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B.G. Konopelchenko

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B.G.Konopelchenko

Institute of Nuclear Physics
630090, Novosibirsk, U S S R

A b s t r a c t

A method of the construction of the auxiliary linear problems suitable for the inverse spectral transform method is considered. An algebraic form of the compatibility conditions for these linear problems is discussed for the three-dimensional space.

1. The starting point of the inverse spectral transform method is the representation of the nonlinear differential equation as the compatibility condition of the certain set of auxiliary linear problems (see e.g. [1]). The algebraic forms of these compatibily conditions are the well known Lax pair [2], the commutativity condition $[L_1, L_2] = L_1 L_2 - L_2 L_1 = 0$ [3], Manakov's L-A-B triad [4] or Zakharov's algebraic system [5]. Recently Manakov and Zakharov have proposed a new method for the construction of the multidimensional auxiliary linear problems based on the nonlocal Riemann conjugation problem [6].

Here we present the general formulation of this nonlocal Riemann problem method in the generic multidimensional case and consider the possible algebraic forms of the compatibility conditions in multidimensions.

2. The starting point of the Manakov-Zakharov method [6] is the nonlocal Riemann problem

$$\Psi_2(\lambda, x) = \int_\Gamma d\lambda' \Psi_1(\lambda', x) R(\lambda', \lambda; x) \qquad (1)$$

where $\lambda \in \mathbb{C}$, $x = (x_1, \ldots, x_d)$ and Ψ_2, Ψ_1 are boundary values of the analytic function on the contour Γ and $R(\lambda', \lambda; x)$ is the certain matrix function. It is assumed that the function R obeis the equations $(\partial_{x_i} \equiv \frac{\partial}{\partial x_i})$

$$\partial_{x_i} R(\lambda', \lambda; x) = I_i(\lambda') R(\lambda', \lambda; x) - R(\lambda', \lambda; x) I_i(\lambda) \qquad (2)$$
$$(i = 1, \ldots, d)$$

where $I_i(\lambda)$ are certain matrix functions and $[I_i(\lambda'), I_k(\lambda)] = 0$. Then the operators $\mathcal{D}_i$ $(\mathcal{D}_i f \stackrel{def}{=} \partial_{x_i} f + f I_i(\lambda))$ are introduced and with the use of (1) and (2) the set of operators L_i of the form $L_i = \sum_{n_i} q^i_{n_1, \ldots, n_d}(x) \mathcal{D}_1^{n_1} \cdots \mathcal{D}_d^{n_d}$ which have no singularities on λ is constructed. The compatibility of the linear system $L_i \Psi = 0$ $(i = 1, \ldots, K)$ is equivalent to the nonlinear equation. Some concrete examples have been considered in [6].

3. We would like to propose a scheme which naturally leads to the conjugation problem (1) and its generalizations. We start with the formal expansion problem

$$\Psi_2(\lambda,x)=\int d^d\lambda'\,\Psi_1(\lambda',x)\,R(\lambda',\lambda;x) \qquad (3)$$

where $\lambda=(\lambda_1,\dots,\lambda_d)$, $x=(x_1,\dots,x_d)$, Ψ_2 and Ψ_1 are matrix functions and $R(\lambda',\lambda;x)$ is the certain matrix function. Note that all λ_i are independent variables. We assume that $R(\lambda',\lambda,x)$ satisfies the equations

$$\partial_{x_i}R(\lambda',\lambda;x)=\lambda_i' R(\lambda',\lambda;x)-R(\lambda',\lambda;x)\lambda_i \qquad (4)$$
$$(i=1,\dots,d)$$

where $[\lambda_i,\lambda_k]=0$. Denote $\mathcal{D}_i f \stackrel{def}{=} \partial_{x_i} f+f\lambda_i$. The problem which we are interesting in is to construct the operators L_i of the form $L_i=\sum_{n_1,\dots,n_d} u^i_{n_1\dots n_d}(x)\,\mathcal{D}_1^{n_1}\dots\mathcal{D}_d^{n_d}$ which have no singularities at $\lambda_i\to\infty$ $(i=1,\dots,d)$. Similar to [6] one has $L_i(\lambda)\Psi_2(\lambda,x)=\int d^d\lambda'\,L_i(\lambda')\Psi_1(\lambda',x)R(\lambda',\lambda;x)$.

It is not difficult to see that it is not possible to construct such operator L if all the variables $\lambda_1,\dots,\lambda_d$ are independent ones. Indeed, let the highest order terms in L is $\varphi(\mathcal{D}_1,\dots,\mathcal{D}_d)$. So $L\Psi=\varphi(\partial_{x_1},\dots,\partial_{x_d})\Psi+\Psi\varphi(\lambda_1,\dots,\lambda_d)+\Delta$ where Δ contains lower order terms in $\mathcal{D}_1,\dots,\mathcal{D}_d$. The term $\Psi\varphi(\lambda_1,\dots,\lambda_d)$ cannot be excluded at all.

This consideration shows also that the only way to construct the operator L without singularities at $\lambda_i\to\infty$ is to impose some constraint on the variables $\lambda_1,\dots,\lambda_d$.

Let the variables $\lambda_1,\dots,\lambda_d$ are constrainted by the algebraic equation

$$\varphi(\lambda_1,\dots,\lambda_d)=C=const \qquad (5)$$

where $\varphi(\lambda_1,\dots,\lambda_d)$ is some polynomial.

Proposition. If the variables $\lambda_1,\dots,\lambda_d$ in (3) obey the constraint (5) then the operator L which have no singularity at $\lambda_i\to\infty$ is of the form $L=\varphi(\mathcal{D}_1,\dots,\mathcal{D}_d)+\Delta$.

Indeed $L\Psi=\varphi(\mathcal{D}_1,\dots,\mathcal{D}_d)\Psi+\Delta\Psi=\varphi(\partial_{x_1},\dots,\partial_{x_d})\Psi+\Psi\varphi(\lambda_1,\dots,\lambda_d)+\widehat{\Delta}$. The highest singularities on $\lambda_1,\dots,\lambda_d$ which are collected to the term $\Psi\varphi(\lambda_1,\dots,\lambda_d)$ are annihilated due to the constraint (5). The singularities of the lower order are annihilated by the procedure described in [6].

An example: $\varphi(\lambda_1,\dots,\lambda_d)=\lambda_1^2+\lambda_2^2+\dots+\lambda_d^2=const$. The corresponding operator L is $L=\partial_{x_1}^2+\dots+\partial_{x_d}^2+\sum_{i=1}^{d}u_i\partial_{x_i}+W$.

Let now we have few constraints

$$\varphi_\alpha(\lambda_1,\dots,\lambda_d)=C_\alpha=const \qquad (\alpha=1,\dots,n). \quad (6)$$

So we are able to construct n operators L_α without singularities. Let us consider the system of n linear equations

$$L_\alpha\Psi=0 \qquad (\alpha=1,\dots,n). \quad (7)$$

A necessary condition of the compatibility of this system is the existence of non-empty cross-section of the surfaces (6). If the cross-section of the surfaces (6) is empty then the system (7) has no nontrivial solution.

Let the constraints (6) are independent (i.e. all functions $\varphi_\alpha(\lambda_1,\dots,\lambda_d)$ are algebraically independent) and their cross-section S has a generic complex dimension $d-n$. Let us parametrize (uniformize) this cross-section S by $d-n$ variables $\mu_1,\dots,\mu_{d-n}$. The common solutions of the corresponding system (7) depend on these uniformized variables μ_α. For these common solutions of (7) the relation (3) is equivalent to the following

$$\Psi_2(\mu,x)=\int d^{n-d}\mu'\,\rho(\mu')\,\Psi_1(\mu',x)\,R(\mu',\mu;x) \qquad (8)$$

where $\mu=(\mu_1,\dots,\mu_d)$ and $\rho(\mu')$ is the certain measure on the manifold S. One can consider (8) as the $n-d$ -dimensional generalization of the nonlocal Riemann conjugation problem (1), namely as the problem of construction of the (may be, analytic) function Ψ whose boundary values Ψ_1,Ψ_2 on the $n-d-1$ -dimensional surface Γ are related by (8) with the certain matrix function $R(\mu',\mu;x)$. Unfortunately by now the problem (8) has been effectively solved only in the one-dimensional case $n-d=1$ (see e.g.[6]). In this case we arrive to (1) ($\mu_1\equiv\lambda$) and one construct the linear system (7) by the method given in [6]. The possibility of the generalization of (1) to the multidimensional manifolds has been discussed by S.V.Manakov.

4. At the threedimensional space (d = 3) we have two independent constraints $\varphi_\alpha(\lambda_1,\lambda_2,\lambda_3)=C_\alpha$ $(\alpha=1,2)$ and $\dim\mathcal{S}=1$ in the generic case.

Let us consider few illustrative examples.

A. For the constraints of the form

$$\varphi_1=A_1(\lambda_3)\lambda_1-B_1(\lambda_3)=0, \qquad (9)$$
$$\varphi_2=A_2(\lambda_3)\lambda_2-B_2(\lambda_3)=0$$

where A_i, B_i are polynomial, one has two operators

$$L_i=\bar{A}_i(\partial_{x_3})\partial_{x_i}-\bar{B}_i(\partial_{x_3}) \qquad (i=1,2) \qquad (10)$$

where $\bar{A}_i(\partial_{x_3})$ and $\bar{B}_i(\partial_{x_3})$ are differential operators over x_3, such that $\bar{A}_i\xrightarrow[x\to\infty]{}A_i(\partial_{x_3})$, $\bar{B}_i\xrightarrow[x\to\infty]{}B_i(\partial_{x_3})$. The system (9) defines the one-dimensional manifold Γ which can be obviously parametrized by the single variable λ_3 and $\lambda_1=\frac{B_1(\lambda_3)}{A_1(\lambda_3)}$, $\lambda_2=\frac{B_2(\lambda_3)}{A_2(\lambda_3)}$. It is the case of one marked variable x_3 which has been considered in [5,6].

B. The second example is

$$\varphi_1=\lambda_1^2-\sigma^2\lambda_2^2=const, \qquad (11a)$$
$$\varphi_2=\lambda_3+P(\lambda_1,\lambda_2)=0 \qquad (11b)$$

where $\sigma^2=\pm 1$ and $P(\lambda_1,\lambda_2)$ is an arbitrary polynomial. The corresponding operators are

$$L_1=\partial_{x_1}^2-\sigma^2\partial_{x_2}^2+u_1\partial_{x_1}+u_2\partial_{x_2}+W, \qquad (12)$$
$$L_2=\partial_{x_3}+\bar{P}(\partial_{x_1},\partial_{x_2})$$

where $u_1(x), u_2(x), W(x)$ are functions and $\bar{P}(\partial_{x_1},\partial_{x_2})$ is the differential operator such that $\bar{P}\xrightarrow[x\to\infty]{}P(\partial_{x_1},\partial_{x_2})$. The operators of the form (12) and corresponding hierarchies of integrable equations have been considered in [7-10,4,5]. The uniformized variable for (11) can be choosen as $\mu=\lambda_1+\sigma\lambda_2$. The systems integrable by (12) and invariant under the rotations on the plane (x_1,x_2) correspond to $P=P(\lambda_1^2-\sigma^2\lambda_2^2)$. For

such integrable systems in virtue of (11a) one has λ_3 = const that corresponds to the trivial evolution law in the variable x_3 and to the linearizable systems. The example of such rotatinally invariant system is

$$\frac{\partial W}{\partial x_3}+\gamma\Delta W+\gamma\sum_{k=1}^{2}u_k\partial_{x_k}W=0, \qquad (13)$$
$$\frac{\partial u_k}{\partial x_3}+2\gamma\partial_{x_k}W=0 \qquad k=1,2$$

that corresponds to

$$L_1=\Delta+\sum_{k=1}^{2}u_k\partial_{x_k}+W, \qquad (14)$$
$$L_2=\partial_{x_3}+L_1-\gamma W$$

where $\Delta=\partial_{x_1}^2+\partial_{x_2}^2$, γ is the constant, σ^2 = -1, and P = $\lambda_1^2+\lambda_2^2$. The system (13) is linearizable by the introduction of the gauge variable $g(x)$: $u_k=2\partial_{x_k}\ln g$ and $W=-g^{-1}\Delta g$. Namely, the system (13) is equivalent to the heat equation for g: $\partial_{x_3}g+\gamma\Delta g=0$. In a similar manner one can consider instead of (11) the case $\varphi_1=\varphi_1(\lambda_1,\lambda_2)=C_1$.

C. Let $\lambda_1,\lambda_2,\lambda_3$ are the matrix valued commuting variables and constraints are

$$\varphi_{ik}\psi=(\lambda_i^{(0)}-\lambda_k^{(0)})\psi\lambda_i\lambda_k+\psi(\lambda_kA_i-\lambda_iA_k)=0 \qquad (15)$$
$$i\neq k,\quad i,k=1,2,3$$

where $[A_i,A_k]=0$ $(i,k=1,2,3)$ and $\lambda_i^{(0)}$ are constants. The corresponding operators L_{ik} are

$$L_{ik}=(\lambda_i^{(0)}-\lambda_k^{(0)})\partial_{x_i}\partial_{x_k}+u_{ik}^{i}(x)\partial_{x_i}+u_{ik}^{k}(x)\partial_{x_k}+W_{ik}(x). \qquad (16)$$
$$(i\neq k,\ i,k=1,2,3)$$

Since $\varphi_{12}\psi\lambda_3-\varphi_{13}\psi\lambda_3+\varphi_{23}\psi\lambda_1=0$ then only two of the constraints (15) are independent. Their cross-section Γ is the one-dimensional one and possesses the rational uniformization $\lambda_i=\frac{A_i}{\lambda-\lambda_i^{(0)}}$, $(i=1,2,3)$, $\lambda\in\mathbb{C}$. The description of the rational curves by the quadrics has been discussed in [11].

The construction of the operators L_{ik} of the form (16) (up to the redefinition $u_{ik}^{i}\to(\lambda_i^{(0)}-\lambda_k^{(0)})u_{ik}^{i}$) which start with

[12]. The use of this uncertainty allows one [10] to prove the existence of the matrix commutativity representations $[\bar{L}_1, \bar{L}_2] = 0$ in addition to the known Manakov's triad representations for the integrable systems considered in [7-9]. This is also valid for the systems of the type (12) with $L_i = \sum_{n,m=0}^{n+m=N} u_{nm}(x_1, x_2, x_3)\, \partial_{x_1}^n \partial_{x_2}^m$ [10].

For the systems which contain the variables x_1, x_2, x_3 more symmetrically the condition (19) can be also represented in the equivalent more symmetric form. For example for the three-dimensional chiral fields type equations (17) associated with the operators L_{ik} (16) (divided by $\lambda_i^{(0)} - \lambda_k^{(0)}$) the compatibility condition is equivalent to the system of operator equalities

$$[L_{ik}, L_{nk}] = \alpha_{ink} L_{in} + \beta_{ink} L_{ik} + \gamma_{ink} L_{nk} \qquad i,k,n = 1,2,3 \quad i \neq k \neq n \tag{22}$$

where there is no summation over repearted indices and

$$\alpha_{ink} = \frac{\partial U^n_{nk}}{\partial x^k} - \frac{\partial U^i_{ik}}{\partial x^i} + [U^i_{ik}, U^n_{nk}],$$

$$\beta_{ink} = \frac{\partial U^k_{nk}}{\partial x^i} - \frac{\partial U^i_{ik}}{\partial x^n} + [U^i_{ik}, U^k_{nk}],$$

$$\gamma_{ink} = \frac{\partial U^n_{nk}}{\partial x^i} - \frac{\partial U^k_{ik}}{\partial x^k} + [U^k_{ik}, U^n_{nk}].$$

Note that the noncommutative algebraic representations of the compatibility conditions different from (20) and (22) have been considered in the other contexts in [13, 14].

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Б.Г.Конопельченко

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