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DYNAMICAL MECHANISMS OF NOISE

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НОВОСИБИРСК

Dynamical Mechanisms of Noise*)

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ABSTRACT

Using simple examples, a comparative analysis of the two different mechanisms of intrinsic noise in a dynamical system is presented. The first, traditional, one with a discrete spectrum is related to a very big number of freedoms $N\to\infty$, and frequencies $N_\omega\to\infty$, of a completely integrable system, both classical as well as quantum. The additional statistical hypothesis on the statistical ensemble, and restrictions of the initial conditions are required. The second, new mechanism of noise with a continuous spectrum is based upon the exponential local instability of motion in a nonintegrable classical or semiclassical system, and it does not depend on N. This mechanism allows a purely dynamical approach to the statistical laws, and opens a new domain for their application to simple ($N\sim 1$) systems. However, in quantum mechanics only the first mechanism is operative as $N_\omega\to\infty$, independently of N, but also under restrictions on the initial conditions. Some developments of the random matrix theory are discussed, which describe the global structure of chaotic eigenstates of a quantum system.

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A distinctive feature of a «statistical system», as the object of the statistical mechanics, is the principal role of random fluctuations, or the noise, of some nature. It is just noise that, on the one hand, prevents the prediction of a system's complete highly intricate dynamical evolution, and, on the other hand, greatly simplifies the statistical (incomplete) description of the most significant process of a *nonrecurrent* (*aperiodic*) *relaxation* which leads to a statistical steady-state, in the simplest case. Clearly, the study of noise mechanisms is of a great interest for both the fundamental physics and various applications. This is especially important for «intermediate systems» like heavy and middle nuclei, and still more for the atoms whose dynamical as well as statistical description are very complicated. I mention, in passing, that the main difficulty of an «eternal» problem, the hydrodynamic turbulence, is of the same nature.

Until recently, only one dynamical mechanism of noise had been considered (let us term it the *traditional* one), related to a huge number N of freedoms in a dynamical (hamiltonian) system. Even though at any finite N the motion of such a system is quasiperiodic, i. e. has a discrete spectrum, the trajectory becomes so intricate that it imitates a «random» process fairly well. This mechanism has been studied up to now in detail, and it can be substituted in practical calculations by an external random noise with some prescribed statistical properties, the former driving the dynamical system in question. In this approach the system may have any number of freedoms, all the complexity of statistical fluctuations being related to the extrinsic noise, particularly, to some thermal bath of a given

temperature. I mention that such an approach has led to the formation of a whole new branch of mathematics (of the probability theory), the stochastic differential equations.

Recent intensive studies have revealed a completely different dynamical mechanism of noise which does not depend on the number of freedoms, and which is determined by a strong motion instability, completely unnecessary for the traditional mechanism. This new phenomenon has been termed the *dynamical chaos*. Sometimes, it is also called stochasticity, yet one should not miss the principal difference of the latter from the phenomenon described by stochastic equations.

The main topic of my talk is the discussion of interdependence among the two noise mechanisms in classical and quantum mechanics.

1. THERMODYNAMIC LIMIT, OR RELAXATION IN CONTINUOUS SPECTRUM

Below we restrict ourselves to the discussion of simplest models only, in order to emphasize the principal questions of noise mechanisms. Consider a classical many-dimensional linear oscillator with Hamiltonian

$$H=\sum_{n=0}^{N} E_n+\varepsilon\sum_{m,n}\alpha_{mn}q_m q_n\,,$$
$$E_n=\frac{1}{2}(p_n^2+\omega_n^2 q_n^2)\,. \tag{1.1}$$

Here $\alpha_{mn}=\alpha_{nm}$, and ε is a small perturbation parameter, so the model is a system of weakly coupled one-dimensional oscillators. As it well known, this system is completely integrable, and for any given coefficient matrix of the potential energy

$$U_{mn}=\frac{\omega_n^2}{2}\delta_{mn}+\varepsilon\alpha_{mn} \tag{1.2}$$

the solution is readily available via diagonalization of the matrix. An instructive analysis of this apparently simple problem is given in Ref. [1]. However, we may put a more complicated question on a *generic* (most probable) behaviour of a many-dimensional linear oscillator. This question can be answered by means of a statistical theory in which matrix elements U_{mn} are assumed to be random numbers taken from a certain statistical ensemble. Notice that interpretation of the random as the generic has received recently a rigorous mathematical justification in Kolmogorov's algorithmic theory (see, e. g., Ref. [2]).

Even though for any particular realization of matrix (1.2) the quantities α_{mn} do not depend on time, their random dispersion is just the origin of a permanently fluctuating noise as $N\to\infty$.

Notice that $U_{mn}=H_{mn}$ in Eq. (1.2) may be also considered as matrix elements of the Hamiltonian operator for some quantum system with discrete spectrum. Thus, a far-reaching analogy between classical and quantum problems arises which will be dicussed in detail in Section 5 below. The most fascinating and surprising point of this analogy is, perhaps, in that a very particular classical system (linear oscillator!) proves to be related to a whole broad class of quantum systems.

A generic description of such systems is provided by the statistical theory of random matrices (RMT) (see Refs [3—5]). In case of real H_{mn} the so called Gaussian Orthogonal Ensemble (GOE) is usually assumed, which is invariant in respect to any rotation of the basic vectors. In other words, the generic statistical properties of such a system are independent of the choice of basis and, hence, are completely determined by matrix H_{mn}, and by the corresponding classical model. This important property forms the ground of Dyson's version for RMT [3]. He criticizes the assumption on H_{mn} randomness as physically unclear. However, Dyson's approach persues, actually, the same end, namely, to describe generic properties of a quantum system.

Coming back to the classical problem, assume, first, a natural (but noninvariant) normalization

$$\alpha_{mn}\sim\Delta\omega_n^2\sim\omega_n\Delta\ll\omega_n^2 \tag{1.3}$$

to get rid of ω_n^2 average, insignificant to the problem. Here Δ is an effective full interval of unperturbed frequencies ω_n related to their density $\tau(\omega)$ by the expression

$$\tau=\frac{dn}{d\omega_n}=\frac{N}{\Delta}. \tag{1.4}$$

Density τ has dimensionality of time, and it is going to play an important role in what follows.

Let us separate the many-dimensional oscillator into two unequal parts, a «test» oscillator $n=0$ (we shall drop this subscript below), and the «thermal bath» $n=1, ..., N$. Diagonalize the bath, i. e. transform it to the normal modes

$$Q_n = b_{nm}\, q_m\,; \qquad m, n = 1, ..., N. \tag{1.5}$$

Here b_{mn} is an orthogonal matrix that is

$$b_{nm}\, b_{nm'} = \delta_{mm'}\,. \tag{1.6}$$

We arrive at Bogolyubov's problem [6] (1945 year) on statistical relaxation in a classical system with Hamiltonian

$$H = E + \sum_{n=1}^{N} (E_n + \varepsilon \alpha_n Q_n q). \tag{1.7}$$

It is easily verified by using normalization (1.6) that new $\langle \alpha_n^2 \rangle = 4 \langle \alpha_{mn}^2 \rangle$.

In paper [6] Bogolyubov rigorously proved that under certain conditions the test oscillator approaches statistical steady-state of the thermal bath. If initially $E(0) = 0$, the relaxation takes especially simple form:

$$\rho(E, t) = \frac{\omega}{2\pi T} e^{-E/T}; \qquad T(t) = T_0 (1 - e^{-\gamma t}) \tag{1.8}$$

where ρ is phase density of the ensemble of test oscillators, and where T_0 is bath temperature. The relaxation rate

$$\gamma = \frac{\pi}{2}\, \varepsilon^2 J(\omega) \tag{1.9}$$

is determined by the perturbation spectrum density

$$J(\omega) = \frac{\alpha^2(\omega)\, \tau(\omega)}{\omega^2}. \tag{1.10}$$

Another particular case of relaxation is also interesting, namely, $E(0) = E_0 \gg T_0$; $T(0) = 0$, that is the oscillator is initially in a dynamical state. Then, regardless of the initial oscillator phase,

$$\langle E \rangle = E_0 e^{-\gamma t} \gg T_0\,; \qquad T = T_0 (1 - e^{-\gamma t}) \tag{1.11}$$

where brackets $\langle ... \rangle$ denote the ensemble (phase) average.

Curiously, the dissipation of the energy of coherent oscillations into heat is controlled by a peculiar conservation law

$$T_0 \langle E \rangle + E_0 T = T_0 E_0 = \text{const}\,. \tag{1.12}$$

Consider now the conditions for relaxation [6]. The most important one is the transition to the thermodynamic limit in thermal bath

$$N \to \infty; \qquad \varepsilon \to 0; \qquad \frac{\gamma}{\Delta} \sim \varepsilon^2 N = \text{const} \ll 1\,. \tag{1.13}$$

In the last estimate the normalization (1.3) is used. In thermodynamic limit the perturbation spectrum for the test oscillator becomes continuous, hence the term *«relaxation in continuous spectrum»*! The importance of this condition is in that only continuous spectrum provides permanent relaxation, i. e. an *aperiodic* process distinctive of statistical mechanics.

Would the relaxation always be exponential? It depends on the relative rate γ/Δ, and on function $J(\omega)$. It is required in Ref. [6] that in thermodynamic limit the density $J(\omega)$ should be a continuous (nonsingular) function of ω. Then, for the sufficiently small γ, $J(\omega) \approx \text{const}$ within the interaction domain $\Delta\omega \sim \gamma$. As rigorously proved in Ref. [6] the relaxation is exponential under this condition. Otherwise, a nonexponential relaxation may occur [1].

This can be demonstrated in the following simplified way (see also Ref. [1]). Consider the motion equation for the test oscillator implied by Hamiltonian (1.7) with an additional friction force $-\mu \dot{q}$ as the reaction of thermal bath. Then, the expansion $q = b_n Q_n$ in bath normal modes is described by the expression

$$b_n^2 = \frac{\varepsilon^2 \alpha_n^2}{(\omega^2 - \omega_n^2)^2 + \mu^2 \omega_n^2}.$$

If $\alpha_n^2 \approx \text{const}$, it is a Lorentzian spectrum due to the resonant denominator. Moreover, as the normal modes of the full system (thermal bath + probe oscillator) are close to Q_n as $N \to \infty$, the relation $b_n b_n = 1$ holds (1.6), and $\mu = \gamma$. The time average $\bar{E} = \langle E_n \rangle\, (\omega_n \approx \omega) = T_0$. The exponential relaxation is, thus, a direct implication of Lorentzian spectrum

$$b_n^2 \approx \frac{\gamma}{2\pi\tau}\, \frac{1}{(\Delta\omega)^2 + \frac{\gamma^2}{4}} \approx \frac{\varkappa}{2\pi}\, \frac{1}{(\Delta n)^2 + \frac{\varkappa^2}{4}}. \tag{1.14}$$

Here $\Delta\omega = \omega - \omega_n$; $\Delta n \approx \tau \cdot \Delta\omega$ the detuning from resonance in the number of normal modes, and a new parameter

$$\varkappa = \gamma\tau \tag{1.15}$$

characterizes the number of bath oscillators effectively interacting with the test oscillator during relaxation.

Another important condition for relaxation is a restriction of the system initial conditions excluding the excitation of a few normal modes. In the limiting case of just a single mode the energy of the test oscillator would remain constant. Clearly, as $N \to \infty$ the probability of such initial conditions is rapidly vanishing.

Notice, that under the latter condition the relaxation proceeds symmetrically in both directions of time ($t \to \pm\infty$) because Bogolyubov's model (1.7) as well as (1.1) are time-reversible. In this connection it is worthwile to mention that Loschmidt's paradox on ostensibly irreversible relaxation of a dynamically reversible system is, actually, the confusion of two different notions, the relaxation *reversibility* upon velocity *reversal*, and *nonrecurrence* of phase density $\rho(t)$ in a closed system without any external interference.

In Ref. [6] the distribution of thermal-bath oscillators was assumed to be Gibbsian. However, this is not crucial in that the equilibrium temperature may depend of frequency: $T_0 \to T_0(\omega) = \langle E_n \rangle_{\omega_n = \omega}$ [7].

In conclusion of this Section I emphasize again that the traditional noise mechanism requires the actual interaction of a rel..xing system with a large number $N \to \infty$ of other freedoms, that is it needs a very many-dimensional «thermal bath» in a «typical» state specified by some statistical ensemble.

2. DYNAMICAL CHAOS IN CLASSICAL MECHANICS

Now, consider a totally different mechanism of noise, the *dynamical chaos*, which can been discovered or, better to say, comprehended in recent extensive studies (see, e. g. Refs [8, 9]). The most striking distinction of the new mechanism is its independence, in principle, of the number of freedoms of a closed system.

To begin with, consider a couple of «simple» examples. The first one is the motion of comet Halley driven by Jupiter's perturbation [10], the famous three-body problem. The second, related, example, is the classical photoeffect in Hydrogen [11]. The simplest model

for the both is described by the same two-dimensional canonical map (see Refs [10, 12]):

$$\begin{aligned} w_{n+1} &= w_n + \mu F(x_n), \\ x_{n+1} &= x_n + w_{n+1}^{-3/2} \end{aligned} \tag{2.1}$$

which specifies the change in energy w (in appropriate units) in comet (electron), and of perturbation phase x of Jupiter (monochromatic electric field) per one revolution of comet (electron); μ is small perturbation parameter proportional to Jupiter's mass (field strength); $F(x)$ the periodic function ($F(x) = \sin(2\pi x)$ in the second example).

Model (2.1) has 1.5 freedoms only as it is one-dimensional oscillator driven by an external (given) periodic perturbation. Nevertheless, under certain conditions, the motion of this model becomes highly irregular and unpredictable, described by some diffusion equation. Hence the term «diffusive photoeffect» in the second example.

This mechanism of noise is explained by a strong local instability of motion which, in turn, is described by map (2.1) linearized about some (generally chaotic) trajectory x_n^0, w_n^0:

$$\begin{aligned} \delta w_{n+1} &= \delta w_n + \mu \frac{dF(x_n^0)}{dx_n} \delta x_n, \\ \delta x_{n+1} &= \delta x_n - \frac{3}{2} (w_{n+1}^0)^{-5/2} \delta w_{n+1}. \end{aligned} \tag{2.2}$$

Here $(\delta x_n, \delta w_n)$ is the tangent vector of length l. The local instability is characterized by Lyapunov's exponent

$$\begin{aligned} \Lambda &= \lim_{n \to \infty} \frac{1}{n} \ln \frac{l}{l_0} > 0, \\ l &\sim l_0 e^{\Lambda n}. \end{aligned} \tag{2.3}$$

The inequality, that is «exponential» instability, is necessary and sufficient condition for dynamical chaos (see Ref. [2]). It must hold on a set of initial conditions (x_0^0, w_0^0) of dimensionality bigger than one to exclude the case of an isolated unstable periodic trajectory. Moreover, the motion should be bounded, at least in some variables (in x in Eq. (2.1), for example).

In model (2.1) the inequality $\Lambda > 0$ is always realized for a sufficiently small w. It is easily verified from the relation

(see Ref. [8])

$$\frac{\delta x_{n+1}}{\delta x_n} = 1 - \frac{3}{2}(w^0_{n+1})^{-5/2}\frac{dF(x^0_n)}{dx_n}.$$

According to the above mentioned algorithmic theory of Kolmogorov the chaotic trajectory is unpredictable in that the related information per unit time

$$\frac{I(t)}{|t|} \to \Lambda; \quad t \to \pm\infty \tag{2.4}$$

does not decrease as $|t| \to \infty$ [2]. It implies that for each new time interval one needs a new information which cannot be extracted from measurement, to arbitrarily high but finite accuracy $\varepsilon \to +0$, of any preceeding section of trajectory.

Obviously, over any finite time interval the prediction of chaotic trajectory is quite possible, and is controlled by the *randomness parameter* [14] (see also Ref. [15])

$$\mathscr{R} = \frac{\Lambda |t|}{|\ln \varepsilon|}. \tag{2.5}$$

Prediction corresponds to a finite domain of *temporary determinism* ($\mathscr{R} \leqslant 1$) which turns, as $\mathscr{R}$ increases, to the infinite domain of *asymptotic randomness* ($\mathscr{R} \gg 1$). For example, even though the trajectory of comet Halley is predicted to a very high accuracy on a few month time interval and still satisfactorily on one revolution (75 years), it is completely prohibitive to predict if the comet would be ejected from the Solar system at the nearest intersection of its orbit with that of Jupiter in about 100 revolutions.

A still simpler example of dynamical chaos is described by one-dimensional (!) map of the unit interval on itself (see Refs [9, 13]):

$$\varphi_{n+1} = k\varphi_n \mod 1 \tag{2.6}$$

with any integer $k > 1$. A different representation of the latter model is complex map

$$z_{n+1} = z_n^k; \quad z = e^{2\pi i \varphi}. \tag{2.7}$$

Lyapunov's exponent for this model is $\Lambda = \ln k > 0$. Curiously, the solution of difference Eqs (2.6, 2.7) is available in explicit form

$$z_n = z_0^{(k^n)}; \quad \varphi_n = \varphi_0 k^n \mod 1. \tag{2.8}$$

Yet, it doesn't help at all to get rid of chaos which is born upon transition from regularly increasing but nonphysical angle (phase) φ to the physical *direction* of z vector determined by φ *modulo 1*.

Another example of same sort, the map (see Ref. [9])

$$x_{n+1} = 4x_n(1 - x_n) \tag{2.9}$$

also admits the explicit solution

$$x_n = \sin^2(2^n\varphi_0) \tag{2.9}$$

which shows its relation to maps (2.6, 2.7).

The reason why a very simple system may exhibit so complicated behaviour is in that the system must «provide» strong (exponential) instability only. The true origin of dynamical chaos is in *continuety* of phase space in classical mechanics. Exactly fixed (formally!) initial conditions contain already an infinite amount of information which is «pulling out» by the mechanism of local instability. In any dynamical system the initial conditions completely determine the whole trajectory, of course, yet in an unstable system almost any of those trajectories proves to be chaotic [2].

The spectrum of dynamical chaos is always continuous or, to be more precise, it is bound to have a continuous component independently of N. Also, there is no need in additional statistical hypotheses, like the choice of α_{mn}-matrix ensemble, which would provide generic behaviour of a system. Actually, a *minimal hypothesis* still remains, namely, the motion initial conditions should not belong to a certain set of zero measure, including, particularly, the everywhere dense set of unstable periodic trajectories. This hypothesis is the limiting case of the restriction on initial conditions in the traditional noise mechanism (Section 1).

The importance of dynamical chaos is, first of all, in the extension of statistical mechanics onto a completely new region of simple (small N) dynamical systems. The statistical laws here are generally rather different, of course, from those in the thermodynamic limit which simply does not exist here. For example, even the microcanonical distribution is not necessarily reached if the chaotic component comprises a part of energy surface only. Another example: a homogeneous diffusion in chaotic system may be both superslow [16] as well as superfast [17] that is the dispersion of distribution

function $\sigma^2 \propto t^\beta$ grows not necessarily in proportion to time $(0<\beta<2)$.

The dynamical chaos has many applications in various domains of classical mechanics and other branches of science. The time, when this phenomenon looked exotic and made confusion, has passed.

Nowadays, the dynamical chaos is come across literally everywhere. However, if one tries to consider it as a fundamental law in physics a «little» difficulty arises, namely, there is no classical mechanics in the Nature. It is only approximation, the limit of quantum mechanics. An important question arises if the dynamical chaos is possible in a quantum system?

3. QUANTUM CHAOS, OR RELAXATION IN DISCRETE SPECTRUM

Chaos in quantum mechanics seems, at the first glance, to be inevitable in view of the probabilistic interpretation of a basic quantum quantity, the state vector ψ. However, this chaos appears only in a very specific process, the measurement, which is, in a sence, «foreign» to the quantum mechanics as it involves the «invasion» of the classical device, observer and the like.

Is dynamical chaos possible in the proper quantum dynamics, i. e. in the time evolution of ψ state vector? It turns out that it is not because the energy (and frequency) spectrum of a bounded quantum motion is always discrete. This was clear already to Krylov [18] in late fourties and is related ultimately to a fundamental property of quantum phase space, namely, to its discreteness (see Refs [19, 7]). On the other hand, the correspondence principle (also fundamental!) requires some transition to the classical chaos. This apparent paradox was discussed, and has been resolved in Ref. [19] by introducing characteristic time scales of the quantum evolution. The most important one is the diffusion scale (t_d) depending on the average density η of energy (or quasienergy) levels in a quantum system:

$$t_d \sim \hbar\eta . \qquad (3.1)$$

Quantity η is determined by those eigenstates only which are actually present (excited) in a given initial state of the system. This is

especially essential for the quasienergy whose full level density is infinite, as a rule, because its values are taken modulo $2\pi\hbar/T$ where T is the period of external perturbation.

Thorough numerical simulations [19—21] reveal that in quantum system a process close to the classical diffusion does occur for $t \lesssim t_d$, and under additional condition

$$V\eta \gtrsim 1 \qquad (3.2)$$

where V is the perturbation matrix element. The latter inequality implies a strong mixing of unperturbed states, or the breakdown of the quantum perturbation theory. In the opposite limiting case $(V\eta \ll 1)$ the unperturbed states persist no matter what would be the system behaviour in the classical limit. In the problem of quantum chaos the condition (3.2) had been first formulated by Shuryak [22], and was used than in many other papers (sometimes without any reference to [22]).

I emphasize that inequality (3.2) alone does not imply the quantum chaos (diffusion). For example, in case of one isolated nonlinear resonance the mixing of unperturbed states is of a regular nature, and the eigenfunctions are restricted within the resonance domain [22, 23]. The quantum chaos requires also the chaos in the classical limit [19].

In any event, the numerical experiments show that for $t \gg t_d$ the quantum diffusion completely stops, and only some stationary oscillations are left (not to be confused with a stationary quantum state, described by the eigenfunction!). This peculiar quantum phenomenon, first observed in Ref. [20], has been termed the *quantum localization* (of classical dynamical chaos). It may be considered as a dynamical counterpart of Anderson's statistical localization in solids [30]. I emphasize that the quantum localization can occur in spite of a strong mixing of many unperturbed states with close energies, that is under condition $V\eta \gg 1$ (3.2) due to coherence of a ψ wave. This is why any time-dependent noise delocalizes quantum system destroying its coherence [31, 32].

The limiting localization under $V\eta \ll 1$, i. e. the persistence of the unperturbed states, is called sometimes *perturbative localization*.

In case of quantum localization the swing of almost-periodic oscillations decreases toward the classical limit. Notice that so called «Poincare's recurrence time» of almost-periodic function sharply increases with the recurrence accuracy, and greatly exceeds t_d.

The physical meaning of diffusion scale t_d is directly connected with the time-energy uncertainty relation because the spectrum discreteness comes into play on time $t \gg \hbar/\Delta E = \hbar\eta \sim t_d$ only. If in the classical limit the motion has a continuous spectrum (classical chaos), $\eta \to \infty$, and t_d indefinitely grows.

But a similar (time-frequency) uncertainty relation holds in the classical mechanics. Hence, the mean frequency density τ (1.4) does also determine some characteristic time scale of classical evolution at finite N. Thus, a similarity is suggested [7] between the thermodynamic limit in the classical model ($N, \tau \to \infty$) and the classical limit in a quantum system ($\eta, t_d \to \infty$). The most important distinction is in that the limiting continuous spectrum in a quantum system depends not only on its number of freedoms (which may be small as in classical dynamical chaos) but also on a big number of frequencies $N_\omega \propto \eta$ which determine the quantum dynamics at any N. In the traditional (quantum) statistical mechanics the growth of $N_\omega \sim N$ with N only is taken into account. However, with chaos in the classical limit, the number $N_\omega \to \infty$ independently of N. This is just the main significance of the classical dynamical chaos in the problem of quantum chaos in quasiclassical region. I mention that arising of chaos in a classical system may be interpreted as a limit transition $N_\omega \to \infty$ at finite N, due to divergence of the perturbation theory series.

At any finite N, i. e. arbitrarily far in quasiclassical region (but not in the classical limit!), the quantum chaos is but a simulation of the true dynamical chaos. Hence, the term «*quantum pseudochaos*» [7]. In view of a finite time scale t_d for the quantum diffusion the quantum chaos is called also «*transient*, or *temporary chaos*» [19, 7].

Interestingly, the quantum diffusion and relaxation are rather different in nature from those in classical dynamical chaos as there is no strong instability of motion in the former case. This has been inferred from the reversibility of quantum diffusion («echo» phenomenon) in numerical simulation in spite of computation errors [21, 24]. Even continuous component of the spectrum in the diffusive photoeffect does not cause any strong instability, nor it prevents the reversibility [24]. I mention that for an unbounded quantum motion only one (rather exotic) model is so far known with the true dynamical chaos in the configurational space of angle variables, due to an exponential growth of the conjugated action variables [25].

However, the dynamical chaos is quite possible in a semiclassical system where some freedoms (if only one!) are described classically. The simplest example is a two-level quantum system interacting with a single mode of classical electromagnetic field [26]. A more interesting case is the three-level system coupled to two modes of the classical field. Here the dynamical chaos may occur at arbitrarily small (!) coupling parameter [27]. In a simple classical model this curious degeneracy phenomenon had been discovered by Ford and Lunsford [38], and then studied by many others [39, 27] (see also Fig. 2.11 in Ref. [9] and its explanation in Ref. [40]).

An important application of the theory of semiclassical dynamical systems is the process of measurement in a quantum system by the classical device. There exists a hypothesis [28] that irreversible (or, more precisely, nonrecurrent, see Section 1 above) collapse of ψ-function is caused just by the dynamical chaos arising in such a system.

4. CLASSICAL RELAXATION IN DISCRETE SPECTRUM

Coming back to the classical model, let us try to aswer a fundamental question in traditional statistical mechanics: is there any relaxation at finite N, and, if so, in what does it differ from that in the thermodynamic limit? To the best of my knowledge, this problem has not yet been studied in any detail, apart from a few brief remarks in literature (see, i. g., the end of paper [29]). In the context of quantum chaos this question was discussed in Ref. [7] using a simple example of linear oscillator (1.1).

Since for $t \ll N/\Delta = \tau$ the classical relaxation proceeds as if in a continuous spectrum, that is as if in the limit $N \to \infty$ (uncertainty relation), the result crucially depends on the ratio of scale τ to relaxation time $t_R \sim \gamma^{-1}$ (1.9), i. e. on the quantity $\varkappa = \gamma\tau$ (1.15) which may be called the *discreteness parameter* (of the spectrum).

If this parameter is big ($\varkappa \gg 1$) there is enough time for the relaxation to get accomplished. For example, the average energy of a test oscillator $\bar{E} \approx T_0$ will be close, independently of its initial value $E(0)$, to the temperature of the thermal bath. However, unlike the relaxation in continuous spectrum (in the thermodynamic limit), the *phase* average $\langle E \rangle$ of the ensemble of test oscillators would suffer residual almost-periodic oscillations because the test oscillator effectively interacts with a finite number ($\sim\varkappa$) of thermal-bath oscillators. Residual relative fluctuations are apparently of the order

$\varkappa^{-1/2}\to 0$ as $\varkappa\to\infty$. Notice that similar fluctuations would persist in the thermodynamic limit for a *finite* ensemble of $\varkappa$ test oscillators.

In the opposite limit ($\varkappa\ll 1$) the test oscillator interacts essentially with one bath oscillator only, that of the nearest frequency. In case $E(0)=0$, for example, the time-averaged energy $\bar{E}$ is given by

$$\frac{\bar{E}}{T_0}=\frac{\varepsilon^2\alpha_n^2}{2\omega_n^2(\Delta\omega_n)^2}=\frac{\varkappa_n}{\pi(\tau_n\Delta\omega_n)^2} \tag{4.1}$$

where $\Delta\omega_n=\omega-\omega_n$. Typically, $\tau\Delta\omega\sim 1$, and $\bar{E}\sim\varkappa T_0$. This crude estimate just corresponds to the picture of limiting relaxation $\dot{T}=\gamma T_0$ (1.11) during time interval $\tau\ll\gamma^{-1}$. In this case the fluctuations of $\bar{E}$ are very big owing to those of detuning $\nu=\tau\cdot\Delta\omega$. Moreover, ensemble average $\langle\bar{E}\rangle$ may be even much bigger than $\varkappa T_0$. Assuming, for example, homogeneous distribution in ν over interval $(\nu_m, 1)$, where $\nu_m\sim\sqrt{\varkappa}$ (the full energy exchange with the bath oscillator, see Eq. (4.1)), the estimate $\langle\bar{E}\rangle\sim\sqrt{\varkappa}\ T_0$ has been obtained in Ref. [7]. However, the «repulsion» of the bath eigenfrequencies ω_n, quite similar to that of quantum levels (see Section 5 below), results in a decrease of probability for small ν in proportion to ν. Then $\langle\bar{E}\rangle/T_0\sim\varkappa|\ln\varkappa|$ which is roughly of the order of the first estimate.

Notice that under transition to thermodynamic limit (1.13) the discreteness parameter $\varkappa=\gamma N/\Delta\to\infty$.

In conclusion of this Section I mention that in case of dissipation in model (1.1) with oscillation damping time $T\ll\tau$ the condition for relaxation takes the form $\varkappa_1=\gamma T\gtrsim 1$, while average $\langle E\rangle\sim\varkappa_1 T_0\ll T_0$ within the interval $\tau^{-1}\ll\gamma\ll T^{-1}$. This simply means that the time for relaxation is here restricted by the damping.

5. A GLOBAL RANDOM MATRIX THEORY FOR CHAOTIC EIGENSTATES

In Section 1 above we saw that an ensemble of matrices (1.2) represents both a very simple classical model (1.1) and a class of quantum systems. In order to employ this analogy still further we, first, transform the classical matrix (1.2) as follows. Get rid of unimportant but big mean frequency $\langle\omega_n\rangle$, and redefine $\omega_n\to\langle\omega_n\rangle+\omega_n$. Putting $\langle\omega_n\rangle=1$ we obtain

$$U_{mn}=H_{mn}=\omega_n\delta_{mn}+\varepsilon\alpha_{mn} \tag{5.1}$$

with the natural normalization $\alpha_{mn}\sim\omega_n\sim\Delta\ll 1$ (cf. Eq. (1.3)). Now, quantities ω_n are of both signs, and $\langle\omega_n\rangle=0$. Also, $\hbar=1$ below.

Consider some quantum system which in the classical limit is fully chaotic, i. e. without any stable regions. Then, the Wigner function of any initial state would be approaching, within a time interval $t\sim t_d\sim\tau$ (3.1), the microcanonical distribution on an energy surface. From the classical model the relaxation time is of the order

$$t_R\sim\gamma^{-1}=(2\pi\varepsilon^2\alpha^2\tau)^{-1} \tag{5.2}$$

where Eqs (1.9, 1.10), and a new form of matrix H_{mn} are used.

As we already know (Section 4) the result of relaxation depends on the discreteness parameter

$$\varkappa=\gamma\tau=2\pi(\varepsilon\alpha\tau)^2\sim\frac{t_d}{t_R}\sim\varepsilon^2N^2. \tag{5.3}$$

In the latter estimate the natural normalization $\alpha\sim\Delta$ is used. If $\varkappa\gg 1$ ($t_d\gg t_R$) the relaxation process approaches microcanonical distribution. Hence, the eigenfunctions, or to be more precise, their Wigner's functions would be also close to microcanonical distribution, i. e. they will be ergodic on *energy «surface»*. On the other hand, if $\varkappa\ll 1$ the relaxation would stop at $t\sim t_d\ll t_R$ with Wigner's eigenfunctions localized within small domains of energy surface.

Literally, the energy surface exists in the classical limit only. In the quantum system it has the form of an energy layer of finite width. The structure of this layer is described by the eigenfunctions (1.14). Particularly, the layer width

$$\Delta\omega\approx\gamma;\quad \Delta n\approx\varkappa. \tag{5.4}$$

In the classical limit $\Delta n\to\infty$; $\Delta\omega/\Delta\to 0$, or

$$\varkappa\sim(\varepsilon N)^2\to\infty;\quad \frac{\gamma}{\Delta}\sim\varepsilon^2N\to 0. \tag{5.5}$$

Notice the difference from the thermodynamic limit (1.13) in classical model where $\gamma/\Delta\sim\varepsilon^2N=\text{const}$ owing to $\Delta, \gamma=\text{const}$. In quantum system $\gamma/\Delta\sim\hbar\gamma/E$ where E is system's energy.

Conditions (5.5) shows that the perturbation parameter ε is to

satisfy, in quasiclassical region, the double inequality

$$\frac{1}{N^2} \ll \varepsilon^2 \ll \frac{1}{N}. \qquad (5.6)$$

The right-hand inequality implies that the eigenfunctions (1.14) are always localized in the full N-dimensional basis of the model. I will term this distinctive feature of the global eigenfunction structure the *transverse localization* (in respect to energy surface).

Transverse localization is not present in the current statistical theory of complex quantum systems, the random matrix theory (RMT). Indeed, the basic RMT principle after Dyson [3] is statistical invariance in respect to any rotation of the full N-dimensional basis, or the ergodicity of eigenvectors in any such a basis. From the above consideration we see that this principle holds within the energy layer only, i. e. under condition $N \ll \varkappa$ which violates the right-hand inequality (5.6). Thus, the existing RMT is a *local* statistical theory of quantum systems as was already pointed out in Ref. [35].

On the contrary, the theory under discussion in present talk may be termed as a *global* one (GRMT) because it describes the structure of the whole energy layer provided that $N \gg \varkappa$ (5.6). The principal parameter of such a theory is just $\varkappa$ which determines the physical *dimensionality of eigenvectors* as contrasted to an arbitrary dimensionality N of the RMT model.

In complex nuclei $\varkappa \sim 10^6$ is really huge, and the global Lorentzian structure of eigenfunctions (1.14) appears only in the so called «strenght function» (see, e. g., Ref. [36]). However, in complex atoms the global structure is much more important since $\varkappa \sim 10$ as a single, so far, example demonstrates [35].

Parameter $\varkappa$, which has been obtained above by using a classical model, can be derived also within the framework of quantum mechanics, of course. However, this is not a simple task because for $\varkappa \gg 1$ the quantum perturbation theory is inapplicable. Nevertheless a rough estimate can be obtained from the relation for an energy level shift $\delta E \sim V^2 \eta$ in the second order of perturbation theory. Indeed, $\varkappa \sim \delta E \cdot \eta \sim (V\eta)^2 \sim (\varepsilon N)^2$ which agrees with Eq. (5.3). More accurate quantum calculations [34] result exactly in relation (5.3), with some additional assumptions though, particularly, on the Lorentzian spectrum shape (for a simple model derivation see Ref. [36]).

An important question in GRMT: which statistical properties of the local RMT do persist, and which ones change? and how? In Ref. [33] is shown that RMT (Wigner — Dyson's) distribution for nearest level spacing (the level «repulsion») is reached already under condition

$$\varkappa \gg 1. \qquad (5.7)$$

The same conclusion can be drawn from the results of the first numerical simulation in Ref. [37] (Fig. 13). Notice different normalization of matrices in Refs [33, 37] which was missed in Ref. [7]. However, the question if condition (5.7) is sufficient for the restoration also of long-term correlations of many levels as described by Dyson's Δ-statistics (the spectrum «regidity») remains open to the best of my knowledge.

Another interesting question is the nature of asymmetry between the matrices $\omega_n \delta_{mn}$ and α_{mn} in Eq. (5.1). Why the perturbation α_{mn} is so powerful? Notice that instead of fixed matrix $\omega_n \delta_{mn}$ one may use a random ensemble produced by random rotations of the basis which would not affect the GOE statistics of α_{mn}.

Interestingly, that the motivation in Ref. [34] for the study of a combined matrix ensemble like Eq. (5.1) was taking account of a regular component of nuclear dynamics (the «order») as represented by the fixed set of ω_n in addition to «chaos» of α_{mn}. In my view, the universal origin of such an order is simply the energy conservation resulting in the transverse localization of eigenfunctions, and in the formation, in the classical limit, of energy surfaces.

Studying of combined ensembles, as a RMT natural intrinsic development, certainly leads, under condition (5.6), to an essentially new, global, theory of quantum chaotic eigenstates. Yet, it is just a beginning! A peculiar property of model (5.1), used in such a theory, is in that both the ergodicity border $\varkappa \sim 1$ (5.3) and the mixing border (3.2) ($V\eta \sim \varepsilon\alpha\tau \sim \sqrt{\varkappa} \sim 1$) coincide in order of magnitude. This is certainly not a generic case [19—21, 25, 41]. The above-mentioned peculiarity of model (5.1) is due to homogeneity of perturbation matrix α_{mn}, so that the mixing of states is independent of their relative position on energy surface. Hence the (longitudinal) localization on energy surface turns out to be always perturbative, or limiting (see Section 3 above). In other words, the eigenstates of model (5.1) are either perturbatively localized or ergodic. The (longitudinally) localized eigenstates would correspond to a randomly

«porous» function b_n in Eq. (1.14) with many $b_n \approx 0$ (cf. Ref. [35]). To represent such a structure some further modification of matrix ensemble (5.1) is required. I conjecture that it can be achieved either by using perturbation matrices with rapidly vanishing off-diagonal elements α_{mn}, or by choosing matrices α_{mn} also randomly «porous», or both.

In any event, the homogeneous perturbation α_{mn} cannot persist in quasiclassical region as quantity $|m-n|$ is proportional to the frequency of classical perturbation.

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