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ON THE DIAGONAL GAUGE IN THREE-DIMENSIONAL GRAVITY

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НОВОСИБИРСК

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On the Diagonal Gauge in Three-Dimensional Gravity

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ABSTRACT

The functional measure and corresponding effective action in three-dimensional gravity are studied in the gauge in which the metric is diagonal.

1. INTRODUCTION

To get any idea of what is quantum gravity it is useful to refer to the low-dimensional models. In particular, the Faddeev—Popov determinant in the two-dimensional gravity found by Polyakov in the conformal gauge [1] provides a non-trivial dynamics of the quantized theory [2].

In this note we calculate the Fadeev—Popov determinant in three-dimensional Euclidean gravity for metric written in the diagonal gauge:

$$ds^2 = e^{2F_1}(dx^1)^2 + e^{2F_2}(dx^2)^2 + e^{2F_3}(dx^3)^2. \qquad (1)$$

Locally, this gauge is always possible: we first transform metric tensor $g_{\alpha\beta}$ to principal axes at the given point, then we make it's three non-diagonal components vanish also in the vicinity of this point by proper small transformations of three coordinates x^α. The problem of global diagonalization of the metric is more complicated, but it is not important for us here.

2. THE ACTION

Consider for definiteness the Einstein action. In the diagonal gauge

$$S = -\frac{\mu}{2}\int G g^{1/2} d^3x, \quad G = 2e^{-2F_1}F_{2,1}F_{3,1} + \text{perm}, \quad g \equiv \det g_{\alpha\beta}. \qquad (2)$$

Here $\mu=\text{const}$ and $(...)_{,\alpha}\equiv\partial(...)/\partial x^\alpha$. Variation of (2) with respect to F_α leads only to the diagonal components of the Einstein equations; in the linear approximation

$$G_{11}\equiv R_{11}-\frac{1}{2}g_{11}R\equiv F_{3,22}+F_{2,33}=0,\ \text{perm}. \tag{3}$$

However, since not all of the Einstein equations are independent (due to $G^\alpha_{\beta;\alpha}=0$) the non-diagonal components

$$G_{23}\equiv R_{23}-\frac{1}{2}g_{23}R\equiv -F_{1,23}=0,\ \text{perm} \tag{4}$$

follow from (3) and appropriate boundary conditions of the type

$$G_{23}|_{x^2=-\infty}=G_{23}|_{x^3=-\infty}=0,\ \text{perm}. \tag{4a}$$

Such the conditions should be also implied in the corresponding definition of the functional integral for gravity.

3. THE MEASURE

Classical dynamics of three-dimensional general relativity is locally trivial—empty space-time is flat [3]. Nontriviality of quantum dynamics stems from the gauge nature of gravity and from the necessity to extract in proper way the measure on the equivalence classes of metrics $g_{\alpha\beta}$ from the measure Dg on the space of all the metrics in the functional integral [4]*:

$$\langle\Phi(g)\rangle=\int\frac{Dg}{\Omega}\Phi(g)\exp(-s(g)). \tag{5}$$

To take into account physically equivalent metrics no more than once we have divided the measure Dg in (5) by the volume Ω of the diffeomorphism group Diff R^3 tangent space to which is spanned by infinitesimal translations η^α:

$$x'^\alpha=x^\alpha-\eta^\alpha(x),\quad \delta_\eta g_{\alpha\beta}=g_{\alpha\gamma}\eta^\gamma_{,\beta}+g_{\beta\gamma}\eta^\gamma_{,\alpha}+g_{\alpha\beta,\gamma}\eta^\gamma. \tag{6}$$

Introduce invariant norms in the functional spaces of $\delta g_{\alpha\beta}$ and η^α [1, 6]:

$$\|\delta g\|^2=\int g^{1/2}d^3x(g^{\alpha\gamma}g^{\beta\delta}+Cg^{\alpha\beta}g^{\gamma\delta})\delta g_{\alpha\beta}\delta g_{\gamma\delta}, \tag{7a}$$

$$\|\eta\|^2=\int g^{1/2}d^3x\,\eta^\alpha\eta^\beta g_{\alpha\beta},\quad C=\text{const}. \tag{7b}$$

Corresponding measures are normalized as

$$\int D\delta g\exp\left(-\frac{1}{2}\|\delta g\|^2\right)=\int D\eta\exp\left(-\frac{1}{2}\|\eta\|^2\right)=1. \tag{8}$$

The measure $D\delta g$ on tangent space is identified with the measure of interest Dg [6]. Physical variation of the metric $\delta_F g_{\alpha\beta}$ in the gauge (1) is parametrized by δF_α. The total variation is

$$\delta g_{\alpha\beta}=(\delta_F+\delta_\eta)g_{\alpha\beta},\quad \delta g_{11}=2g_{11}\delta F'_1,\quad \delta F'_1\equiv\delta F_1+\eta^1_{,1}+\eta^\alpha F_{1,\alpha},$$
$$\delta g_{12}=g_{11}\eta^1_{,2}+g_{22}\eta^2_{,1},\ \text{perm}, \tag{9}$$

so the norm is

$$\|\delta g\|^2=\int g^{1/2}d^3x\left[\frac{2}{g_{11}g_{22}}\delta g_{12}^2+\text{perm}+4\sum_{\alpha=1}^{3}\delta F'^2_\alpha+4C\left(\sum_{\alpha=1}^{3}\delta F'_\alpha\right)^2\right]. \tag{10}$$

Replacing integral $\int g^{1/2}d^3x[...]$ by the sum over points we get the measure normalized up to a constant:

$$Dg=D\eta DF\left[\det M\prod_x g^{-1/2}(x)\right],$$

$$D\eta=\prod_x g^{1/2}(x)d^3\eta(x),\qquad DF=\prod_x d^3F(x), \tag{11}$$

$$M=\begin{pmatrix}0 & \exp(\varphi_1)\,\partial_3 & \exp(-\varphi_1)\,\partial_2\\ \exp(-\varphi_2)\,\partial_3 & 0 & \exp(\varphi_2)\,\partial_1\\ \exp(\varphi_3)\,\partial_2 & \exp(-\varphi_3)\,\partial_1 & 0\end{pmatrix},$$

$$\varphi_1=F_2-F_3,\ \text{perm},\qquad \partial_\alpha\equiv\frac{\partial}{\partial x^\alpha}.$$

Det means determinant in the operator sense. Make two remarks concerning (11). First, the metric diagonality condition does not uniquely fix the frame: $\delta_\eta g_{\alpha\beta}=0$ for $\alpha\neq\beta$ does not necessarily mean $\eta^\alpha=0$, i. e. M has zero modes. The latter are excluded by imposing boundary conditions

$$\eta^\alpha|_\infty\equiv|_{|x^1|+|x^2|+|x^3|=\infty}=0 \tag{12}$$

* Due to the bottomlessness of the action (2) the Euclidean path integral (5) is defined by integrating over complex-valued fields [5]; see below.

(i. e. fixing the frame at infinity). Second, the factors like $\prod_x C$, $C=\text{const}$ are omitted in Dg in (11). By locality and general covariance these factors are equivalent to including some cosmological term in the effective action [6]. If invariant vacuum expectation values are of interest, integration over $D\eta$ is cancelled by Ω. The factor [...] in Dg in (11) can be written in the form

$$\left[\det M_0 \prod_x g^{-1/2}(x)\right] \det MM_0^{-1}, \qquad M_0=\begin{pmatrix} 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}. \qquad (13)$$

Here the factor [...] is proportional to

$$\prod_{x,\alpha} \exp(-F_\alpha) \det \partial_\alpha \sim \prod_{x,\alpha} \exp(-F_\alpha)(dx^\alpha)^{-1} = \prod_{x,\alpha} (ds)^{-1} \qquad (14)$$

which is 1 up to an adjustment of cosmological term.

4. DISCRETIZATION

So we are left with $\det MM_0^{-1}$. The matrix MM_0^{-1} takes the form

$$(MM_0^{-1})_{\alpha\beta} = \delta_{\alpha\beta} + a_{\alpha\beta}(x)\,\partial_\beta \partial_\alpha^{-1} \qquad (15)$$

(there is no summation over α, β) where

$$a_{\alpha\beta}+\delta_{\alpha\beta} = \begin{pmatrix} \mathrm{ch}\,(F_2-F_3) & \mathrm{sh}\,(F_3-F_2) & \mathrm{sh}\,(F_2-F_3) \\ \mathrm{sh}\,(F_3-F_1) & \mathrm{ch}\,(F_3-F_1) & \mathrm{sh}\,(F_1-F_3) \\ \mathrm{sh}\,(F_2-F_1) & \mathrm{sh}\,(F_1-F_2) & \mathrm{ch}\,(F_1-F_2) \end{pmatrix}. \qquad (16)$$

Then

$$\ln\det MM_0^{-1} = \mathrm{Tr}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \times$$

$$\times \sum_{\alpha_1,\dots,\alpha_n} a_{\alpha_1\alpha_2}\,\partial_{\alpha_2}\partial_{\alpha_1}^{-1}\, a_{\alpha_2\alpha_3}\,\partial_{\alpha_3}\partial_{\alpha_2}^{-1} \dots a_{\alpha_n\alpha_1}\,\partial_{\alpha_1}\partial_{\alpha_n}^{-1}. \qquad (17)$$

We first evaluate the trace on regular lattice with spacing h along each coordinate axis. In terms of invariant geodesic length Δs between neighbouring sites this lattice is inhomogeneous one. Such the «naive» discretization can in some cases lead to wrong continuum limit and, in particular, it does not reproduce the conformal trace

anomaly in two-dimensional gravity [7]. However, there is a distinctive circumstance in our case. Below we shall establish the additivity property of (17) for M defined on subspaces $V(Q_m)$ of functions $\eta(x)$ vanishing outside some lattice subsets Q_m (see (26)). This immediately generalizes our analysis to cover the piecewise-regular lattice with $\Delta s \approx \text{const}$ (this lattice is regular in the regions Q_m sufficiently small for that the metric in them might be considered flat; see Fig. 1). It is just the point our treatment differs from that of two-dimensional gravity on the lattice [7]. In the latter case we deal, roughly, with the determinant of the Laplace operator Δ on

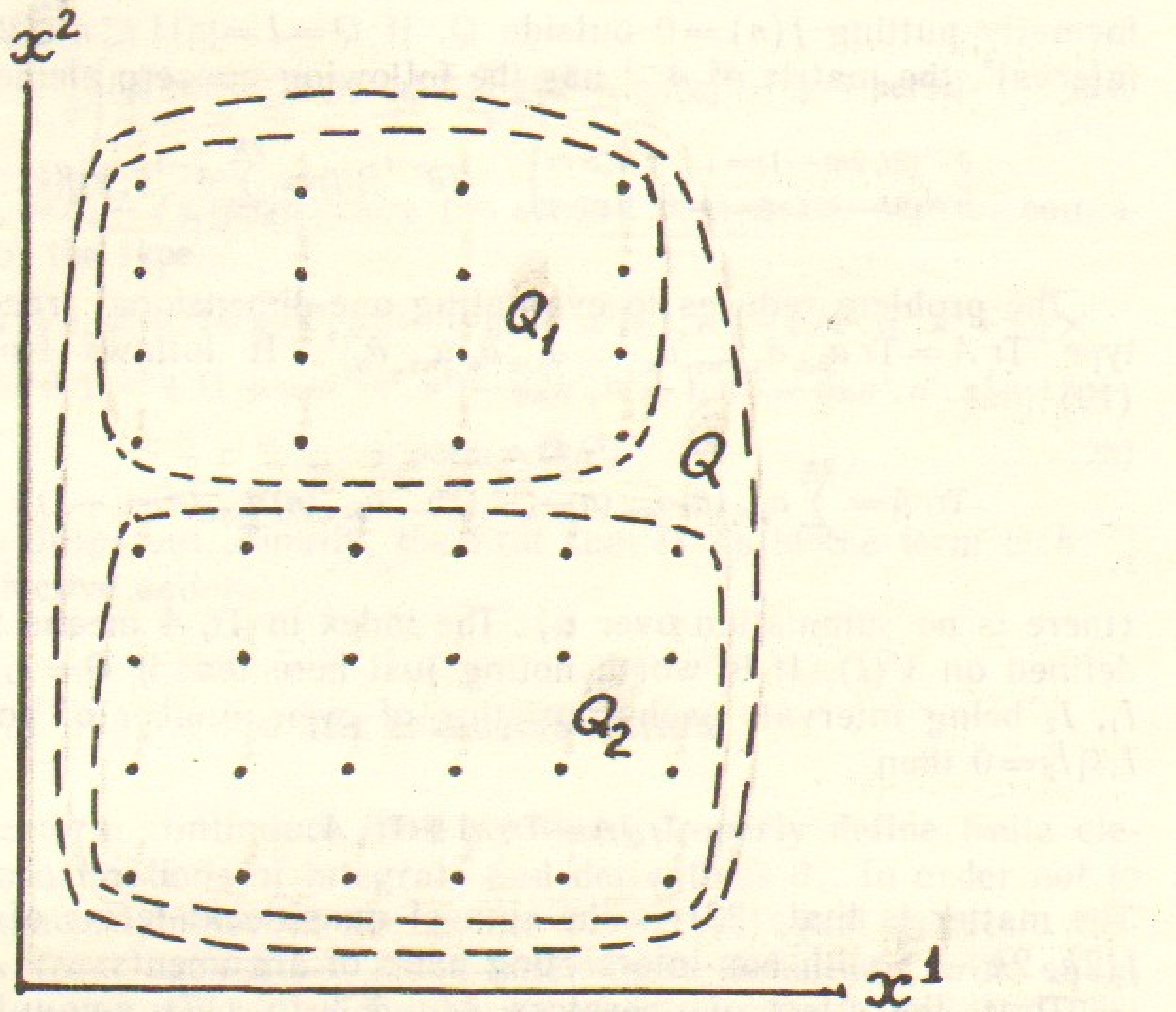

Fig. 1. The piecewise-regular lattice: it's spacing h^α in the region Q_1 and Q_2 is different; $Q=Q_1\cup Q_2$.

curved manifold [1]. Explicitly writing the determinant on the lattice it is not difficult to convince ourselves that boundary effects between nighbouring subsets Q_m break the additivity property for $\ln\det\Delta$. Therefore introducing piecewise-regular lattice makes the analysis quite involved. In any case, the answer will depend non-trivially on the metric. It is just the way the anomaly in two dimensions might show up within our approach.

5. THE DETERMINANT

Consider vector space $V(Q)$ of functions $f(n)$ defined on a set Q of the points n of one-dimensional lattice Z taken along any of the coordinate axes x^α. Introduce differentiation operator ∂ on $V(Q)$

$$(\partial f)(n)=f(n+1)-f(n-1) \tag{18}$$

formally putting $f(n)=0$ outside Q. If $Q=I=\{n \mid 1\leqslant n\leqslant 2N\}$ is an interval*, the matrix of ∂^{-1} has the following nonzero elements:

$$\left.\begin{array}{ll}\partial^{-1}(2l,2m-1)=1, & l\geqslant m\\ \partial^{-1}(2l-1,2m)=-1, & l\leqslant m\end{array}\right\} \qquad (\partial^{-1}f)(i)=\sum_{k=1}^{2N}\partial^{-1}(i,k)f(k). \tag{19}$$

The problem reduces to evaluating one-dimensional traces of the type $\operatorname{Tr} A=\operatorname{Tr} a_{\beta_1\alpha}\partial_\alpha a_{\alpha\gamma_1}\partial_\alpha^{-1}\ldots a_{\beta_m\alpha}\partial_\alpha a_{\alpha\gamma_m}\partial_\alpha^{-1}$. It follows from (18), (19) that

$$\operatorname{Tr}_I A=\sum_{n=1}^{2N} a_{\beta_1\alpha}(n)\,a_{\alpha\gamma_1}(n-(-1)^n)\ldots a_{\beta_m\alpha}(n)\,a_{\alpha\gamma_m}(n-(-1)^n) \tag{20}$$

(there is no summation over α). The index in $\operatorname{Tr}_I A$ means that A is defined on $V(I)$. It is worth noting just here that if $Q=I_1\cup I_2$ with I_1, I_2 being intervals each consisting of even number of points and $I_1\cap I_2=0$ then

$$\operatorname{Tr}_Q A=\operatorname{Tr}_{I_1}A+\operatorname{Tr}_{I_2}A. \tag{21}$$

The matter is that (20) is the sum of quasi-local terms of the type $f(2k,\,2k-1)$ with non-intersecting pairs of arguments.

Thus, the effect of operators $\partial_\alpha\ldots\partial_\alpha^{-1}$ in (17) amounts to the shift

$$a_{\alpha\beta}(n)\to a_{\alpha\beta}^{(\alpha)}(n)\equiv a_{\alpha\beta}(n-(-1)^n) \tag{22}$$

in the x^α direction. As a result, (17) can be summed up to give

* The requirement for the number of points of I be even is necessary for ∂^{-1} to exist on $V(I)$.

$$\ln\det MM_0^{-1}=\sum_{\bar n}\ln\det|\delta_{\alpha\beta}+a_{\alpha\beta}^{(\alpha)}(\bar n)|=$$

$$=\sum_{\bar n}\ln\operatorname{ch}\Phi(\bar n)=\sum_{\bar n}\frac{1}{2}\Phi^2(\bar n)+O(\Phi^4), \tag{23}$$

$$\Phi=F_2^{(1)}-F_3^{(1)}+F_3^{(2)}-F_1^{(2)}+F_1^{(3)}-F_2^{(3)}.$$

Here $\bar n=(n^1,\,n^2,\,n^3)$. In the assumptions of sufficient smoothness of F_α the term $O(\Phi^4)$ is $O(h^4)$. It's contribution to the effective action vanishes as $O(h)$ in the continuum limit. The remaining sum can be rewritten as

$$\frac{1}{2}\sum_{\bar n}\left[(\varphi_1^{(1)}-(\varphi_1)^2+\text{perm}\right]+\sum_{\bar n}[(\varphi_2^{(2)}-\varphi_2)(\varphi_3^{(3)}-\varphi_3)+\text{perm}] \tag{24}$$

where $\varphi_1=F_2-F_3$, perm. Here the second sum is the sum of combinations of the type

$$[\varphi_2(n^1,n^2+1,n^3+1)+\varphi_2(n^1,n^2,n^3)-\varphi_2(n^1,n^2+1,n^3)-\varphi_2(n^1,n^2,n^3+1)]\times$$
$$\times[\varphi_3(n^1,n^2+1,n^3+1)+\varphi_3(n^1,n^2,n^3)-\varphi_3(n^1,n^2+1,n^3)-\varphi_3(n^1,n^2,n^3+1)]+$$
$$+\text{perm}=O(h^4) \tag{25}$$

and also drops out. Finally, the first sum leads to the term $O(h^{-1})$ in the effective action.

6. THE EFFECTIVE ACTION

To perform continuum limit we must properly define finite element approximations of integrals and derivatives ∂_α. In order not to violate space symmetries the points placed at equal geodesic distance Δs from each other must enter these definitions with equal weight, i. e. we must consider the piecewise-regular lattice. But first consider non-intersecting subsets Q_m of the regular Z^3 lattice with Q being the union of them. An additional, not very restrictive requirement is that intersection of each Q_m with any coordinate line of the lattice be an union of intervals each consisting of even number of points. Then

$$\ln\det{}_Q MM_0^{-1}=\sum_m\ln\det{}_{Q_m}MM_0^{-1}, \tag{26}$$

which immediately follows from (21). This formula can be taken as a consistent definition of the determinant on the piecewise-regular lattice. To make use of it we introduce inhomogenecus lattice spacing $h^{\alpha}=e^{-F_{\alpha}}\Delta s$ (see Fig.1; Q_m are taken small enough to assume an approximation $F_{\alpha}=\text{const}$ inside each of them). Then we get the following Euclidean effective action:

$$S_{eff}=-\int g^{1/2}d^3x\left\{e^{-2F_1}\left[\mu F_{2,1}F_{3,1}+\frac{1}{\varepsilon^2}(F_2-F_3)^2_{,1}\right]+\text{perm}+\lambda\right\}, \quad (27)$$

where $\varepsilon^2=2\Delta s\to 0$, λ is the cosmological constant.

Description of the system with the action (27) (with «incorrect» sign) can be given a sense by performing path integration over complex-valued fields [5]: $F_{\alpha}\to F_{\alpha}+\pi i$. This corresponds to quantizing conformal degree of freedom with negative metric [8]. Then S_{eff} reverts it's sign while ds^2 remains unchanged. In this treatment the dominant contribution to the functional integral is given (at least, in Gaussian approximation) by those fields which realize the absolute minimum of the redefined S_{eff}:

$$(F_2-F_3)_{,1}=0, \text{ perm.} \quad (28)$$

This constraines $F_{\alpha}-F_{\beta}$ to be $\xi_{\alpha}(x^{\alpha})-\xi_{\beta}(x^{\beta})$ with ξ_{α} being an arbitrary function of x^{α}. Therefore we can made $F_1=F_2=F_3=F$ by proper reparametrization $x'=x'(x'^{1})$, perm. preserving the diagonality of $g_{\alpha\beta}$. In the nonlinear theory redefining the fields

$$f_1=(F_2-F_3)/\varepsilon, \text{ perm}, \quad f_1+f_2+f_3=0 \quad (29)$$

we get at $\varepsilon\to 0$:

$$S_{eff}=\int d^3x[\mu e^F(\bar{\partial}F)^2+e^F(f^2_{1,1}+f^2_{2,2}+f^2_{3,3})+\lambda e^{3F}]. \quad (30)$$

7. DISCUSSION

The above calculation illustrates the main idea about three-dimensional gravity: there is the dynamical mechanism connected with the Faddeev-Popov ghost effect which enforces this theory be conformally Euclidean one. At least the latter property is the necessary requirement for the functional measure on the equivalence classes be free from ultraviolet infinities. Of course, the results of calculation can depend on the regularization procedure used as well as the functional integral itself in the nonrenormalizable field theory does. A nice possibility to sum up ghost diagram series into exact analytical expression is connected with factorization of the diagrams into the products of one-dimensional integrals over p_1, p_2, p_3. UV regularization which does not violate this property is necessarily reduced to the product of one-dimensional regularizations. Of these only Z^3 lattice regulator makes an apparent physical sense although another regularizations (e. g. Gaussian exponential momenta cut-off) also lead to the local $O(\Lambda)$ term in S_{eff} in first approximation quadratic in $F_{\alpha}-F_{\beta}$ (Λ is cut-off parameter with the dimension of mass). Note that continuum invariant regulators like that of Pauli—Willars or the stochastic one [9] effective in the quantum field theory do not remove divergencies in the problem at hand: roughly, they merely cancel contributions with large $|\det M_0|\sim|p_1p_2p_3|$ in the momenta space.

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