \qquad (7)$$

The properties of this distribution depend on the value of the parameter $\varkappa=\gamma/a$. In the case $\varkappa\ll 1$ the typical value $\Gamma\sim\gamma/N\sim\varkappa D\ll D$, so that for $|E_m-E_n|\gg\varkappa D$ eq. (7) reduces to the product of independent distributions (3) and (5) for energies and widths respectively. However for $|E_m-E_n|<\varkappa D$ the finiteness of widths removes the level repulsion. On the other hand, the probability of coincidence of two complex eigenvalues is suppressed quadratically in accordance to the lack of T-invariance in decaying systems.

In the opposite limit of $\varkappa\gg 1$ the above mentioned (see the last paragraph of the previous section) property of the matrix $\hat{\Gamma}$ becomes essential. In particular, for the one-channel case one level only has a large width whereas the remaining widths are small. Generally, in the case of k channels $(k<N)$ k «collective» rapidly decaying states are formed. Just such a picture has been observed earlier in numerical simulation of nuclear reactions [10].

5. The main density of states of an unstable system is

$$\rho(E,\Gamma)=\langle\sum_{n=1}^{N}\delta(E-E_n)\,\delta(\Gamma-\Gamma_n)\rangle=$$

$$=N\int\prod_{n=2}^{N}dE_n\,d\Gamma_n\,\mathscr{P}(E,E_2,\dots,E_N;\Gamma,\Gamma_2,\dots,\Gamma_N). \qquad (8)$$

The direct calculation of the integral in eq. (8) is here even more difficult than in the case of the GOE [1]. As usually, it is more convenient to start with the calculation of the expectation value

$$g(E,\Gamma)=\frac{1}{N}\langle\mathrm{Tr}\,(\mathscr{E}-\hat{\mathscr{H}})^{-1}\rangle\equiv\frac{1}{N}\langle\mathrm{Tr}\,\hat{G}(\mathscr{E})\rangle,$$

$$\mathscr{E}=E-\frac{i}{2}\Gamma \qquad (9)$$

of the trace of the Green function $\hat{G}(\mathscr{E})$. The problem of finding $\rho(E,\Gamma)$ is mathematically equivalent to the reconstruction of the two-dimensional charge density from the given electrostatic field. Drawing on this analogy one obtains

$$\rho(E,\Gamma)=\frac{N}{4\pi}\left[\frac{\partial g(E,\Gamma)}{\partial E}-2i\,\frac{\partial g(E,\Gamma)}{\partial\Gamma}\right]. \qquad (10)$$

It means that $\rho(E,\Gamma)$ does not vanish in those points of the plane (E,Γ) where the function $g(E,\Gamma)$ of the complex variable $\mathscr{E}$ is nonanalytic.

One can verify that in the basis where the hermitian part $\hat{H}$ is diagonal

$$\mathrm{Tr}\,\hat{G}(\mathscr{E})=\mathrm{Tr}\,\hat{G}_0(\mathscr{E})+K(\mathscr{E}), \qquad (11)$$

$$K(\mathscr{E})=\frac{i}{2}\,\frac{dR(\mathscr{E})}{d\mathscr{E}}\left[1+\frac{i}{2}R(\mathscr{E})\right]^{-1}, \qquad (12)$$

where the function $R(\mathscr{E})$ is defined in eq. (6a) and $\hat{G}_0(\mathscr{E})=(\mathscr{E}-\hat{H})^{-1}$ is the Green function of a stable system. As it should be, the only singularities of $\mathrm{Tr}\,\hat{G}(\mathscr{E})$ are poles in the points where eq. (6a) is satisfied. The contributions of the poles at $\mathscr{E}=\varepsilon_n$ cancel in the sum of two terms in the r.h.s. of eq. (11). Therefore for $\Gamma>0$ one should take into account in eq. (10) the contribution of $K(\mathscr{E})$ only. After some algebra one obtains

$$\rho(E,\Gamma)=\frac{1}{8}\left\langle\left|\frac{dR(\mathscr{E})}{d\mathscr{E}}\right|^2\delta^{(2)}\left(1+\frac{i}{2}R(\mathscr{E})\right)\right\rangle= \qquad (13a)$$

$$=-\frac{1}{2\pi^2}\,\frac{\partial^2}{\partial\mathscr{E}\,\partial\mathscr{E}^*}\int\frac{d^2\lambda}{|\lambda|^2}\,e^{-2i\,\mathrm{Im}\,\lambda}\times$$

$$\left(\left\langle\exp\left\{-\frac{1}{2}\mathrm{Tr}\ln\left(\hat{I}-i\frac{\gamma}{N}[\lambda^*\hat{G}_0(\mathscr{E})+\lambda\,\hat{G}_0(\mathscr{E}^*)]\right)\right\}\right\rangle_\varepsilon-1\right). \qquad (13b)$$

In eq. (13b) we have used the integral representation of the two-dimensional Dirac δ-function and have carried out the averaging over random variables γ_n (see eq. (5)). Thus the problem is reduced to the averaging over the GOE.

6. In the limit $N\to\infty$ for $\varkappa\gg 1$ we get from (13)

$$\rho(E,\Gamma)=\frac{1}{8}\gamma^2\left|\frac{\partial g_0(\mathscr{E})}{d\mathscr{E}}\right|^2\delta^{(2)}\left(1+\frac{i}{2}\gamma g_0(\mathscr{E})\right) \tag{14}$$

where

$$g_0(\mathscr{E})=\lim_{N\to\infty}\frac{1}{N}\langle \mathrm{Tr}\,\hat{G}_0(\mathscr{E})\rangle_\varepsilon=\frac{1}{2a^2}(\mathscr{E}-\sqrt{\mathscr{E}^2-4a^2}\,). \tag{15}$$

It should be stressed that the density function (14) has the normalization

$$\int_{-\infty}^{\infty}dE\int_{+0}^{\infty}d\Gamma\,\rho(E,\Gamma)=1 \tag{16}$$

while the function (13) is normalized to N (see eq. (8)). The discrepancy is due to the fact that in the limit under consideration only one pole has the nonvanishing imaginary part whereas the remaining $N1$ ones drift to the real axis.

Such a behaviour of the eigenvalues follows from the secular equation (6a) which is equivalent to the two real ones

$$\frac{1}{4}\Gamma\sum_{n=1}^{N}\frac{\gamma_n}{(E-\varepsilon_n)^2+\frac{1}{4}\Gamma^2}=1;\quad \frac{1}{4}\Gamma\sum_{n=1}^{N}\frac{\gamma_n\varepsilon_n}{(E-\varepsilon_n)^2+\frac{1}{4}\Gamma^2}=E. \tag{17}$$

Supposing that $(E-\varepsilon_n)^2\sim a^2\ll\frac{1}{4}\Gamma^2$ (i. e. $\varkappa^2/4\gg 1$), one can convince oneself that the solution $\Gamma\approx w$, $E\approx\frac{1}{w}\sum_{n=1}^{N}\gamma_n\varepsilon_n\to 0$ exists since $\gamma_n\sim w/N$ and the distribution of ε_n around the point $\varepsilon=0$ is symmetric. The first correction to that width is of order $4w/\varkappa^2$ and for other $N-1$ eigenvalues $\Gamma_n\sim 4w/\varkappa^2N\to 0$ because of the condition $\sum_{n=1}^{N}\Gamma_n=w$. Just that «collective» solution $\Gamma=w=\gamma$, $E=0$ is the only one contributing to the asymptotic density function (14). As to

other states, they are distributed over the interval $(-2a,2a)$ of the real axis according to the standard semicircle rule [1].

Quite similarly one can take an average of the S-matrix. In the one-channel case

$$S(E)=1-i\,T(E)=\left[1-\frac{i}{2}R(\mathscr{E})\right]\left[1+\frac{i}{2}R(\mathscr{E})\right]^{-1}. \tag{18}$$

In the same approximation as above one obtains

$$\langle S(E)\rangle=\frac{1-\frac{i}{2}\gamma g_0(E)}{1+\frac{i}{2}\gamma g_0(E)} \tag{19}$$

in agreement with [9].

Thus, we have shown that the level statistics of unstable systems differs significantly from that of stable systems.

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REFERENCES

1. Statistical Theories of Spectra: Fluctuations.— N.Y.: Academic Press (1965).
2. *Brody J.A. et al.* Rev. Mod. Phys., 1981, v.53, p.385.
3. *Berry M.V.* In: Chaotic Behaviour of Deterministic Systems. Amsterdam, North-Holland, 1983, p.171.
4. *Chirikov B.V.* Phys. Rep., 1979, v.52, p.263.
5. *Mello P.A.* Phys. Lett., 1979, v.81B, p.103;
   Mello P.A., Seligman J.H. Nucl. Phys., 1980, v.A344, p.489.
6. *Kobzarev I.Y.* In: Material of VII Winter School of LINP, (1972), p.298.
7. *Agassi D. et al.* Phys. Rep., 1975, v.C22, p.145.
8. *Verbaarschot J.J.M. et al.* Phys. Rev. Lett., 1984, v.52, p.1597.
9. *Weidenmüller H.A.* Ann. Phys., 1984, v.158, p.120.
10. *Kleinwächter, Rotter I.* Phys. Rev., 1985, v.C32, p.1742.
11. *Pastur L.A.* Theor. Math. Phys., 1972, v.10, p.102.

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В.Г. Зелевинский, В.В Соколов

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