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RECURSION AND GROUP STRUCTURES OF THE INTEGRABLE EQUATIONS IN 1+1 AND 1+2 DIMENSIONS

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Recursion and Group Structures of the Integrable Equations in 1+1 and 1+2 Dimensions

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ABSTRACT

A bilocal approach to the construction of the integrable equations and their general Backlund transformations connected with one- and two-dimensional spectral problems is discussed. Three different, but equivalent, ways of calculations are considered. The principal role of the bilocal adjoint representation of the spectral problem in such constructions, in particular, in the calculation of the bilocal «recursion» operator is emphasized. Multidimensional spectral problems are briefly discussed.

1. INTRODUCTION

Nonlinear evolution equations integrable by the inverse scattering transform (IST) method [1—4] posses a number of remarkable recursion and group-theoretical properties. The most convenient and beautiful method of describing these structures is to use the so called recursion operator [5—7, 2—4]. This recursion operator method has been effectively applied to a number of one-dimensional spectral problems (see e. g. [2—8]). Starting with the given spectral problem

$$\partial_x \Psi(x, \lambda) = U(x, \lambda)\Psi(x, \lambda) \qquad (1.1)$$

where $\partial_x = \frac{\partial}{\partial x}$, $U(x, \lambda)$ is the matrix parametrized by the certain independent fields, λ is a spectral parameter and uses the generalized adjoint representation of (1.1) which is of the form

$$\partial_x \Phi(x, \lambda) = U'(x, \lambda)\Phi(x, \lambda) - \Phi(x, \lambda) U(x, \lambda) \qquad (1.2)$$

where $\Phi(x, \lambda)$ is the so called squared eigenfunction or tensor product, one gets the recursion operator Λ which has a principal property

$$\Lambda(x)\Phi_I(\lambda) = \lambda \Phi_I(x, \lambda) \qquad (1.3)$$

where Φ_I is the irreducible independent dynamical part of Φ. Using (1.2) and (1.3) one finds (see e. g. [8]) the general Backlund transformations (BTs) $P \to P'$ for the problem (1.1). They are of the form

$$\sum_\alpha B_\alpha(\Lambda^+, t)\, K_\alpha(P', P) = 0 \qquad (1.4)$$

where Λ^+ is the operator adjoint to the operator Λ, B_α are arbitrary functions entire on Λ^+, K_α are certain quantities and $P(x, t)$ is a set of independent fields from $U(x, \lambda)$. General form of the nonlinear evolution equations integrable by (1.1) is the following

$$\frac{\partial P(x,t)}{\partial t} = \sum_\alpha \Omega_\alpha(L^+, t)\mathscr{L}_\alpha(P) \qquad (1.5)$$

where $L^+ \overset{def}{=} \Lambda^+|_{P'=P}$, $\mathscr{L}_\alpha \overset{def}{=} K_\alpha|_{P'=P}$ and Ω_α are arbitrary functions entire on L^+. Symmetry transformations for equations (1.5) in the infinitesimal form are

$$\delta P(x, t) = \sum_\alpha \omega_\alpha(L^+)\mathscr{L}_\alpha \qquad (1.6)$$

where ω_α are arbitrary entire functions.

All the quantities K_α, $\mathscr{L}_\alpha$ and operators Λ^+, L^+ are different for different spectral problems but the general forms (1.4) — (1.6) of BTs, integrable equations and their symmetries are common for all of them [8]. Emphasize that the recursion operator Λ^+ and the quantities K_α, $\mathscr{L}_\alpha$ in (1.4) — (1.6) are defined and act on the same space as the initial potential $P(x, t)$, i. e. they are local ones. This is the characteristic feature of the one-dimensional spectral problems of the form (1.1).

The generaltzation of the results (1.4) — (1.6) to the two-dimensional spectral problems, in particular, to the $N \times N$ matrix problem

$$\partial_x \Psi + A\, \partial_y \Psi + P(x, y, t)\Psi = 0 \qquad (1.7)$$

has been given in [9, 10]. Within the approach used in [9, 10] the general BTs are of the form

$$\sum_\alpha \sum_{n=0}^{\infty} b_{\alpha n}(t)\, \hat{\Lambda}_n^+ K_{\alpha n}(P', P) = 0 \qquad (1.8)$$

where $b_{\alpha n}(t)$ are arbitrary functions and $\hat{\Lambda}_n^+$ are the operators calculated by certain recurrent relations. The general form of the integrable equations is the following

$$\frac{\partial P(x,y,t)}{\partial t} = \sum_\alpha \sum_{n=0}^{\infty} \omega_{\alpha n}(t)\, L_n^+ \mathscr{L}_{n\alpha}(P) \qquad (1.9)$$

where $\omega_{\alpha n}(t)$ are arbitrary functions. All the operators and quantities which are contained in (1.8), (1.9) are local ones, i. e. they are defined and act on the same space as the $P(x, y, t)$. The results (1.8), (1.9) [9, 10] have demonstrated that the usual (standard) recursion operator does not exist for the two-dimensional spectral problems. A general theorem of a nonexistence of the usual local recursion operator for the nonlinear evolution equations in $1+d$ ($d \geqslant 2$) dimensions has been proved in [11].

The importance and relevance of the bilocal objects, i. e. the quantities which are defined on the wider space than the initial field $P(x, y, t)$, for the two-dimensional spectral problems have been pointed out in [12]. The true and correct analog of (1.2) for the two-dimensional spectral problems is a so called bilocal adjoint representation. For example for the problem (1.7) it is of the form

$$\partial_x \Phi(x, y', y) + A\,\partial_{y'}\Phi(x, y', y) + \partial_y \Phi(x, y', y)A + \\ + P'(x, y')\Phi(x, y', y) - \Phi(x, y', y)P(x, y) = 0 \qquad (1.10)$$

where $\Phi(x, y', y)$ is bilocal tensor product $\Phi(x, y', y) \overset{def}{=} = \hat{F}'(x, y') \otimes \check{F}(x, y)$ of the solution $\hat{F}'(x, y')$ of the pproblem (1.7) with the potential $P'(x, y')$ and the solution $\check{F}(x, y)$ of the problem adjoint to (1.7). The bilocal adjoint representation (1.10) is the starting point of the bilocal approach which has been proposed in [13]. This approach essentially simplifies the calculations of the general integrable equations and their BTs for the two-dimensional spectral problems [13]. But the advantage of the bilocal approach has not been realized in [13] in a full measure since the principal bilocal adjoint representation has been used in [13] mainly for the calculation of the corresponding local operators and quantities (defined on the diagonal $y' = y$) which are contained in formulas (1.8), (1.9).

A very important development of the theory of recursion structures of the $1+2$ dimensional integrable equations has been initiated recently by Fokas and Santini [14—16]. In the paper [14] they have observed that the Kadomtsev — Petviashvili (KP) hierarchy can be represented in a more compact form (compare with [10, 13]) if one uses a bilocal «recursion» operator Δ_{12} and proceed to the diagonal $y_1=y_2$ at the very end. Then they have considered [15] the 2×2 matrix two-dimensional spectral problem (1.7) with the corresponding Devay—Stewartson hierarchy and have studied the properties and structures of the corresponding bilocal (extended) quantities [15, 16]. The reasons which have lead of the authors of the papers [14—16] to their constructions are distinguished from those which have lead to (1.10) [12, 13]. Nevertheless the observation which have been done in [14, 15] is a very important [17] for the whole bilocal approach proposed in [13].

In the present paper we demonstrate that the introduction of the projection onto the diagonal $y'=y$ really at the very end, but not at the half step before as in [13], enables one to give a logically complete formulation of the bilocal approach to the 1+2 dimensional integrable equations. The recursion operator method in a form which is adequate to the 1+2 dimensional equations is considered. It is shown that the recursion operator method in the bilocal formulation allows one to construct the infinite-dimensional groups of general BTs, the hierarchies of integrable equations and their symmetry groups in the form which manifest their bilocal recursion structure for a wide class of the two- and one-dimensional spectral problems. The bilocal adjoint representation of the type (1.10) plays a principal role in all these constructions.

We will discuss the three different ways of calculations. The first way is the logically completed formulation of the initial bilocal approach of [13]. Within this approach the general BTs $P\to P'$ associated with the given two-dimensional spectral problem is of the form [17, 18]

$$\Delta\sum_{\alpha}\sum_{n=0}^{\infty} b_{\alpha n}(t)\,\hat{\Lambda}_n^+\,K_{\alpha n}(P',P)=0 \tag{1.11}$$

where the operator $\hat{\Lambda}_n^+(x,y',y)$ and quantities $K_{\alpha n}$ are bilocal ones, i. e. they depend on $P'(x,y',t)$ and $P(x,y,t)$ and Δ denotes the operation of the projection onto the diagonal $y'=y$: $\Delta Q(x,y',y)\overset{\text{def}}{=}Q(x,y',y)\,|_{y'=y}$. The general form of the integrable equations is

$$\frac{\partial P(x,y,t)}{\partial t}=\Delta\sum_{\alpha}\sum_{n=0}^{\infty}\omega_{\alpha n}(t)\hat{L}_n^+\,\mathcal{L}_{n\alpha} \tag{1.12}$$

where $L_n^+\overset{\text{def}}{=}\Lambda_n^+\,|_{P'=P(x,y',t)}$ and $\mathcal{L}_{n\alpha}\overset{\text{def}}{=}K_{n\alpha}\,|_{P'=P(x,y',t)}$. The bilocal ope-

rators $\Lambda_n^+(x,y',y)$ are calculated by the recurrent relation [17, 18]

$$\hat{\Lambda}_n^+(x,y',y)=\left(\hat{\Lambda}_1^+(x,y',y)+\frac{\partial}{\partial y'}\right)^{n-1}\hat{\Lambda}_1^+(x,y',y) \tag{1.13}$$
$$n=2,3,\ldots$$

where $\frac{\partial}{\partial y'}$ in (1.13) acts only on $\hat{\Lambda}_1^+(x,y',y)$ and the operator $\hat{\Lambda}_1^+(x,y',y)$ is calculated directly from (1.10). The operators $\hat{\Lambda}_n^+(x,y',y)$ are bilocal ones, i. e. they depend on $P'(x,y',t)$ and $P(x,y,t)$ but all of them contain only the operator $\partial_{y'}+\partial_y$ and not $\partial_{y'}$ and ∂_y separately. Hence these operators permit the direct projection onto the diagonal $y'=y$ and the action of the projection operator Δ on the whole «products» in (1.11) is equivalent to the action of Δ on the each factor in (1.11) and (1.12). As the result the formulas (1.11), (1.12) can be rewritten in the terms of local operators and quantities and we arrive exactly to (1.8) and (1.9).

The second way consists in the introduction of a formal spectral parameter into the two-dimensional spectral problem. For example, instead of (1.7) one should consider the problem

$$\partial_x\tilde{\Psi}(x,y,\lambda)+A\partial_y\tilde{\Psi}(x,y,\lambda)+P(x,y,t)\tilde{\Psi}(x,y,\lambda)=\lambda A\tilde{\Psi}(x,y,\lambda) \tag{1.14}$$

The parameter λ is a formal one since under the redefinition $\tilde{\Psi}=e^{\lambda y}\Psi$ the problem (1.14) is converted to (1.7). The corresponding bilocal adjoint representation is

$$\partial_x\Phi(x,y',y,\lambda)+A\partial_{y'}\Phi(x,y',y,\lambda)+\partial_y\Phi(x,y',y,\lambda)A+$$
$$+P'(x,y')\Phi(x,y',y,\lambda)-\Phi(x,y',y,\lambda)P(x,y)=\lambda[A,\Phi(x,y',y,\lambda)] \tag{1.15}$$

where $\Phi(x,y',y,\lambda)\overset{\text{def}}{=}\hat{F}'(x,y',\lambda)\otimes\check{F}(x,y,\lambda)$. The relation (1.15) gives

$$\Lambda(x,y',y)\Phi_l(\lambda)=\lambda\Phi_l(x,y',y,\lambda) \tag{1.16}$$

where Φ_l is the irreducible (independent) part of Φ. The operator $\Lambda(x,y',y)$ is just the bilocal recursion operator. With the use of such operator the general BTs can be represented in the form [17, 18],

$$\Delta\sum_{\alpha}B_\alpha(\Lambda^+,t)\,K_\alpha(P',P)=0 \tag{1.17}$$

where $B_\alpha(\Lambda^+,t)$ are arbitrary functions entire on the operator Λ^+

adjoint to Λ. The general form of the integrable equations is [17, 18]

$$\frac{\partial P(x,y,t)}{\partial t} = \Delta \sum_{\alpha} \Omega_{\alpha}(L^{+},t)\mathcal{L}_{\alpha} \tag{1.18}$$

where $L^{+} \stackrel{def}{=} \Lambda^{+}|_{P'=P(x,y',t)}$, $\mathcal{L}_{\alpha} \stackrel{def}{=} K_{\alpha}|_{P'=P(x,y',t)}$, $\Omega_{\alpha}(L^{+},t)$ are arbitrary functions entire on L^{+} and their symmetry transformation are

$$\delta P(x,y,t) = \Delta \sum_{\alpha} \omega_{\alpha}(L^{+})\mathcal{L}_{\alpha} \tag{1.19}$$

where ω_{α} are arbitrary entire functions.

The BTs, integrable equations and symmetry transformations (1.17) — (1.19) are the most close in their form to the corresponding results (1.4) — (1.6) for the one-dimensional problems. The principal difference is that all the quantities in (1.17) — (1.19) are bilocal ones. Emphasize that the bilocal recursion operators $\Lambda^{+}(x,y',y)$ and $\mathcal{L}^{+}(x,y',y)$ are really bilocal (extended) operators and they do not permit the direct projection onto the diagonal $y'=y$.

There exists the other way of deriving the formulas (1.17) — (1.19). This third way consists in the using the nonlocal gauge transformations

$$\Psi(x,y) \to \Psi'(x,y) = \int dy\, G(x,y',y)\,\Psi(x,y) \tag{1.20}$$

for the two-dimensional spectral problems where $G(x,y',y)$ is the matrix valued operator. This approach is the nonlocal two-dimensional generalization of the one-dimensional approach [19—21] to BTs via gauge transformations. The relevance of nonlocal gauge transformations (1.20) to the two-dimensional spectral problems has been pointed out in [12].

The gauge operator $G(x,y',y)$, as it is not difficult to show, obeis an equation of the form (1.10). Choosing G in the form $G(x,y',y) = \sum_{k=0}^{n} \partial_y^k\, \delta(y'-y)\, V_k(x,y',y)$, expanding over $\partial_y^k\, \delta(y'-y)$ and solving the obtained equations with respect to V_{kF}, we obtain the bilocal recursion operator $\Lambda(x,y',y)$ and the BTs (1.17) with $B_{\alpha} = b_{\alpha n}(\Lambda^{+})^{n}$. The Anzatz for the gauge operator $G(x,y',y)$ in the form of the series on the n-th order derivatives of the Dirac delta-function $\delta(y'-y)$ has been proposed in [15] (for the infinitesi-

mal gauge transformation (1.20) for the problem (1.7) at $N=2$). Namely by this way the Kadomtsev — Petviashvili (KP) and Devay — Stewarson (DS) hierarchies in the form (1.18) and their symmetry transformations (1.19) have been calculated for the first time in [15]. Note that performing the integration in this Fokas — Santini Ansatz we get the local Ansatz $G(x,y',y) = \delta(y'-y) \sum_{k=0}^{n} W_k(x,y)\,\partial_y^k$ which has been used in [22].

The two different forms (1.11) — (1.12) and (1.17) — (1.18) of the general BTs and integrable equations associated with the two-dimensional spectral problems are equivalent to each other. This can be checked straighforwardly with the use of the important relation

$$\Lambda^{+}(x,y',y) = -\partial_{y'} - \hat{\Lambda}_1^{+}(x,y',y). \tag{1.21}$$

The form (1.17) — (1.18) is, of course, more convenient and beautiful.

The form (1.17) — (1.19) of general BTs, integrable equations and their symmetries in 1+2 dimensions manifestly reflect the fact that they are generated by the single bilocal operator Λ^{+} (L^{+}). It will be recalled that the usual (standard) recursion operator which exists in 1+1 dimensions and has been considered in [2—8, 11] is the operator which converts the vector fields given by the r.h.s. of evolution equation into the vector fields ($L^{+}\delta U = \delta U$). Emphasize that the Fokas — Santini type bilocal recursion operator $L^{+}(x,y',y)$ is not such standard recursion operator due to the presense of the projection (degenerate) operator Δ in (1.17) — (1.19). Nevertheless such a bilocal recursion operator is a true two-dimensional analog of the one-dimensional recursion operator. It is the main result of the papers [14, 15]. The bilocal recursion operators play a fundamental role in the description of the recursion and group properties of the nonlinear integrable equations in 1+2 dimensions.

The recursion operator method in the bilocal formulation is applicable also to the one-dimensional spectral problems. It is well known that the local recursion operator with the property (1.3) can be constructed not for all one-dimensional spectral problems. The simplest (rather trivial) example is the $N \times N$ matrix problem

$$(A\,\partial_x + P(x,t))\,\Psi = \mu \Psi. \tag{1.22}$$

The local recursion operator for the problem (1.22) does not exist.

But the recursion operator method in the bilocal form effectively works. Using the corresponding bilocal (on x' and x) adjoint representation for (1.22)

$$A\,\partial_{x'}\Phi(x',x,\mu)+\partial_x\Phi(x',x,\mu)A+P'(x')\Phi(x',x,\mu)-$$
$$-\Phi(x',x,\mu)P(x)=0 \qquad (1.23)$$

one finds the bilocal recursion operator and constructs the general BTs and integrable equations in the bilocal form. Similar bilocal approach is effectively applicable to the other one-dimensional spectral problems, in particular, to the nonlocal Riemann—Hilbert spectral problem.

Note that the bilocal adjoint representation of the type (1.23) plays a fundamental role in the whole IST method. For example, the kernel $K(x',x,\mu)$ of the dressing transformation $\Psi(x,\mu)\to\to\Psi'(x',\mu)=\int dx\,K(x',x,\mu)\,\Psi(x,\mu)$ for the problem (1.22) obeis equation (1.23). So the bilocal approach can be considered as a special version of the general Zakharov—Shabat dressing method [23, 1].

We also will discuss the possible extensions of the bilocal approach under consideration to the multidimensional spectral problem.

The paper is organized as follows. In section 2 the one-dimensional problem (1.22) and the Rieman—Hilbert spectral problem are considered. Section 3 is devoted to the two-dimensional problem (1.7). The spectral problem $\partial_y\Psi+\sum_{k=0}^{n}U_k(x,y,t)\,\partial_x^k\,\Psi=0$ is discussed in section 4. Recursion operators, BTs and integrable equations for the problem $(\partial_x^2-\sigma^2\partial_y^2)\Psi+\varphi(x,y,t)\,(\partial_x+\sigma\partial_y)\Psi+U(x,y,t)\Psi=0$ are constructed in section 5. In section 6 the multidimensional generalizations are discussed.

2. ONE-DIMENSIONAL SPECTRAL PROBLEMS

The recursion operator method is applicable to a number of one-dimensional spectral problems of the type (1.1). The principal possibility of construction of the local recursion operator is connected with the fact that (1.1) contains a pure derivative ∂_x. For example, for the well known $N\times N$ matrix problem

$$\partial_x\Psi+P(x,t)\Psi=\mu A\Psi \qquad (2.1)$$

the adjoint representation is a local one (it is of the form (1.15) with $\partial_y=0$) and as a result one has the local recursion operator $\Lambda(x)$ which contains ∂_x and ∂_x^{-1} [24—27, 8]. The term $\partial_x\Psi$ in (2.1) defines also the bilinear form $\langle\chi\Psi\rangle=\int_{-\infty}^{+\infty}dx\,\mathrm{tr}\,(\chi(x)\Psi(x))$ with respect to which we calculate the adjoint operators.

The situation changes crucially if one considers the $N\times N$ matrix problem

$$(A\,\partial_x+P(x,t))\Psi=\mu\Psi. \qquad (2.2)$$

The problem (2.2) is obviously equivalent to the problem of the type (2.1) (by the gauge transformation $\Psi\to A^{-1}\Psi$). Nevertheless the problem (2.2) itself can be treated by the bilocal approach. The starting point of the bilocal approach is the bilocal adjoint representation of (2.2) which is of the form

$$A\,\partial_{x'}\Phi^{in}(x',x,\mu)+\partial_x\Phi^{in}(x',x,\mu)A+$$
$$+P'(x')\Phi^{in}(x',x,\mu)-\Phi^{in}(x',x,\mu)P(x)=0 \qquad (2.3)$$

where

$$(\Phi^{in}(x',x,\mu))_{kl}\stackrel{def}{=}\hat{F}'_{kn}(x',\mu)\,\check{F}_{il}(x,\mu)$$

and

$$A\partial_{x'}\hat{F}'(x',\mu)+P'(x',t)\,\hat{F}'(x',\mu)=\mu\hat{F}'(x',\mu),$$
$$-\partial_x\check{F}(x,\mu)A+\check{F}(x,\mu)\,P'(x,t)=\mu\check{F}(x,\mu). \qquad (2.4)$$

We will consider for simplisity the case of diagonal matrix A $(A_{ik}=\delta_{ik}a_k)$, $P_{ii}=0$ and $P(x,t)\xrightarrow[|x|\to\infty]{}0$.

Multiplying (2.3) by the diagonal matrix-operator $B(\partial_x,t)$ and integrating, in the manner standard for the recursion operator method [8], we obtain the following fundamental relation

$$\int_{-\infty}^{+\infty}dx\,dx'\,\delta(x'-x)\,\mathrm{tr}\,\{B(\partial_x,t)(P'(x')\Phi^F(x',x)-\Phi^F(x',x)P(x))\}=0 \qquad (2.5)$$

where $\Phi_F\stackrel{def}{=}\Phi-\Phi_D$, $(\Phi_D)_{ik}\stackrel{def}{=}\delta_{ik}\Phi_{ii}$. Representing $B(\partial_x,t)$ in the form $B(\partial_x,t)=\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty}b_{\alpha n}(t)\,H_\alpha\,\partial_x^n$ where the matrices H_α $(\alpha=1,\dots,N)$ form a basis of the algebra of diagonal matrices, we rewrite (2.5) as follows

$$\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} b_{\alpha n}(t)\,\langle H_\alpha P'\partial^n\Phi_F^F - PH_\alpha(-\partial')^n\Phi_F^F\rangle = 0 \qquad (2.6)$$

where $\partial \equiv \partial_x$, $\partial' \equiv \partial_{x'}$ and $\langle\ldots\rangle \overset{def}{=} \int_{-\infty}^{+\infty} dx\,dx'\,\delta(x'-x)\,\mathrm{tr}(\ldots)$.

Now we must express $\partial_x^n\,\Phi_F^F(x',x)$ and $\partial_{x'}^n\,\Phi_F^F(x',x)$ recurrently via $\Phi_F^F(x',x)$. To do this we again use the adjoint representation (2.3). The projection of (2.3) onto the matrix diagonal gives $\Phi_D^F(x',x) = -d^{-1}(P'\Phi_F^F - \Phi_F^F P)_D$ where $d = A(\partial_{x'}+\partial_x)$. Substituting this Φ_D^F into the off-diagonal part of (2.3) we obtain ($\mathrm{ad}_A\Phi \overset{def}{=} [A,\Phi]$)

$$\Lambda(x',x)\Phi_F^F \overset{def}{=} \mathrm{ad}_A^{-1}\{A\partial_{x'}\Phi_F^F(x',x) + \partial_x\Phi_F^F(x',x)A + (P'(x')\Phi_F^F(x',x) - \Phi_F^F(x',x)P(x))_F - P'(x')d^{-1}(P'\Phi_F^F - \Phi_F^F P)_D + d^{-1}(P'\Phi_F^F - \Phi_F^F P)_D P(x)\} = 0. \qquad (2.7)$$

The relation (2.7) gives

$$\partial_{x'}\Phi_F^F(x',x) = \hat\Lambda_1(x',x)\Phi_F^F, \quad \partial_x\Phi_F^F(x',x) = \check\Lambda_1(x',x)\Phi_F^F \qquad (2.8)$$

where $\hat\Lambda_1 = \partial_{x'} - \Lambda(x',x)$ and $\check\Lambda_1 = \partial_x + \Lambda(x',x)$. Note that the operators $\hat\Lambda_1$ and $\check\Lambda_1$ contain only the total derivative $\partial_{x'}+\partial_x$ in contrast to the operator $\Lambda(x',x)$.

Together with (2.8) one has

$$\partial_{x'}^n\Phi_F^F(x',x) = \hat\Lambda_n(x',x)\Phi_F^F, \quad \partial_x^n\Phi_F^F(x',x) = \check\Lambda_n(x',x)\Phi_F^F, \quad n = 1,2,3,\ldots \qquad (2.9)$$

where the operators $\hat\Lambda_n$ are calculated by the recurrent relation

$$\hat\Lambda_n(x',x) = \hat\Lambda_{n-1}(x',x)\,\hat\Lambda_1(x',x) + \frac{\partial\hat\Lambda_{n-1}(x',x)}{\partial x'}, \qquad n = 2,3,\ldots$$

For the operators $\hat\Lambda_n^+$ adjoint to $\hat\Lambda_n$ with respect to the bilinear form $\langle\ldots\rangle$ one has

$$\hat\Lambda_n^+(x',x) = \hat\Lambda_1^+(x',x)\,\hat\Lambda_{n-1}^+ + \frac{\partial\hat\Lambda_{n-1}^+(x',x)}{\partial x'}$$

and therefore

$$\hat\Lambda_n^+(x',x) = \left(\hat\Lambda_1^+(x',x) + \frac{\partial}{\partial x'}\right)^{n-1}\hat\Lambda_1^+(x',x), \quad n = 2,3,\ldots \qquad (2.10)$$

where the operator $\frac{\partial}{\partial x'}$ in (2.10) acts only on $\hat\Lambda_1^+(x',x)$.

Analogously

$$\check\Lambda_n^+(x',x) = \left(\check\Lambda_1^+(x',x) + \frac{\partial}{\partial x}\right)^{n-1}\check\Lambda_1^+(x',x), \quad n = 2,3,\ldots \qquad (2.11)$$

The use of (2.9) — (2.11) allow one to transform (2.6) into the equality

$$\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} b_{\alpha n}(t)\langle H_\alpha P'\check\Lambda_n\Phi_F^F - PH_\alpha(-1)^n\hat\Lambda_n\Phi_F^F\rangle = \int dx\,\mathrm{tr}\{\Phi_F^F(x,x)\times$$
$$\times\int dx'\,\delta(x'-x)\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} b_{\alpha n}(t)\,(\check\Lambda_n^+ H_\alpha P' - (-1)^n\hat\Lambda_n^+ PH_\alpha)\} = 0. \qquad (2.12)$$

The equality (2.12) is fulfilled if

$$\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} b_{\alpha n}(t)\,\Delta(\check\Lambda_n^+ H_\alpha P' - (-1)^n\hat\Lambda_n^+ PH_\alpha) = 0 \qquad (2.13)$$

where $P' \equiv P(x',t)$ and Δ denotes the projection onto the diagonal $x'=x$: $\Delta Q(x',x) = Q(x',x)\big|_{x'=x}$.

The transformations (2.13) are the general BTs for the problem (2.2). Considering the infinitesimal displacement in time ($P'(x',t) = P(x',t) + \varepsilon\frac{\partial P(x',t)}{\partial t}$, $b_{\alpha n} = \delta_{n,0} - \varepsilon\omega_{\alpha n}(t)$, $\varepsilon\to 0$) one gets the general form of the integrable equations

$$\frac{\partial P(x,t)}{\partial t} = \Delta\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty}\omega_{\alpha n}(t)\,(\check L_n^+ H_\alpha P' - (-1)^n\hat L_n^+ PH_\alpha)$$

where $L_n^+ \overset{def}{=} \Lambda_n^+\big|_{P'=P(x',t)}$ and $\omega_{\alpha n}(t)$ are arbitrary functions.

Following the second way described in the introduction one should consider instead of (2.2) the problem

$$A\,\partial_x\tilde{\Psi}+P(x,t)\tilde{\Psi}=\lambda A\tilde{\Psi} \tag{2.14}$$

where λ is the formal parameter. The adjoint representation for (2.14) is given by

$$A\,\partial_{x'}\Phi^{in}(x',x,\lambda)+\partial_x\Phi^{in}(x',x,\lambda)A+P'(x')\Phi^{in}(x',x,\lambda)-$$
$$-\Phi^{in}(x',x,\lambda)P(x)=\lambda[A,\Phi^{in}(x',x,\lambda)] \tag{2.15}$$

where

$$\Phi(x',x,\lambda)\overset{def}{=}\hat{F}'(x',\lambda)\otimes\check{F}(x,\lambda)$$

and

$$A\partial_{x'}\hat{F}'(x',\lambda)+P'(x')\hat{F}'(x',\lambda)=\lambda A\hat{F}'(x',\lambda),$$
$$-\partial_x\check{F}(x,\lambda)+\check{F}P(x)=\lambda\check{F}(x,\lambda)A.$$

Multiplying (2.15) by an arbitrary diagonal matrix $B(\lambda,t)$ and integrating, one obtain instead of (2.5) the following fundamental relation

$$\sum_{\alpha=1}^{N}\langle B_\alpha(\lambda,t)(H_\alpha P'\Phi^F(\lambda)-PH_\alpha\Phi^F(\lambda))\rangle=0 \tag{2.16}$$

The irreducible form of the adjoint representation (2.15) for $\Phi_F^F(x',x,\lambda)$ is

$$\Lambda(x',x)\Phi_F^F(\lambda)=\lambda\Phi_F^F(x',x,\lambda) \tag{2.17}$$

where the operator Λ is given by (2.7). With the use of (2.17) one gets

$$\text{l.h.s. of (2.16)}=\sum_{\alpha=1}^{N}\langle H_\alpha P'B_\alpha(\Lambda,t)\Phi_F^F-PH_\alpha B_\alpha(\Lambda,t)\Phi_F^F\rangle=$$
$$=\langle\Phi_F^F\cdot\Delta\sum_{\alpha=1}^{N}B_\alpha(\Lambda^+,t)(H_\alpha P'-PH_\alpha)\rangle=0 \tag{2.18}$$

where the operator $\Lambda^+(x',x)$ is the operator adjoint to the operator $\Lambda(x',x)$ given by (2.7), i. e.

$$\Lambda^+(x',x)\cdot=\mathrm{ad}_A^{-1}A\partial_x\cdot+\mathrm{ad}_A^{-1}\partial_{x'}\cdot A+$$

$$+(P(x)\,\mathrm{ad}_A^{-1}\cdot-\mathrm{ad}_A^{-1}\cdot P'(x'))_F-P(x)d^{-1}(P\mathrm{ad}_A^{-1}\cdot-\mathrm{ad}_A^{-1}\cdot P')_D+$$
$$+d^{-1}(P\mathrm{ad}_A^{-1}\cdot-\mathrm{ad}_A^{-1}\cdot P')_D\,P'(x'). \tag{2.19}$$

Finally the relation (2.18) gives

$$\sum_{\alpha=1}^{N}\Delta B_\alpha(\Lambda^+,t)(H_\alpha P'-PH_\alpha)=0 \tag{2.20}$$

where the operator Λ^+ is given by (2.19).

The formula (2.20) is the main result of the recursion operator method. The transformations (2.20) form an infinite-dimensional group (Backlund—Calogero group) of general BTs $P\to P'$ for the one-dimensional spectral problem (2.2).

The consideration of (2.20) for the infinitesimal displacement in time $(P'(x',t)=P(x',t)+\varepsilon\frac{\partial P(x',t)}{\partial t},\ B_\alpha=1-\varepsilon\Omega_\alpha(L^+,t),\ \varepsilon\to 0)$ gives

$$\frac{\partial P(x,t)}{\partial t}=\sum_{\alpha=1}^{N}\Delta\Omega_\alpha(L^+,t)(H_\alpha P'-PH_\alpha) \tag{2.21}$$

where $L^+(x',x)\overset{def}{=}\Lambda^+(x',x)|_{P'=P(x',t)}$ and $\Omega_\alpha(L^+,t)$ are arbitrary entire functions on L^+.

Formula (2.21) gives the general form of the nonlinear evolution equations in 1+1 dimensions (t,x) integrable by the $N\times N$ problem (2.2). It is not difficult to show that the BTs and integrable equations in the bilocal form (2.20) and (2.21) are gauge equivalent to the corresponding BTs and integrable equations in a local form for the problem $\partial_x\Psi+A^{-1}P(x,t)\Psi=\lambda A^{-1}\Psi$.

At last the same results (2.20), (2.21) can be obtained by the using the nonlocal gauge transformations

$$\Psi(x,\mu)\to\Psi'(x',\mu)=\int dx\,G(x',x)\Psi(x,\mu) \tag{2.22}$$

for the problem (2.2) where $G(x',x)$ is an $N\times N$ matrix-valued operator of the form $G(x',x)=\sum_{k=1}^{n}\partial_x^k\,\delta(x'-x)\cdot V_k(x',x)$.

The equivalence of BTs (2.13) and BTs (2.20) is proved rather simply.

Note that the operators $\check{\Lambda}_n^+$ in (2.13) act only on the functions $H_\alpha P'(x',t)$ which depend only on x' and the operators $\hat{\Lambda}_n^+$ act only on the functions on x $(P(x,t)H_\alpha)$. On the subspaces of such functions $Z'(x')$ and $Z(x)$ the actions of the operators $\check{\Lambda}_n^+$ and $\hat{\Lambda}_n^+$ are equivalent to the following

$$\check{\Lambda}_n^+(x',x)Z'=(\check{\Lambda}_1^+(x',x)+\partial_x)^n Z'=(\Lambda^+(x',x))^n Z',$$
$$(-1)^n\hat{\Lambda}_1^+(x',x)Z=(-\hat{\Lambda}_1^+(x',x)-\partial_{x'})^n Z=(\Lambda^+(x',x))^n Z \qquad (2.23)$$

where the operator Λ^+ is given by (2.19). In virtue of (2.23) the relation (2.13) is equivalent to the relation (2.20) with $B_\alpha(\Lambda^+,t)=\sum_{n=0}^{\infty} b_{\alpha n}(t)(\Lambda^+)^n$.

The operator equality

$$\Lambda^+(x',x)=-\partial_{x'}-\hat{\Lambda}_1^+(x',x)=\partial_x+\check{\Lambda}_1^+(x',x) \qquad (2.24)$$

can be verified straightforwardly with the use of the definitions of $\hat{\Lambda}_1^+$, $\check{\Lambda}_1^+$ and Λ^+. Starting with the relation (2.20) and using (2.24) we arrive to the relation (2.13). Thus the BTs (2.13) and BTs (2.20) completely coincide.

In similar manner one can treat the other one-dimensional spectral problems which contain a nonpure derivative ∂_x. Benjamin—Ono equation and intermediate long waves equations have been considered by Fokas and Santini [28]. Here we will discuss briefly the nonlocal Riemann—Hilbert spectral problem

$$E_{x,\eta}\Psi-U\Psi=\lambda A\Psi \qquad (2.25)$$

which has been proposed in [29]. Here $E_{x,\eta}=\exp i\eta\,\partial_x$, $U=\sqrt{1+qr}+\begin{pmatrix}0 & q(x,t)\\ r(x,t) & 0\end{pmatrix}$ and $A=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. By the gauge transformation $\Psi\to\hat{\Psi}=g^{-1}\Psi$ where $E_{x,\eta}\,g=\sqrt{1+qr}\,g$, the problem (2.25) is converted to the problem

$$\sqrt{1+qr}\;D_{x,\eta}\hat{\Psi}(x)-P(x)\hat{\Psi}(x)=\lambda A\hat{\Psi}(x) \qquad (2.26)$$

where $D_{x,\eta}=E_{x,\eta}-1$ and $P=\begin{pmatrix}0 & q(x,t)\\ r(x,t) & 0\end{pmatrix}$. The problem (2.26) is more convenient for our purpose. The problem adjoint to (2.26) is

$$\sqrt{1+qr}\;D_{x,-\eta}\check{\Psi}(x)-\check{\Psi}(x)P(x)=\lambda\check{\Psi}(x)A. \qquad (2.27)$$

The adjoint representation of (2.26) for $\Phi(x',x,\lambda)=\hat{\Psi}'(x',\lambda)\otimes\check{\Psi}(x,\lambda)$ is of the form

$$\sqrt{1+q'r'}\;D_{x',\eta}\,\Phi(x',x,\lambda)-\sqrt{1+qr}\;D_{x,-\eta}\,\Phi(x',x,\lambda)-$$
$$-P'(x')\Phi(x',x,\lambda)+\Phi(x',x,\lambda)P(x,t)=\lambda[A,\Phi(x',x,\lambda)]. \qquad (2.28)$$

After the excluding Φ_D the adjoint representation (2.28) gives

$$\Lambda(x',x)\Phi_F(\lambda)=\lambda\Phi_F(x',x,\lambda) \qquad (2.29)$$

where the bilocal recursion operator $\Lambda(x',x)$ is of the form

$$\Lambda(x',x)=\frac{1}{2}A(d-P'(x')d^{-1}(P'\cdot-\cdot P)+d^{-1}(P'\cdot-\cdot P)P(x)) \qquad (2.30)$$

where $d=\sqrt{1+q'(x')r'(x')}\;D_{x',\eta}-\sqrt{1+q(x)r(x)}\;D_{x,-\eta}$. The author is grateful to P. Santini for the stimulating discussions of the problem (2.25).

The bilocal recursion operator method is applicable also to the problem $\sum_{n=0}^{N}U_n(x)\,\partial_x^n\Psi=\lambda\Psi$ with $U_N(x)\neq\text{const}$ and, in particular, to the problem $\rho^2(x)\,\partial_x^2\Psi=\lambda\Psi$ which produces the well known Harry—Dym hierarchy equations.

Note in conclusion that the bilocal approach can be applied to the one-dimensional problems of the form (1.1) and, in particular, to (2.1) or to the Schrodinger spectral problem $(\partial_x^2+U(x,t))\Psi=\lambda\Psi$ too. But formally bilocal recursion operators and other quantities contain only $\partial_x+\partial_{x'}$ and they are really local ones. As a result we arrive to the usual local formulation of the recursion operator method.

3. MATRIX TWO-DIMENSIONAL SPECTRAL PROBLEM

Now we will consider the two-dimensional spectral problems and start with the $N\times N$ problem (1.7), i. e.

$$\partial_x\hat{\Psi}+A\partial_y\hat{\Psi}+P(x,y,t)\hat{\Psi}=0 \qquad (3.1)$$

where A is a diagonal matrix $(A_{ik}=\delta_{ik}a_i,\ a_i\neq a_k)$, $P(x,y,t)$ is the $N\times N$ matrix such that $P_{ii}=0$ and $P(x,y,t)\xrightarrow[x,y\to\infty]{}0$. The problem

(3.1) is a natural generalization of the one-dimensional problem (2.2) (with $\mu=0$) with the change $x \to y$ and an addition of the pure derivative over x. The addition of the pure derivative into the spectral problem will not change all the bilocal constructions essentially. So the results for the problem (3.1) are similar in their form to the corresponding results for its one-dimensional counter part (2.2). Nevertheless we present here all the constructions for the problem (3.1) completely in order to do the similarities and differences between one- and two-dimensional cases more transparent.

As in the one-dimensional case the starting point of the bilocal approach is the bilocal adjoint representation of (3.1) which is of the form (1.10), i. e.

$$\partial_x \Phi^{in}(x,y',y) + A\partial_{y'}\Phi^{in}(x,y',y) + \partial_y \Phi^{in}(x,y',y)A + \\ + P'(x,y')\Phi^{in}(x,y',y) - \Phi^{in}(x,y',y)P(x,y) = 0 \quad (3.2)$$

where $\Phi(x,y',y) \stackrel{def}{=} \hat{F}'(x,y') \otimes \check{F}(x,y)$ (in components $(\Phi^{in}(x,y',y))_{kl} = \hat{F}'_{kn}(x,y')\check{F}_{il}(x,y)$, $i,k,n,l=1,\ldots,N$) and

$$\partial_x \hat{F}'(x,y') + A\partial_{y'}\hat{F}'(x,y') + P'(x,y')\hat{F}'(x,y') = 0$$

and

$$-\partial_x \check{F}(x,y) - \partial_y \check{F}(x,y)A + \check{F}(x,y)P(x,y) = 0.$$

Multiplying (3.2) by the diagonal matrix operator $B(\partial_y, t)$, integrating and assuming that $\int dy\, dy'\, \delta(y-y')(\partial_{y'}+\partial_y)\, Q(y',y) = 0$, we obtain the fundamental relation

$$\langle B(\partial,t)P'\Phi^F - B(-\partial',t)\Phi^F P\rangle = 0 \quad (3.3)$$

where $\langle\ldots\rangle \stackrel{def}{=} \int dx\, dy\, dy'\, \delta(y'-y)\,\mathrm{tr}(\ldots)$, $\partial \equiv \partial_y$, $\partial' \equiv \partial_{y'}$, $\Phi_F \stackrel{def}{=} \Phi - \Phi_D$, $(\Phi_D)_{ik} \stackrel{def}{=} \delta_{ik}\Phi_{ii}$. Since $B(\partial_y,t) = \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} b_{\alpha n}(t) H_\alpha \partial_y^n$ where the matrices H_α form a basis for the diagonal matrices, $b_{\alpha n}(t)$ are arbitrary functions, the relation (3.3) is equivalent to the following

$$\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} b_{\alpha n}(t)\,\langle H_\alpha P'\partial^n\Phi_F^F - PH_\alpha(-\partial')^n\Phi_F^F\rangle = 0. \quad (3.4)$$

The adjoint representation (3.2) gives us also the recursion operator. Indeed, from (3.2) one has $\Phi_D^F(x,y',y) = -d^{-1}(P'\Phi_F^F - \Phi_F^F P)_D$ where $P' \equiv P(x,y',t)$, and $d = \partial_x + A(\partial_{y'}+\partial_y)$. As a result the irreducible form of the adjoint representation (3.2) for $\Phi_F^F(x,y',y)$ is the following

$$\Lambda(x,y',y)\Phi_F^F \stackrel{def}{=} \mathrm{ad}_A^{-1}(\partial_x\Phi_F^F + A\partial_{y'}\Phi_F^F + \partial_y\Phi_F^F A + \\ +(P'(x,y')\Phi_F^F - \Phi_F^F P(x,y))_F - \\ - P'(x,y')d^{-1}(P'\Phi_F^F - \Phi_F^F P)_D + d^{-1}(P'\Phi_F^F - \Phi_F^F P)_D P(x,y)) = 0. \quad (3.5)$$

From the (3.5) it follows that

$$\partial_{y'}\Phi_F^F(x,y',y) = \hat{\Lambda}_1(x,y',y)\Phi_F^F, \\ \partial_y\Phi_F^F(x,y',y) = \check{\Lambda}_1(x,y',y)\Phi_F^F \quad (3.6)$$

where $\hat{\Lambda}_1 = \partial_{y'} - \Lambda(x,y',y)$ and $\check{\Lambda}_1 = \partial_y + \Lambda(x,y',y)$. Note that the operators $\hat{\Lambda}_1$ and $\check{\Lambda}_1$ contain only the total derivative $\partial_{y'}+\partial_y$ in contrast to the operator $\Lambda(x,y',y)$.

Together with (3.6) one has

$$\partial_{y'}^n\Phi_F^F(x,y',y) = \hat{\Lambda}_n(x,y',y)\Phi_F^F, \\ \partial_y^n\Phi_F^F(x,y',y) = \check{\Lambda}_n(x,y',y)\Phi_F^F, \\ n=1,2,3,\ldots \quad (3.7)$$

where the operators $\hat{\Lambda}_n$ are calculated by the recurrent relation

$$\hat{\Lambda}_n(x,y',y) = \hat{\Lambda}_{n-1}(x,y',y)\hat{\Lambda}_1 + \frac{\partial\hat{\Lambda}_{n-1}(x,y',y)}{\partial y'}, \quad n=2,3,\ldots$$

For the operators $\hat{\Lambda}_n^+$ adjoint to $\hat{\Lambda}_n$ with respect to the bilinear form $\langle\ldots\rangle$ one has

$$\hat{\Lambda}_n^+(x,y',y) = \hat{\Lambda}_1^+(x,y',y)\hat{\Lambda}_{n-1}^+ + \frac{\partial\hat{\Lambda}_{n-1}^+(x,y',y)}{\partial y'}$$

and therefore

$$\hat{\Lambda}_n^+(x,y',y) = \left(\hat{\Lambda}_1^+(x,y',y) + \frac{\partial}{\partial y'}\right)^{n-1}\hat{\Lambda}_1^+(x,y',y), \quad (3.8) \\ n=1,2,3,\ldots$$

where the operator $\frac{\partial}{\partial y'}$ in (3.8) acts only on $\hat{\Lambda}_1^+(x,y',y)$.

Analogously

$$\check{\Lambda}_n^+(x,y',y)=\left(\check{\Lambda}_1^+(x,y',y)+\frac{\partial}{\partial y}\right)^{n-1}\check{\Lambda}_1^+(x,y',y), \tag{3.9}$$

$$n=1,2,3,\ldots$$

The use of (3.7) — (3.9) gives

$$\text{l.h.s. of (3.4)}=\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty}b_{\alpha n}(t)\,\langle H_\alpha P'\check{\Lambda}_n\Phi_F^F-PH_\alpha(-1)^n\hat{\Lambda}_n\Phi_F^F\rangle=$$

$$=\int dx\,dy\,\mathrm{tr}\{\Phi_F^F(x,y,y)\times$$

$$\times\int dy'\,\delta(y'-y)\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty}b_{\alpha n}(t)\,(\check{\Lambda}_n^+H_\alpha P'-(-1)^n\hat{\Lambda}_n^+PH_\alpha)\}=0 \tag{3.10}$$

The equality (3.10) is fulfilled if

$$\sum_{\alpha=1}^{N}\sum_{n=0}^{\infty}b_{\alpha n}(t)\Delta(\check{\Lambda}_n^+H_\alpha P'-(-1)^n\hat{\Lambda}_n^+PH_\alpha)=0 \tag{3.11}$$

where $P'\equiv P(x,y',t)$ and Δ denotes the projection onto the diagonal $y'=y$: $\Delta Q(y',y)\overset{def}{=}Q(y',y|_{y'=y}$.

The relation (3.11) gives the general BTs for the problem (3.1) in the form (1.11). Let us transform them to the form (1.17).

Note, that the operators $\check{\Lambda}_n^+$ in (3.11) act only on the functions $H_\alpha P'(x,y',t)$ which depend only on y' and the operators $\hat{\Lambda}_n^+$ act only on the functions on y $(P'(x,y,t)H_\alpha)$. On the subspaces of such functions $Z'(x,y')$ and $Z(x,y)$ the actions of the operators $\check{\Lambda}_n^+$ and $\hat{\Lambda}_n^+$ are equivalent to the following

$$\check{\Lambda}_n^+(x,y',y)Z'=(\check{\Lambda}_1^+(x,y',y)+\partial_y)^nZ'=(\Lambda^+(x,y',y))^nZ', \tag{3.12}$$

$$(-1)^n\hat{\Lambda}_n^+(x,y',y)Z=(-\hat{\Lambda}_1^+(x,y',y)-\partial_{y'})^nZ=(\Lambda^+(x,y',y))^nZ, \tag{3.13}$$

$$n=1,2,3,\ldots$$

where the operator $\Lambda^+(x,y',y)$ is the operator adjoint to the operator $\Lambda(x,y',y)$ given by (3.5), i. e.

$$\Lambda^+(x,y',y)\cdot=\mathrm{ad}_A^{-1}\partial_x\cdot+\mathrm{ad}_A^{-1}A\partial_y\cdot+\mathrm{ad}_A^{-1}\partial_{y'}\cdot A+$$
$$+(P(x,y)\,\mathrm{ad}_A^{-1}\cdot-\mathrm{ad}_A^{-1}\cdot P'(x,y'))_F-P(x,y)\,d^{-1}(P\mathrm{ad}_A^{-1}\cdot-\mathrm{ad}_A^{-1}\cdot P')_D+$$
$$+d^{-1}(P\mathrm{ad}_A^{-1}\cdot-\mathrm{ad}_A^{-1}\cdot P')_D\,P'(x,y'). \tag{3.14}$$

As a result the relation (3.11) is equivalent to

$$\sum_{\alpha=1}^{N}\Delta B_\alpha(\Lambda^+,t)(H_\alpha P'-PH_\alpha)=0 \tag{3.15}$$

where the operator Λ^+ is given by (3.14) and $B_\alpha(\Lambda^+,t)=$ $=\sum_{n=0}^{\infty}b_{\alpha n}(t)(\Lambda^+)^n$.

The transformations (3.15) form an infinite-dimensional group (Backlund—Calogero group) of general BTs $P\to P'$ for the two-dimensional spectral problem (3.1).

The consideration of (3.15) for infinitesimal displacement in time $\left(P'(x,y',t)=P(x,y',t)+\varepsilon\frac{\partial P(x,y',t)}{\partial t},\quad B_\alpha=1-\varepsilon\Omega_\alpha(L^+,t),\quad \varepsilon\to 0\right)$ as usually gives

$$\frac{\partial P(x,y,t)}{\partial t}=\Delta\sum_{\alpha=1}^{N}\Omega_\alpha(L^+,t)(H_\alpha P'-PH_\alpha) \tag{3.16}$$

where $L^+(x,y',y)\overset{def}{=}\Lambda^+(x,y',y)|_{P'=P(x,y',t)}$ and $\Omega_\alpha(L^+,t)$ are arbitrary entire functions on L^+.

Formula (3.16) gives the general form of the nonlinear evolution equations in 1+2 dimensions (t,x,y) integrable by the $N\times N$ problem (3.1).

The second way of construction of the transformations (3.15) and integrable equations (3.16) is the following.

The redifinition of the wave function in (3.1) $\Psi=e^{-\lambda y}\tilde{\Psi}$ transforms into the problem

$$\partial_x\tilde{\Psi}+A\partial_y\tilde{\Psi}+P(x,y,t)\tilde{\Psi}=\lambda A\tilde{\Psi} \tag{3.17}$$

where λ is the parameter. Similar to (3.2) one finds the adjoint representation for (3.17). It is of the form

$$\partial_x\Phi^{in}(x,y',y,\lambda)+A\partial_{y'}\Phi^{in}(x,y',y,\lambda)+\partial_y\Phi^{in}(x,y',y,\lambda)A+$$
$$+P'(x,y')\Phi^{in}(x,y',y,\lambda)-\Phi^{in}(x,y',y,\lambda)P(x,y)=$$
$$=\lambda[A,\Phi^{in}(x,y',y,\lambda)] \tag{3.18}$$

where

$$\Phi(x,y',y,\lambda)\overset{def}{=}\hat{\tilde{F}}'(x,y',\lambda)\otimes\check{\tilde{F}}(x,y,\lambda)$$

and

$$\partial_{x'}\hat{\hat{F}}' + A\partial_{y'}\hat{\hat{F}}' + P'(x,y')\hat{\hat{F}}' = \lambda A \hat{\hat{F}}',$$
$$-\partial_x \check{\check{F}} - \partial_y \check{\check{F}} A + \check{\check{F}} P = \lambda \check{\check{F}} A.$$

Multiplying (3.18) by an arbitrary diagonal matrix $B(\lambda, t)$ and integrating, one obtains instead of (3.3) the following fundamental relation

$$\sum_{\alpha=1}^{N} \langle B_\alpha(\lambda, t)(H_\alpha P' \Phi_F^F(\lambda) - PH_\alpha \Phi_F^F(\lambda))\rangle = 0. \quad (3.19)$$

The irreducible form of the adjoint representation (3.18) for $\Phi_F^F(x, y', y, \lambda)$ is

$$\Lambda(x, y', y)\Phi_F^F(\lambda) = \lambda \Phi_F^F(x, y', y, \lambda) \quad (3.20)$$

where the operator Λ is given by (3.5). With the use of (3.20) one gets:

$$\text{l.h.s. of } (3.19) = \sum_{\alpha=1}^{N} \langle H_\alpha P' B_\alpha(\Lambda, t)\Phi_F^F - PH_\alpha B_\alpha(\Lambda, t)\Phi_F^F \rangle =$$

$$= \langle \Phi_F^F \cdot \Delta \sum_{\alpha=1}^{N} B_\alpha(\Lambda^+, t)(H_\alpha P' - PH_\alpha)\rangle = 0 \quad (3.21)$$

where the operator Λ^+ is given by (3.14). As a result we again obtain the relation (3.15).

In virtue of the operator equality $\Lambda^+ = -\partial_{y'} - \hat{\Lambda}_1^+ = \partial_y + \check{\Lambda}_1^+$ the BTs in the forms (3.11) and (3.15) completely equivalent.

At last the third way of deriving the formulas (3.15), (3.16) consists in the using the nonlocal gauge transformations (1.20)

$$\Psi(x, y) \to \Psi'(x, y') = \int dy\ G(x, y', y)\Psi(x, y) \quad (3.22)$$

for the problem (3.1) with the Anzatz

$$G(x, y', y) = \sum_{k=0}^{n} \partial_y^k \delta(y' - y)\, V_k(x, y', y)$$

(for $N=2$ see [15]).

Namely by this third way the bilocal recursion operator Λ^+ and the hierarhy of the integrable equations in the form (3.16) in the particular case $N=2$ have been derived for the first time in [15].

The transformations (3.15) with the time-independent B_α form an infinite-dimensional group of auto-BTs for equations (3.16). The BC group of transformations (3.15) contains also an infinite-dimensional abelian symmetry group of equations (3.16). In the infinitesimal form these symmetry transformations are

$$\delta P(x, y, t) = \Delta \sum_{\alpha=1}^{N} \omega_\alpha(L^+)(H_\alpha P' - PH_\alpha) \quad (3.23)$$

where $\omega_\alpha(L^+)$ are arbitrary entire functions. Note that the quantities $\mathscr{L}_\alpha = H_\alpha P' - PH_\alpha$ are starting symmetries in the terminology of the paper [15].

All three ways of calculations give, of course, the equivalent results. The different forms of BTs and integrable equations may be useful for the different purposes.

The first way of construction of BTs (3.15) and integrable equations (3.16) is the logically complete formulation of the initial bilocal approach proposed in [13]. The operators Λ_n^+ and L_n^+ contain only the operator $\partial_{y'} + \partial_y$ but not $\partial_{y'}$ and ∂_y separately. Hence these operators permit the direct projection onto the diagonal $y' = y$. As a result the formula (3.11) can be rewritten in the terms of local operators and quantities and we obtain the general BTs in the form given in [9].

The second way is the most close to the usual one-dimensional procedure (see e. g. [8]). Emphasize that the bilocal recursion operator $\Lambda^+(x, y', y)$ does not permit the direct projection onto the diagonal $y' = y$.

4. TWO-DIMENSIONAL DIFFERENTIAL ZAKHAROV—SHABAT SPECTRAL PROBLEM

The scalar two-dimensional Zakharov—Shabat problem

$$\partial_y \Psi + \partial_x^N \Psi + U_{N-2}(x, y, t)\partial_x^{N-2}\Psi + \ldots + U_0(x, y, t)\Psi = 0 \quad (4.1)$$

has been considered previously in [10, 13]. In order to construct the general BTs and general integrable equations in the forms (1.11) and (1.12) one needs only the minor modifications of the formulas derived in [13]. Really the only thing one should do is to convert the local operators Λ_n and quantities K_n found in [13] into the bilocal ones (it is simply the change $U'_k(x,y)\to U'_k(x,y')$) and apply the projection Δ onto the diagonal $y'=y$ at the very end. And as a result we arrive to BTs and equations in the forms (1.11) and (1.12). Note also that the bilocal operators Λ_n^+ are calculated by the recurrent relation (1.13).

Let us consider the simplest and most popular case of the nonstationary Shrodinger problem

$$\partial_y\Psi+\partial_x^2\Psi+U(x,y,t)\Psi=0 \tag{4.2}$$

in more details as the illustration.

The well known 2×2 matrix form of (4.2) is given by (1.7) with $A\begin{pmatrix}0&0\\1&0\end{pmatrix}$ and $P\begin{pmatrix}0&-1\\U&0\end{pmatrix}$. The adjoint representation is (1.10). Multiplying (1.10) by the matrix-operator $B(\partial_y,t)$ of the form

$$B(\partial_y,t)=B_1(\partial_y,t)+B_2(\partial_y,t)\begin{pmatrix}0&-1\\\partial_y&0\end{pmatrix}$$

where B_1 and B_2 are arbitrary functions, we obtain the fundamental relation

$$\langle B(\partial,t)P'\Phi-B(-\partial',t)\Phi P\rangle=0$$

where $\partial\equiv\partial_y$, $\partial'\equiv\partial_{y'}$, and $P=\begin{pmatrix}0&-1\\U&0\end{pmatrix}$. In the matrix components of $\Phi=\begin{pmatrix}\Phi_1&\Phi_2\\\Phi_3&\Phi_4\end{pmatrix}$ this equality is

$$\langle B_1(\partial,t)U'\Phi_2-B_2(\partial,t)U'\Phi_1-$$
$$-B_1(-\partial',t)U\Phi_2+B_2(-\partial',t)U\Phi_4\rangle=0. \tag{4.3}$$

The equality (4.3) with $B\equiv1$ defines the independent dynamical component of Φ. Since $\langle P'\Phi-\Phi P\rangle=\langle(U'-U)\Phi_2\rangle$ then it is Φ_2. The adjoint representation (1.10) also gives

$$\Phi_1=\frac{1}{2}\partial_x^{-1}(-\partial_x^2\Phi_2-(\partial_{y'}+\partial_y)\Phi_2-(U'-U)\Phi_2),$$
$$\Phi_4=\frac{1}{2}\partial_x^{-1}(\partial_x^2\Phi_2-(\partial_{y'}+\partial_y)\Phi_2-(U'-U)\Phi_2) \tag{4.4}$$

and

$$\Lambda(x,y',y)\Phi_2\overset{def}{=}\frac{1}{4}\{\partial_x^2+2\partial_{y'}-2\partial_y+U'+U+\partial_x^{-1}(U'+U)\partial_x+$$

$$+\partial_x^{-1}(\partial_{y'}+\partial_y+U'-U)\partial_x^{-1}(\partial_{y'}+\partial_y+U'-U)\}\Phi_2=0. \tag{4.5}$$

The equation (4.5) is equivalent to the following

$$\partial_{y'}\Phi_2(x,y',y)=\hat\Lambda_1(x,y',y)\Phi_2,$$
$$\partial_y\Phi_2(x,y',y)=\check\Lambda_1(x,y',y)\Phi_2 \tag{4.6}$$

where

$$\hat\Lambda_1=\partial_{y'}-\Lambda(x,y',y)=-\frac{1}{4}\{\partial_x^2-2(\partial_{y'}+\partial_y)+U'+U+$$
$$+\partial_x^{-1}(U'+U)\partial_x+\partial_x^{-1}(\partial_{y'}+\partial_y+U'-U)\partial_x^{-1}(\partial_{y'}+\partial_y+U'-U)\} \tag{4.7}$$

and $\check\Lambda_1=\hat\Lambda_1-(\partial_{y'}+\partial_y)$. The equalities (4.6) also give

$$\partial_{y'}^n\Phi_2(x,y',y)=\hat\Lambda_n(x,y',y)\Phi_2,$$
$$\partial_y^n\Phi_2(x,y',y)=\check\Lambda_n(x,y',y)\Phi_2, \tag{4.8}$$
$$n=1,2,3,\ldots$$

where

$$\hat\Lambda_n(x,y',y)=\hat\Lambda_{n-1}(x,y',y)\hat\Lambda_1+\frac{\partial\hat\Lambda_{n-1}(x,y',y)}{\partial y'}\quad(n=2,3,\ldots).$$

For functions B_1 and B_2 of the form $B_\alpha=\sum_{n=0}^{\infty}b_{\alpha n}(t)\,\partial_y^n$ $(\alpha=1,2)$ in virtue of (4.4) and (4.8) the equality (4.3) is equivalent to the following

$$\langle\sum_{n=0}^{\infty}b_{1n}(t)(U'\check\Lambda_n-(-1)^nU\hat\Lambda_n)\Phi_2+$$
$$+\sum_{n=0}^{\infty}b_{2n}(t)\frac{1}{2}\{U'\partial_x^{-1}(\partial_x^2+\partial_{y'}+\partial_y+U')\check\Lambda_n\Phi_2-$$
$$-\sum_{m=0}^{n}C_m^nU'\left(\frac{\partial}{\partial y}\right)^{n-m}U\cdot\check\Lambda_m\Phi_2+(-1)^nU\partial_x^{-1}(\partial_x^2-\partial_{y'}-\partial_y+U)\hat\Lambda_n\Phi_2-$$
$$-(-1)^n\sum_{m=0}^{n}C_m^nU\left(\frac{\partial}{\partial y'}\right)^{n-m}U'\cdot\hat\Lambda_m\Phi_2\}\rangle=0. \tag{4.9}$$

Transferring to the adjoint operators in (4.9) we finally obtain the relation

$$\Delta \sum_{n=0}^{\infty} b_{1n}(t)\,(\check{\Lambda}_n^+ U' - (-1)^n \hat{\Lambda}_n^+ U) +$$

$$+ \Delta \frac{1}{2} \sum_{n=0}^{\infty} b_{2n}(t) \Big\{ \check{\Lambda}_n^+ (-\partial_x^2 + \partial_{y'} + \partial_y - U')\,\partial_x^{-1} U' -$$

$$-(-1)^n \hat{\Lambda}_n^+ (\partial_x^2 + \partial_{y'} + \partial_y + U)\,\partial_x^{-1} U +$$

$$+ \sum_{m=0}^{n} C_m^n \,(\check{\Lambda}_m^+ (\partial_y^{n-m} U)\partial_x^{-1} U' - (-1)^n \hat{\Lambda}_m^+ (\partial_{y'}^{n-m} U')\partial_x^{-1} U)\Big\} = 0 \qquad (4.10)$$

where

$$\hat{\Lambda}_n^+ (x, y', y) = \left(\hat{\Lambda}_1^+ (x, y', y) + \frac{\partial}{\partial y'} \right)^{n-1} \hat{\Lambda}_1^+ (x, y', y),$$

$$\check{\Lambda}_n^+ (x, y', y) = \left(\check{\Lambda}_1^+ (x, y', y) + \frac{\partial}{\partial y} \right)^{n-1} \check{\Lambda}_1^+ (x, y', y), \qquad (4.11)$$

$$n = 2, 3, \ldots$$

$$\check{\Lambda}_1^+ (x, y', y) = \hat{\Lambda}_1^+ (x, y', y) + \partial_{y'} + \partial_y \qquad (4.12)$$

and

$$\hat{\Lambda}_1^+ (x, y', y) = -\frac{1}{4} \{\partial_x^2 + 2(\partial_{y'} + \partial_y) + U' + U + \partial_x (U' + U)\partial_x^{-1} +$$

$$+ (\partial_{y'} + \partial_y + U - U')\partial_x^{-1} (\partial_{y'} + \partial_y + U - U')\partial_x^{-1}\} \qquad (4.13)$$

The transformations (4.10) form an infinite-dimensional BC group for the problem (4.1) in the form (1.11). The general form of the integrable equations (KP hierarchy) is

$$\frac{\partial U(x, y, t)}{\partial t} = \Delta \sum_{n=0}^{\infty} \{ \omega_{1n}(t)\,(\check{L}_n^+ U' - (-1)^n \hat{L}_n^+ U) +$$

$$+ \omega_{2n}(t) (\check{L}_n^+ (-\partial_x^2 + \partial_{y'} + \partial_y - U')\partial_x^{-1} U' -$$

$$-(-1)^n \hat{L}_n^+ (\partial_x^2 + \partial_{y'} + \partial_y + U)\partial_x^{-1} U +$$

$$+ \sum_{m=0}^{n} C_m^n \,(\check{L}_m^+ (\partial_y^{n-m} U)\partial_x^{-1} U' - (-1)^n \hat{L}_m^+ (\partial_{y'}^{n-m} U')\partial_x^{-1} U))\} \qquad (4.14)$$

where $U' \equiv U(x, y', t)$ and ω_{1n}, ω_{2n} are arbitrary functions. The KP hierarchy in the bilocal form similar to (4.14) has been considered in [15, 30].

It is easy to see from (4.11) — (4.13) that all the operators $\hat{\Lambda}_n^+$ and $\check{\Lambda}_n^+$ contain only the total derivative $\partial_{y'} + \partial_y$. Hence these operators permit the direct projection onto the diagonal $y' = y$. Applying the projection operator Δ to all factors in (4.10) and (4.14) we obtain the local forms of BTs and integrable equations.

Now let us consider the second bilocal way for the problem (4.1). The $N \times N$ matrix with the formal spectral parameter λ is

$$\partial_x \chi + A \partial_y \chi + P(x, y, t)\chi = \lambda A \chi \qquad (4.15)$$

where

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & -1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & 0 & -1 \\ U_0 & U_1 & \ldots & \ldots & U_{N-1} & 0 \end{pmatrix}. \qquad (4.16)$$

The adjoint representation of (4.15) is the form

$$\partial_x \Phi(x, y', y, \lambda) + A \partial_{y'} \Phi(x, y', y, \lambda) + \partial_y \Phi(x, y', y, \lambda) A +$$

$$+ P'(x, y')\Phi(x, y', y, \lambda) - \Phi(x, y', y, \lambda) P(x, y) = \lambda [A, \Phi(x, y', y, \lambda)]. \qquad (4.17)$$

Multiplying (4.17) by the matrix B of the form $B(\lambda, t) = \sum_{n=0}^{N-1} B_n(\lambda, t)(-\lambda A - P_\infty)^n$ where $P_\infty = \lim_{x, y \to \infty} P(x, y, t)$, $B_n(\lambda, t)$ are arbitrary entire functions on λ and integrating, we obtain the fundamental relation

$$\langle B(\lambda, t)(P'\Phi(\lambda) - \Phi(\lambda)P) \rangle = 0. \qquad (4.18)$$

The relation (4.18) with $B \equiv 1$ shows that the independent dynamical components of Φ are $\Phi_{1N}, \Phi_{2N}, \ldots, \Phi_{N-1,N}$. The adjoint representation (4.17) enables one to calculate the bilocal recursion operator and to exclude the explicit dependence on λ from (4.18) These calculations are close to those in the one-dimensional case $\partial_y = 0$ [31, 8]. So we present here only the main formulas.

The relation (4.17) gives

$$(-\partial_x - A\partial_{y'} - P'(x, y') + \lambda A)^m \Phi(x, y', y) =$$

$$= \Phi(x, y', y)(A \overleftarrow{\partial}_y + P(x, y) + \lambda A)^m. \qquad (4.19)$$

Projecting the matrix system (4.19) onto the last column and using the obtained recurrent relation, one gets

$$\lambda\mathcal{G}(x, y', y)\Phi_\Lambda(\lambda)=\tilde{\mathcal{F}}(x, y', y)\Phi_\Lambda(\lambda) \qquad (4.20)$$

where $(\Phi_\Lambda)_{ik} \overset{def}{=} \delta_{kN}\Phi_{iN}$ $(i,k=1, ..., N)$ and

$$\tilde{\mathcal{G}}(x, y', y)=\sum_{m=0}^{N}\frac{d}{d\lambda}\{(-\partial_x-A\partial_{y'}-P'(x, y')+\lambda A)^m\}|_{\lambda=0}(\cdot\circ V_m)-1, \quad (4.21)$$

$$\tilde{\mathcal{F}}(x, y', y)=-\sum_{m=0}^{N}(-\partial_x-A\partial_{y'}-P'(x, y')+\lambda A)^m|_{\lambda=0}(\cdot\circ V_m) \qquad (4.22)$$

where $(\Phi\circ V_m)_{ik} \overset{def}{=} \Phi_{ik}\ V_m$ and $V_m \overset{def}{=} U_m+\delta_{m0}\ \partial_y$. The operator $\tilde{\mathcal{G}}$ is degenerated one and there exists a constraint

$$\sum_{k=1}^{N} l_k\,\Phi_{kN}=0 \qquad (4.23)$$

where

$$l_k(x, y', y)=(\partial_{y'}+\partial_y)\delta_{k1}+U'_{k-1}(x, y')-U_{k-1}(x, y)-$$
$$-\sum_{n=1}^{N-k+1}C_{k-1}^{n+k-1}(-\partial_x)^n U_{n+k-1}(x, y) \qquad (4.24)$$

which allows one to express Φ_{NN} via Φ_{1N}, Φ_{2N}, ..., $\Phi_{N-1,N}$; $\Phi_{NN}=-\sum_{k=1}^{N-1} l_k\Phi_{kN}$.

As a result, equation (4.20) is equivalent to the following

$$\lambda\mathcal{G}(x, y', y)\,\Phi^*=\mathcal{F}(x, y', y)\,\Phi^* \qquad (4.25)$$

where $\Phi^* \overset{def}{=} (\Phi_{1N}, \Phi_{2N}, ..., \Phi_{N-1,N})^T$; and $(N-1)\times(N-1)$ matrix operators $\mathcal{G}$ and $\mathcal{F}$ are

$$\mathcal{G}_{nk}(x, y', y)=\sum_{m=0}^{N}\left(\frac{d}{d\lambda}\{(-\partial_x-A\partial_{y'}-P'(x, y')+\lambda A)^m\}|_{\lambda=0}\right)_{n+1,k}\times$$
$$\times(\cdot\circ\nabla_m)-\delta_{n+1,k}, \qquad (4.26)$$

$$\mathcal{F}_{nk}(x, y', y)=-\sum_{m=0}^{N}((-\partial_x-A\partial_{y'}-P'(x, y')+\lambda A)^m|_{\lambda=0})_{n+1,k}(\cdot\circ V_m)-$$

$$-\sum_{m=0}^{N}((-\partial_x-A\partial_{y'}-P'(x, y')+\lambda A)^m|_{\lambda=0})_{nN}(l_k\cdot\circ V_m). \qquad (4.27)$$

The operator $\mathcal{G}$ is lowertriangular one. Hence $\mathcal{G}^{-1}$ is easily calculated. So we finally have

$$\Lambda(x, y', y)\Phi^*(\lambda)=\lambda\Phi^*(x, y', y, \lambda) \qquad (4.28)$$

where

$$\Lambda(x, y', y)=(\mathcal{G}^{-1}\mathcal{F})(x, y', y) \qquad (4.29)$$

and $\mathcal{G}$ and $\mathcal{F}$ are given by (4.26) and (4.27).

The bilocal operator $\Lambda(x, y', y)$ is just the bilocal recursion operator for the general problem (4.1) which we are interesting in.

With the use of (4.28) similar to the one-dimensional case [31, 8] one excludes the explicit dependence on λ in (4.18). Then transferring to the adjoint operators in the relation obtained, one gets

$$\Delta\sum_{n=0}^{N-1}B_n(\Lambda^+, t)\,K_n(U', U)=0 \qquad (4.30)$$

where Λ^+ is the operator adjoint to Λ and K_n are the certain bilocal quantities. We do not present them here due to their cumbersomeness.

The relation (4.30) represents the general BTs for the generic problem (4.1) in the bilocal form. The general form of the integrable equations is

$$\frac{\partial U(x, y, t)}{\partial t}=\Delta\sum_{n=0}^{N-1}\Omega_n(\Lambda^+, t)\,\mathcal{L}_n(U', U) \qquad (4.31)$$

where $U \overset{def}{=} (U_0, ..., U_{N-2})^T$ and $\Omega_n(L^+, t)$ are arbitrary functions entire on L^+.

Let us consider now our simplest standard example, i. e. the case $N=2$. The matrix B is of the form $B(\lambda, t)=B_1(\lambda, t)+$ $+B_2(\lambda, t)\begin{pmatrix}0 & 1\\ -\lambda & 0\end{pmatrix}$ where B_1, B_2 are arbitrary functions. The independent dynamical component of $\begin{pmatrix}\Phi_1 & \Phi_2\\ \Phi_3 & \Phi_4\end{pmatrix}$ is Φ_2. The fundamental relation (4.18) is equivalent to the following $(U_0\equiv U)$

$$\langle B_1(\lambda, t)(U'-U)\,\Phi_2+B_2(\lambda, t)(-U'\Phi_1+U\Phi_4)\rangle=0. \qquad (4.32)$$

The adjoint representation (4.17) gives the relations (4.4) and the bilocal recursion operator $\Lambda(x, y', y)$:

$$\Lambda(x, y', y)\Phi_2(\lambda)=\lambda\Phi_2(x, y', y, \lambda) \qquad (4.33)$$

where $\Lambda(x, y', y)$ is given by (4.5). Using (4.33) and (4.4) and transiting to the adjoint operators, we obtain from (4.32) the relation

$$\Lambda(B_1(\Lambda^+, t)\, K_1+B_2(\Lambda^+, t)\, K_2)=0 \qquad (4.34)$$

where B_1, B_2 are arbitrary functions entire on Λ^+, $K_1=U'-U$, $K_2=\frac{1}{2}\partial_x(U'+U)+\frac{1}{2}(\partial_{y'}+\partial_y+U-U')\partial_x^{-1}(U-U')$ and the recursion operator Λ^+ is

$$\Lambda^+(x, y', y)=\frac{1}{4}\{\partial_x^2+2\partial_y-2\partial_{y'}+U'+U+\partial_x(U'+U)\partial_x^{-1}+$$
$$+(\partial_{y'}+\partial_y+U-U')\partial_x^{-1}(\partial_{y'}+\partial_y+U-U')\partial_x^{-1}\}. \qquad (4.35)$$

The transformations (4.34) form the infinite-dimensional group of general BTs for the problem (4.2).

Correspondingly the general form of the integrable equations is

$$\frac{\partial U(x, y, t)}{\partial t}=\Lambda(\Omega_1(L^+, t)\mathscr{L}_1+\Omega_2(L^+, t)\mathscr{L}_2) \qquad (4.36)$$

where $L^+ \stackrel{def}{=} \Lambda^+|_{U'=U(x, y', t)}$, $\mathscr{L}_\alpha \stackrel{def}{=} K_\alpha|_{U'=U(x, y', t)}$ and Ω_1, Ω_2 are arbitrary functions entire on L^+. Equation (4.36) with $\Omega_1=0$ and $\Omega_2=4L^+$ is the famous KP equation.

The transformations (4.34) with the time-independent functions B_1 and B_2 form an infinite-dimensional group of general auto-BTs for equations (4.36). The symmetry transformations of equations (4.36) in the infinitesimal form are

$$\delta U(x, y, t)=\Lambda(\omega_1(L^+)\mathscr{L}_1+\omega_2(L^+)\mathscr{L}_2) \qquad (4.37)$$

where ω_1 and ω_2 are arbitrary entire functions.

The equivalence of the BTs and integrable equations in the forms (4.10), (4.14) and (4.34), (4.36) can be established with the use of the equality

$$\Lambda^+(x, y', y)=-\partial_{y'}-\hat{\Lambda}_1^+(x, y', y)=\partial_y+\check{\Lambda}_1^+(x, y', y).$$

The third way of construction of the KP hierarchy (4.36) has been considered earlier in [14, 15]. Note that $L^+(x, y_1, y_2)=\Phi_{12}$ and $\Phi_2(x, y', y)=\check{\Psi}(x, y)\,\hat{\Psi}(x, y')$. The quantities $\mathscr{L}_1$ and $\mathscr{L}_2$ are starting Fokas-Santini symmetries.

5. THE SYMMETRIC SECOND ORDER DIFFERENTIAL TWO-DIMENSIONAL SPECTRAL PROBLEM

Here we will consider the spectral problem

$$(\partial_x^2-\sigma^2\partial_y^2+\varphi(x, y, t)(\partial_x+\sigma\partial_y)+U(x, y, t))\,\Psi=0 \qquad (5.1)$$

where $\varphi(x, y, t)$, $U(x, y, t)$ are scalar functions such that $\varphi, U \xrightarrow[x, y\to\infty]{} 0$ and $\sigma^2=\pm 1$. The generic problem $(\partial_x^2-\sigma^2\partial_y^2+\varphi_1\partial_x+\varphi_2\partial_y+U)\Psi=0$ is reduced to (5.1) by the appropriate gauge transformation $\Psi\to g\Psi$. The problem (5.1) and some corresponding integrable equations have been considered in [32—36].

Firstly we represent (5.1) in the 2×2 matrix form [35, 34]

$$\partial_x\chi+\sigma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_y\chi+\begin{pmatrix}0 & -1\\ U & \varphi\end{pmatrix}\chi=0. \qquad (5.2)$$

The adjoint representation of (5.2) is given by (1.10) with $A=\sigma\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $P=\begin{pmatrix}0 & -1\\ U & \varphi\end{pmatrix}$. Multiplying this adjoint representation by the matrix-operator $B(\partial_y, t)$ of the form $B=B_1(\partial_y, t)+B_2(\partial_y, t)\begin{pmatrix}\sigma\partial_y, & -1\\ 0, & -\sigma\partial_y\end{pmatrix}$ where B_1 and B_2 are arbitrary functions and integrating, we obtain in a standard manner the fundamental relation

$$\langle B(-\partial, t)\, P'\Phi-B(\partial', t)\,\Phi P\rangle=0. \qquad (5.3)$$

It follows from (5.3) with $B\equiv 1$ that the independent dynamical components of $\Phi=\begin{pmatrix}\Phi_1 & \Phi_2\\ \Phi_3 & \Phi_4\end{pmatrix}$ are Φ_2 and Φ_4. We denote $\Phi_\Lambda=\begin{pmatrix}\Phi_2\\ \Phi_4\end{pmatrix}$.

We will consider the functions $B_1(\partial_y, t)$ and $B_2(\partial_y, t)$ entire on ∂_y: $B_\alpha(\partial_y, t)=\sum_{n=0}^{\infty} b_{\alpha n}(t)\partial_y^n$, $\alpha=1, 2$. The fundamental relation (5.3), rewritten in the components of Φ, contains Φ_1, Φ_2, Φ_3, Φ_4 and their derivatives over y' and y. It is necessary firstly to express Φ_1, Φ_3 via Φ_2, and Φ_4 (i. e. Φ_Δ) and secondly to express $\partial^n_{y'}\Phi_\Delta(x,y',y)$ and $\partial^n_y\Phi_\Delta(x,y',y)$ via $\Phi_\Delta(x,y',y)$). The adjoint representation alows one to do this. Namely, it gives

$$\begin{aligned}\Phi_1(x,y',y)&=\partial^{-1}_{++}\left(-(\partial_{--}+\varphi'-\varphi)\Phi_4+(U-U')\Phi_2\right),\\ \Phi_3(x,y',y)&=-(\partial_{--}+\varphi'-\varphi)\Phi_4-U'\Phi_2,\end{aligned}\tag{5.4}$$

and

$$\partial_{y'}\Phi^F_\Delta(x,y',y)=\hat\Lambda_1(x,y',y)\Phi^F_\Delta\tag{5.5}$$

where

$$\hat\Lambda_1(y,y')=\frac{1}{2\sigma}\begin{vmatrix}-\partial_{--}+\varphi+\partial^{-1}_{++}(U'-U), & -\partial^{-1}_{++}(2\partial_x+\varphi'-\varphi),\\ I(-2\sigma U'_{y'}+U'\partial_{--}+ & I(-2\sigma\varphi'_{y'}+U-U'+\\ +\partial_{++}U'+U'(\varphi'-\varphi)), & +(\partial_{++}+\varphi')(\partial_{--}+\varphi'-\varphi))\end{vmatrix}\tag{5.6}$$

Here $\partial_{\pm\pm}\overset{def}{=}\partial_x\pm\sigma\partial_{y'}\pm\sigma\partial_y$, $If\overset{def}{=}(\partial_{--}+\varphi'-\varphi)^{-1}f=\exp(\partial^{-1}_{--}(\varphi-\varphi'))\cdot\partial^{-1}_{--}\{\exp(\partial^{-1}_{--}(\varphi'-\varphi))\cdot f\}$ and $U'=U'(x,\ y')$, $\varphi'=\varphi'(x,\ y')$.

It follows from (5.5) that

$$\partial^n_{y'}\Phi^F_\Delta(x,y',y)=\hat\Lambda_n(x,y',y)\Phi^F_\Delta,\quad n=1,2,3,\ldots\tag{5.7}$$

where $\hat\Lambda_n$ are calculated by the recurrent relation

$$\hat\Lambda_n(x,y',y)=\hat\Lambda_{n-1}(x,y',y)\hat\Lambda_1+\frac{\partial\hat\Lambda_{n-1}(x,y',y)}{\partial y'},\quad n=2,3,\ldots$$

For the adjoint operators one has

$$\hat\Lambda^+_n(x,y',y)=\left(\hat\Lambda^+_1(x,y',y)+\frac{\partial}{\partial y'}\right)^{n-1}\hat\Lambda^+_1,\quad n=2,3,\ldots\tag{5.8}$$

where $(I'=I(\varphi\leftrightarrow\varphi'))$

$$\hat\Lambda^+_1=\frac{1}{2\sigma}\begin{vmatrix}\partial_{--}+\varphi+(U-U')\partial^{-1}_{++}, & (\partial_{--}U'+U'\partial_{++}+\\ & +U'(\varphi-\varphi')+2\sigma U'_{y'})I'\\ (-2\partial_x+\varphi'-\varphi)\partial^{-1}_{++}, & (U'-U-(\partial_{--}+\varphi-\varphi')\times\\ & \times(\partial_{++}-\varphi')+2\sigma\varphi'_{y'})I'\end{vmatrix}\tag{5.9}$$

Emphasize that $\frac{\partial}{\partial y'}$ in (5.8) acts only on $\hat\Lambda^+_1$.

Analogously one can show that

$$\partial^n_{y'}\Phi^F_\Delta(x,y',y)=\check\Lambda_n(x,y',y)\Phi^F_\Delta,\quad n=1,2,3,\ldots\tag{5.10}$$

and

$$\check\Lambda^+_n(x,y',y)=\left(\check\Lambda^+_1(x,y',y)+\frac{\partial}{\partial y'}\right)^{n-1}\check\Lambda^+_1,\quad n=1,2,3,\ldots\tag{5.11}$$

Using (5.4) — (5.11), we exclude the explicit dependence on the operators $\partial_{y'}$, ∂_y in (5.3). Then transferring to the adjoint operators, we finally obtain from (5.3) the relation [35]

$$\begin{aligned}&\sum_{n=0}^{\infty}b_{1n}(t)\Delta\left((-1)^n\check\Lambda^+_n\begin{pmatrix}U'\\ \varphi'\end{pmatrix}-\hat\Lambda^+_n\begin{pmatrix}U\\ \varphi\end{pmatrix}\right)+\\ &+\sum_{n=0}^{\infty}b_{2n}(t)\Delta\Bigg\{\sigma(-1)^n\check\Lambda^+_{n+1}\begin{pmatrix}U'\\ \varphi'\end{pmatrix}+\sigma\hat\Lambda^+_{n+1}\begin{pmatrix}-U\\ \varphi\end{pmatrix}+\\ &+\hat\Lambda^+_n\begin{pmatrix}0\\ U\end{pmatrix}+\hat\Lambda^+_n\begin{pmatrix}(U'-U)\partial^{-1}_{++}U'+\varphi'U'\\ (\partial_{--}+\varphi-\varphi')\partial^{-1}_{++}U'-(\partial_{--}+\varphi-\varphi')\varphi'\end{pmatrix}+\\ &+\sum_{m=0}^{n-1}(-1)^{n+1}C^m_n\check\Lambda^+_m\begin{pmatrix}\partial^{n-m}_y U\cdot\partial^{-1}_{++}U'\\ \partial^{n-m}_y\varphi\cdot(\partial^{-1}_{++}U'-\varphi')\end{pmatrix}\Bigg\}=0\end{aligned}\tag{5.12}$$

where $C^m_n=\frac{n!}{m!\,(n-m)!}$.

Transformations (5.12) form an ifinite-dimensional abelian group of general BTs $((U,\ \varphi)\to(U',\ \varphi'))$ for the spectral problem (5.1) in the form (1.11).

The consideration of (5.12) for the infinitesimal displacement in the time: $U'=U(x,y',t)+\varepsilon\frac{\partial U(x,y',t)}{\partial t}$, $\varphi'=\varphi(x,y',t)+\varepsilon\frac{\partial\varphi(x,y',t)}{\partial t}$, $b_{\alpha n}=\delta_{\alpha,1}+\varepsilon\omega_{\alpha n}(t)$, $\varepsilon\to0$) gives [35]

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}U\\ \varphi\end{pmatrix}&=\Delta\sum_{n=0}^{\infty}\Bigg\{\omega_{1n}(t)\left((-1)^n\check L^+_n\begin{pmatrix}U'\\ \varphi'\end{pmatrix}-\hat L^+_n\begin{pmatrix}U\\ \varphi\end{pmatrix}\right)+\\ &+\omega_{2n}(t)\Bigg[\sigma(-1)^n\check L^+_{n+1}\begin{pmatrix}U'\\ \varphi'\end{pmatrix}+\sigma\hat L^+_{n+1}\begin{pmatrix}-U\\ \varphi\end{pmatrix}+\hat L^+_n\begin{pmatrix}0\\ U\end{pmatrix}+\\ &+\check L^+_n\begin{pmatrix}(U'-U)\partial^{-1}_{++}U'+\varphi'U'\\ (\partial_{--}+\varphi-\varphi')\partial_{++}U'-(\partial_{--}+\varphi-\varphi')\varphi'\end{pmatrix}+\\ &+\sum_{m=0}^{n-1}(-1)^{n+1}C^m_n\check L^+_m\begin{pmatrix}\partial^{n-m}_y U\cdot\partial^{-1}_{++}U'\\ \partial^{n-m}_y\varphi\cdot(\partial^{-1}_{++}U'-\varphi')\end{pmatrix}\Bigg]\Bigg\}\end{aligned}\tag{5.13}$$

where $L_n^+ \overset{def}{=} \Lambda_n^+ \Big|_{\substack{U'=U(x,y',t)\\ \varphi'=\varphi(x,y',t)}}$ and $\omega_{\alpha n}(t)$ are abitrary functions.

Formula (5.13) gives the general form of nonlinear evolution systems in $1+2$ dimensions $(t,\ x,\ y)$ integrable by the spectral problem (5.1).

The transformations (5.12) with the time-independent b_{in} form an infinite-dimensional abelian group of general auto-BTs for equations (5.13). The simplest auto BT (5.12) with $b_{in}=0$ ($i=1,\ 2,\ n=1,\ 2,\ 3,\ \ldots$) is of the form

$$b_{10}\begin{pmatrix} U'-U \\ \varphi'-\varphi \end{pmatrix} + b_{20}\Delta\Big\{ -\sigma\hat{\Lambda}_1^+ \begin{pmatrix} U'+U \\ \varphi'-\varphi \end{pmatrix} + \begin{pmatrix} (U'-U)\partial_{++}^{-1})U'+\varphi'U'-\sigma U'_{y'} \\ (\partial_{--}+\varphi-\varphi')(\partial_{++}^{-1}U'-\varphi')+U-\sigma\varphi'_{y'} \end{pmatrix}\Big\}=0. \tag{5.14}$$

The group of general auto BTs' is generated by the two elementary BTs.

The simplest equation (5.13) with $\omega_{1n}=0$, $\omega_{21}=2$ and $\omega_{20}=\omega_{22}=\omega_{23}=\ldots=0$ and $\sigma=1$ is the BLP system [34].

For the functions $\omega_{\alpha n}$ of the form $\omega_{\alpha,2n}=0$ ($\alpha=1,\ 2$) the system (5.13) admits the reduction $\varphi=0$. As a result we obtain the hierarchy of equations the simplest of which ($\omega_{2,3}=4$, $\omega_{2,5}=\ldots=0$, $\sigma^2=\pm 1$) is the Nizhnik — Veselov — Novikov equation [32, 33].

In the one-dimensional limit $\partial_y U=\partial_y\varphi=0$ the BTs (5.12) and integrable equations (5.13) are reduced to BTs and integrable equations associated with the spectral problem $(\partial_x+\varphi+U\partial_x^{-1})\Psi=\lambda\Psi$ (see [37, 38]).

Now let us consider the second way. In this case we must use instead of (5.2) the problem

$$\left(\partial_x+\sigma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_y+\begin{pmatrix} 0 & -1 \\ U & \varphi \end{pmatrix}\right)\tilde{\chi}=\lambda\sigma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\tilde{\chi} \tag{5.15}$$

with the adjoint representation of the form (1.15). Multiplying this adjoint representation by the matrix $B=B_1(\lambda,t)+B_2(\lambda,t)\begin{pmatrix} \sigma\lambda & -1 \\ 0 & -\sigma\lambda \end{pmatrix}$ where B_1 and B_2 are arbitrary functions, we obtain instead (5.3) the following fundamental relation

$$\langle B(\lambda,t)P'\Phi(\lambda)-B(\lambda,t)\Phi(\lambda)P\rangle=0$$

The adjoint representation (1.15) gives the relations (5.4) and the

bilocal recursion operator $\Lambda(x,y',y)$

$$\Lambda(x,y',y)\,\Phi_\Delta^F(\lambda)=\lambda\Phi_\Delta^F(x,y',y,\lambda) \tag{5.17}$$

where

$$\Lambda=\frac{1}{2\sigma}\begin{pmatrix} \partial_{+-}-\varphi-\partial_{++}^{-1}(U'-U), & -1-\partial_{++}^{-1}(\partial_{--}+\varphi'-\varphi) \\ -(\partial_{--}+\varphi'-\varphi)^{-1}(\partial_{-+}U'+ & (\partial_{--}+\varphi'-\varphi)^{-1}(U-U'+ \\ +U'\partial_{+-}+U'(\varphi'-\varphi)), & +(\partial_{--}+\varphi'-\varphi)(\partial_{-+}+\varphi')) \end{pmatrix} \tag{5.18}$$

Using (5.4) and (5.17) we exclude the explicit dependence on λ in (5.16). Then transferring to the adjoint operator, we obtain the general BTs for the problem (5.1) in the form (1.17). They are [17, 18]

$$\Delta(B_1(\Lambda^+,t)K_1+B_2(\Lambda^+,t)K_2)=0 \tag{5.19}$$

where

$$K_1=\begin{pmatrix} U'-U \\ \varphi'-\varphi \end{pmatrix},\quad K_2=\begin{pmatrix} (U'-U)\partial_{++}^{-1}U'-\varphi'U' \\ (\partial_{--}+\varphi-\varphi')(\varphi'-\partial_{++}^{-1}U')-U \end{pmatrix}-\sigma\Lambda^+\begin{pmatrix} U'+U \\ \varphi'-\varphi \end{pmatrix}$$

and the bilocal recursion operator Λ^+ is

$$\Lambda^+=-\frac{1}{2\sigma}\begin{pmatrix} \partial_{+-}+\varphi-(U'-U)\partial_{++}^{-1}, & (\partial_{+-}U'+U'\partial_{-+}-U'(\varphi'-\varphi))\times \\ & \times(\partial_{--}+\varphi-\varphi')^{-1} \\ 1+(\partial_{--}+\varphi-\varphi')\partial_{++}^{-1}, & (U'-U-(\partial_{--}+\varphi-\varphi')\times \\ & \times(\partial_{-+}-\varphi'))(\partial_{--}+\varphi-\varphi')^{-1} \end{pmatrix} \tag{5.20}$$

General form of the integrable equations is the following [17, 18]

$$\frac{\partial}{\partial t}\begin{pmatrix} U \\ \varphi \end{pmatrix}=\Delta(\Omega_1(L^+,t)\mathscr{L}_1+\Omega_2(L^+,t)\mathscr{L}_2) \tag{5.21}$$

where Ω_1 and Ω_2 are arbitrary functions entire on L^+.

Emphasize that again $\Lambda^+=-\partial_{y'}-\hat{\Lambda}_1^+=\partial_y+\check{\Lambda}_1^+$ where Λ^+ is given by (5.20) and $\hat{\Lambda}_1^+$ by (5.9) [35].

The bilocal recursion operator Λ^+ (5.20) and BTs for the problem (5.1) have been derived independently by the gauge transformation (1.20) approach in a recent preprint [36].

The recursion operator method in the bilocal formulation is applicable also to the generic problem $\sum\limits_{n,m=0}^{n+m=N} U_{nm}(x,y)\,\partial_x^n\partial_y^m\Psi=0.$

6. MILTIDIMENSIONAL SPECTRAL PROBLEMS

We see that the recursion operator method in the bilocal form allows one to represent the integrable equations and their BTs in the two different forms (1.11) — (1.12) and (1.17) — (1.18). In the form (1.11) — (1.12) they are defined by the sequence of the operators Λ_n^+ which contain only the total derivative $\partial_{y'}+\partial_y$. Hence the operators Λ_n^+ permit the direct projection onto the diagonal $y'=y$ and the formulas (1.11) — (1.12) can be easy rewritten in the local form.

The bilocal recursion operator $\Lambda^+(x, y', y)$ is the principal ingredient of the BTs and integrable equations in the form (1.17) — (1.18). This operator does not permit the direct projection onto the diagonal $y'=y$ but it seems that the operator $\Lambda^+(x, y', y)$ more adequately reflects the recursion structure of the two-dimensional integrable equations. The forms (1.17) — (1.18) of BTs and integrable equations manifestly indicate that they are generated by the single operator.

For the multidimensional spectral problems the situation changes crucialy. Let us consider the matrix spectral problem

$$(A_1\partial_{x_1}+A_2\partial_{x_2}+\ldots+A_n\partial_{x_n}+P(x_1, \ldots, x_n))\Psi=0 \tag{6.1}$$

where A_i are diagonal matrices and $P_D=0$. The adjoint representation of (6.1) is

$$\sum_{i=1}^{n} A_i\partial_{x_i'}\Phi(x', x)+\sum_{i=1}^{n}\partial_{x_i}\Phi(x', x)A_i+P'(x')\Phi(x', x)-\Phi(x', x)P(x)=0. \tag{6.2}$$

Multiplying (6.2) by an arbitrary diagonal matrix-operator $B=B(\partial_{x_1}, \ldots, \partial_{x_n})$ and integrating, we obtain the fundamental relation

$$\langle B(\partial)P'\Phi_F-B(-\partial')\Phi_F P'\rangle=0 \tag{6.3}$$

where $\langle\ldots\rangle=\int dx\,dx'\delta(x-x')\,\mathrm{tr}(\ldots)$ and $x=(x_1, \ldots, x_n)$.

Within the framework of the first approach one should try to extract from (6.2) the relations of the type

$$\partial_{x_i'}\Phi_F(x', x)=\hat{\Lambda}_{(1)i}(x', x)\Phi_F, \qquad \partial_{x_i}\Phi_F(x', x)=\check{\Lambda}_{(1)i}(x', x)\Phi_F, \qquad i=1, \ldots, n \tag{6.4}$$

where the operators $\Lambda_{(1)i}$ should contain the total operators $\partial_{x_k'}+\partial_{x_k}$ ($k=1, \ldots, n$) but not $\partial_{x_k'}$ and ∂_{x_k} separately. It is not difficult to see that such operators $\Lambda_{(1)i}$ do not exist. Indeed, one can rewrite (6.2) in the form

$$[A_k, \partial_{x_k'}\Phi(x', x)]+(\partial_{x_k'}+\partial_{x_k})\Phi(x', x)A_k+ + \sum_{i\neq k}^{n} A_i\partial_{x_i'}\Phi(x', x)+\sum_{i\neq k}^{n}\partial_{x_i}\Phi(x', x)A_i+ + P'(x')\Phi(x', x)-\Phi(x', x)P(x)=0 \tag{6.5}$$

The relation (6.2) gives

$$\Phi_D(x', x)=-d^{-1}(P'\Phi_F-\Phi_F P)_D \tag{6.6}$$

where $d=\sum_{i=1}^{n} A_i(\partial_{x_i'}+\partial_{x_i})$. Substituting (6.6) into the off-diagonal part of (6.5), we get the equation for $\Phi_F(x', x)$

$$[A_k, \partial_{x_k'}\Phi_F(x', x)]=-(\partial_{x_k'}+\partial_{x_k})\Phi_F(x', x)A_k- - \sum_{i\neq k}^{n} A_i\partial_{x_i'}\Phi_F(x', x)-\sum_{i\neq k}^{n}\partial_{x_i}\Phi_F(x', x)A_i- -(P'\Phi_F-\Phi_F P)_F+P'd^{-1}(P'\Phi_F-\Phi_F P)_D-d^{-1}(P'\Phi_F-\Phi_F P)P. \tag{6.7}$$

Equation (6.7) is the irreducible form of the adjoint representation (6.2). The index k in (6.7) is any from $1, \ldots, n$. But (6.7) contains the total derivative only on x_k, i. e. $\partial_{x_k'}+\partial_{x_k}$ and other derivatives $\partial_{x_i'}$, ∂_{x_i} ($i=1, \ldots, k-1, k+1, \ldots, n$) separately. So the operators of the type $\Lambda_{(1)k}$ are essentially bilocal ones on the variables $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$. This prevents the existence of the needed (completely localizable) operators $\Lambda_{(n)k}$ and stops the further construction within the first way of the bilocal approach described above.

The nontrivial result which can be obtained from (6.3) is the system of equations for the resonantly interacting waves in multidimensions. Indeed, if one consider the very special infinitesimal displacement in time, namely, $P'=P(x')+\varepsilon\frac{\partial P(x', t)}{\partial t}$, $B=1-\varepsilon\times\sum_{\alpha=1}^{N}\sum_{k=1}^{n}H_\alpha\omega_{\alpha(k)}\partial_{x_k}$, ($\varepsilon\to 0$) the relation (6.3) with the use of (6.2) or (6.7) gives

$$\frac{\partial P_{\alpha\beta}(x,t)}{\partial t} - \sum_{k=1}^{n} \Omega_{\alpha\beta(k)} \frac{\partial P_{\alpha\beta}}{\partial x_k} - \sum_{\gamma=1}^{N} (\Omega_{\alpha\gamma} - \Omega_{\gamma\beta}) P_{\alpha\gamma} P_{\gamma\beta} = 0 \tag{6.8}$$

where $\Omega_{\alpha\beta(k)} = \omega_{\alpha(k)} - a_{(k)\alpha}\, \Omega_{\alpha\beta}$ $((A_k)_{il} = \delta_{il}\, a_{(k)i})$ and $\frac{\omega_{\alpha(k)} - \omega_{\beta(k)}}{a_{(k)\alpha} - a_{(k)\beta}} = \Omega_{\alpha\beta}$ $(k=1, \ldots, n)$. The system (6.8) has been derived in [39] in a way very similar to the described here.

Within the second way in the bilocal approach one should introduce the formal spectral parameters into (6.1). One can introduce n different formal parameters in (6.1). Namely, one can consider instead of (6.1) the problems

$$\sum_{i=1}^{n} A_i\, \partial_{x_i} \tilde{\Psi} + P(x)\tilde{\Psi} = \lambda_k A_k \tilde{\Psi}(x, \lambda_k) \tag{6.9}$$

for any $k=1, \ldots, n$. The corresponding adjoint representations are

$$\sum_{i=1}^{n} A_c\, \partial_{x'_i} \Phi(x', x, \lambda_k) + \sum_{i=1}^{n} \partial_{x_i} \Phi(x', x, \lambda_k) A_i + \\ + P'(x')\Phi(x', x, \lambda_k) - \Phi(x', x, \lambda_k) P(x) = \lambda_k [A_k, \Phi(x', x, \lambda_k)], \tag{6.10}$$
$$k = 1, \ldots, n.$$

Excluding Φ_D by (6.6), we obtain

$$\Lambda(x', x)\Phi_F(\lambda_k) = \lambda_k [A_k, \Phi_F(x', x, \lambda_k)], \quad k=1, \ldots, n \tag{6.11}$$

where

$$\Lambda(x', x) = \sum_{i=1}^{n} (A_i\, \partial_{x'_i} \cdot + \partial_{x_i} \cdot A_i) + \\ + (P' \cdot - \cdot P)_F - P' d^{-1}(P' \cdot - \cdot P)_D + d^{-1}(P' \cdot - \cdot P)_D P. \tag{6.12}$$

Thus for given k one has

$$\Lambda_k(x', x)\Phi_F(\lambda_k) = \lambda_k \Phi_F(x', x, \lambda_k) \tag{6.13}$$

where $\Lambda_k = \mathrm{ad}_{A_k}^{-1} \Lambda$.

So, formally one has n different bilocal recursion operators Λ_k $(k=1, \ldots, n)$ for the problem (6.1) but effectively there is only one. The consideration of the nonlocal gauge transformations for the problem (6.1) gives the same result.

Another approach to the multidimensional spectral problems has been discussed in [40].

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Recursion and Group Structures of the Integrable Equations in 1+1 and 1+2 Dimensions

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