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TRANSIENT CHAOS IN A GENERALIZED HENON MAP ON THE TORUS

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ABSTRACT

A generalized Henon system on a torus is considered to investigate some phenomena of transient chaos in weakly dissipative systems. A relationship is numerically verified between the dissipation, lifetime of chaos and area of stable regions, previously established by Chirikov and the first author.

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Recently, much attention has been paid to investigation of different ways in which chaos can arise in strongly deterministic dynamical systems (see e. g. [1]). It is well-known, that the transition from regular to chaotic motion depends on whether the system under consideration is dissipative or Hamiltonian. At present, either Hamiltonian systems or systems with strong dissipation (and small number of degrees of freedom) are quite well investigated. But there is a large region of physical applications for systems with weak dissipation. In such systems the conditions for arising chaos are related, on the one hand, with Hamiltonian chaos, and on the other hand, with the dissipation itself.

The role of dissipation in systems with Hamiltonian chaos was investigated in [2]. As a model for a numerical analysis in [2] there was taken a two-dimensional map of the following type (cf. also [3, 4]):

$$\begin{aligned} p_{t+1} &= \{p_t + kf(x_t) - \varepsilon\cdot(p_t - 1/2)\}, \\ x_{t+1} &= \{x_t + p_{t+1} - 1/2\}, \qquad (1) \\ f(x) &= x^2 - x + 1/6. \end{aligned}$$

The brackets { } indicate that the variables (p, x) are taken modulo 1. In other words, the motion of system (1) is confined to a two-dimensional torus with period one in both variables. In the absence of dissipation ($\varepsilon = 0$) but for large values of k ($k \gg 1$) in the phase space of (1) there are only small regions in which the motion is stable, while outside the motion is chaotic.

It was observed that for weak dissipation ($\varepsilon \ll 1$) this chaotic motion becomes transient in the sence that in the course of time for $t > t_{cr}$ the trajectory (former chaotic) is attracted by some periodic orbit. Such a phenomenon was previously found in [5] and then observed in many dissipative systems with chaotic behaviour including the famous Lorenz system (cf. [6]). As it was shown in [2], if one increases the dissipation, the lifetime of the stochastic component decreases, but then, for $\varepsilon \to \varepsilon_{cr}$, it goes rapidly to infinity. At the same time there appears an attracting set with complicated structure on which the motion again becomes chaotic. From a modern point of view, this set is probably a strange attractor, although this was not strongly proved.

In the paper [2] (cf. also [4]) the dependence of the lifetime t_{cr} on the parameters on the model (1) was numerically investigated. It appeared that for values ε not too near to ε_{cr} the following relation is approximately valid:

$$t_{cr} \cdot S \cdot \varepsilon \approx 1 . \quad (2)$$

Here S denotes the total area of all stable regions of the phase space (in the absence of dissipation), t_{cr} measures the number of iterations of the map (1) up to the moment where the trajectory is «captured» by one of the stable islands. Because there are large fluctuations of t_{cr} in dependence on the initial value (p_0, x_0), the value t_{cr} in (2) should be understand as an average lifetime, i. e. there is taken the average over t_{cr} for many different initial values. Recently, an analogous dependence was found and explained, in [7] for the dissipative Fermi map. Another approach to determine t_{cr} was provided in [8, 9]. There the authors relate the average lifetime to Lyapunov exponents and other characteristics.

Another example where strange attractors appear in two-dimensional maps is the well-known Henon map [10]:

$$x_{n+1} = y_n + 1 - Ax_n^2 , \quad y_{n+1} = Bx_n . \quad (3)$$

One of the main differences between (1) and (3) consists in the fact, that (3) is considered on the whole (x, y)-plane. Therefore the instability of trajectories in the absence of dissipation ($B=1$) yields infinitely extended motions (the trajectories run to infinity), while for the system (1) just the boundedness of the phase volume together with the presence of local instabilities lead to global Hamiltonian

chaos. In the paper [10] the strange attractor was numerically investigated in the case of strong dissipation ($A=1.4$, $B=0.3$). Among other things it was shown that this attracting set has a complicated, Cantor-like structure.

In [11] there was discussed the question to what extent the phenomenon of transient chaos found in the nonlinear map (1) is a typical one. As an example the author considered the Henon map (3) where the variables were taken modulo 2. For small $\varepsilon = 1 - B$ he did not find a degeneration of the chaotic motion in a periodic one. As a conclusion it was stated, that for the Henon map on a torus the transition from Hamiltonian to dissipative chaos goes continuously without transient chaos (and periodic regime). However, the conclusion made in that paper is incorrect and is caused, as it seems to us, by a wrong way to confine the system onto the torus. As a result of the truncation of the map used in [11] it will, strictly speaking, not be Hamiltonian even for $B=1$ (for details see [12]).

In the present paper we investigate the problem of transient chaos for the Henon map on the torus in more detail. Instead of (3) we consider the following modified map:

$$x_{n+1} = y_n + D - Ax_n^2 ; \quad 0 \leqslant D \leqslant 1 ; \quad y_{n+1} = Bx_n \quad (4)$$

and at the same time the truncated system on the torus:

$$x_{n+1} = [y_n + D - Ax_n^2] , \quad y_{n+1} = [Bx_n] . \quad (4')$$

Here the brackets [ ] denote a restriction of (4) onto a torus of period 4. More exactly, system (4′) is obtained from (4) by restricting the variables (x, y) into $-2 \leqslant x, y \leqslant 2$ in such a way that

if $2 \leqslant y_n + D - Ax_n^2 < 6$, then $[y_n + D - Ax_n^2] = y_n + D - Ax_n^2 - 4$;

if $-6 \leqslant y_n + D - Ax_n^2 < -2$, then $[y_n + D - Ax_n^2] = y_n + D - Ax_n^2 + 4$;

and so on.

In the whole paper we fix $A=1$.

First we consider (4) for $B=D=1$. The points $(-1, 1)$, $(1, -1)$ form an orbit of period 2 which is surrounded by small stable regions. Fig. 1 gives an impression of the shape and position of these

Fig. 1. System (4), $A=B=D=1$. The initial point $(1, -0.96)$ is iterated up to the moment when it leaves the square $-2\leqslant x, y\leqslant 2$.

stable islands. To get this figure, system (4) was taken for $A=B=D=1$ and the initial points $(1, -0.96)$ was iterated up to the moment when it leaves the square $-2\leqslant x, y\leqslant 2$ (i. e. it is shown part of one trajectory). Fig. 2 shows two trajectories (more exactly, those parts of the trajectories which are located inside the upper stable region). The outer trajectory is close to the separatrix of the resonances of the 14-harmonics (7 pieses are seen, the other 7 pieses belong to the lower part). Here the initial point $(1, -0.955)$ was iterated 1000 times. The inner trajectory is obtained from $(1, -0.98)$ by 1000 iterations. To get these pictures it does not matter whether we take (4) or (4′) $(A=B=D=1)$.

For system (4′) we observe together with the stable islands also

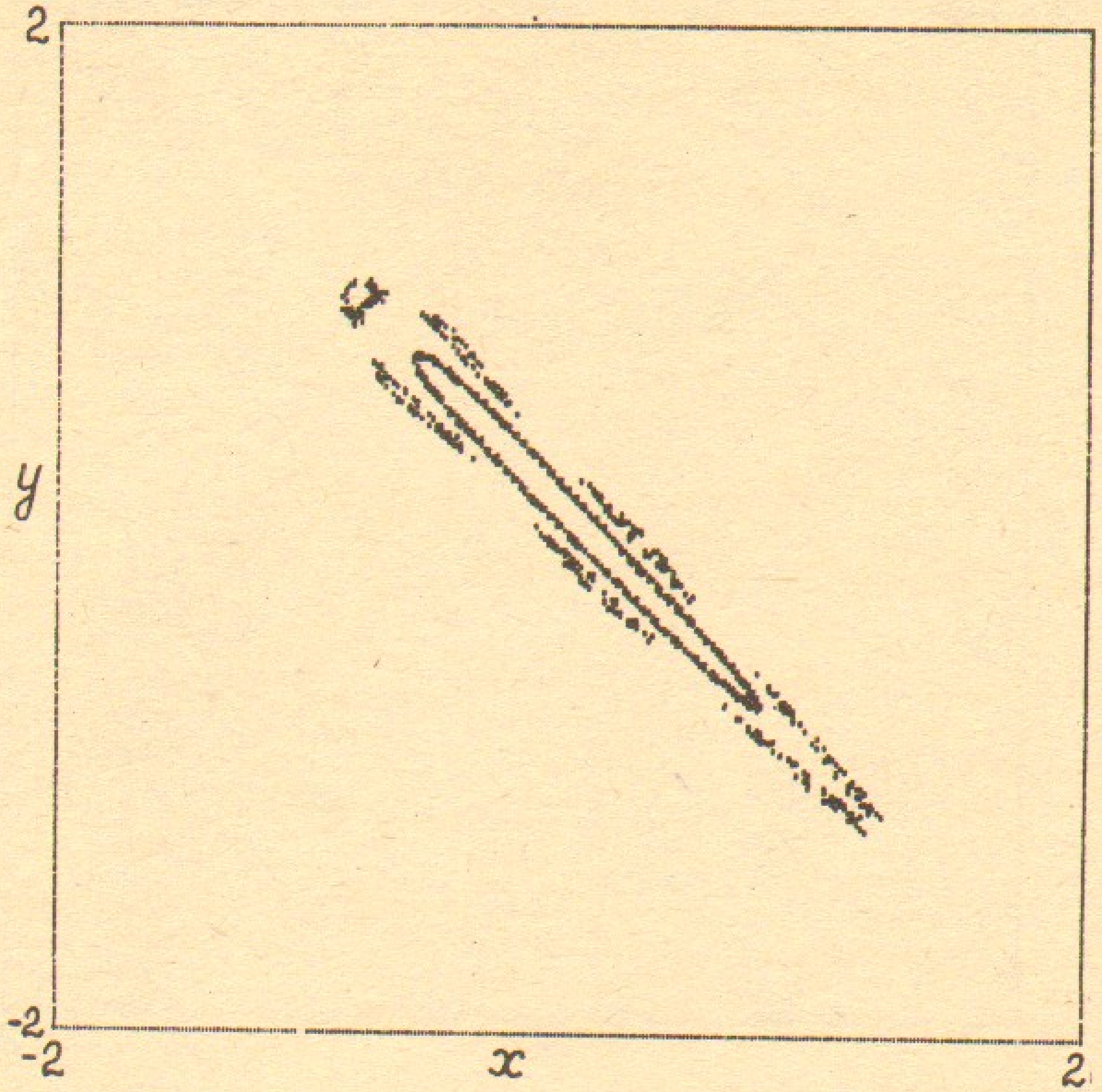

Fig. 2. System (4), $A=B=D=1$; parts of two trajectories inside the upper stable region (cf. Fig. 1). Outer trajectory is close to the separatrix of resonances of 14-harmonics. Initial point $(1, -0.955)$, 1000 iterations. Inner trajectory: initial point $(1, -0.98)$, 1000 iterations.

a stochastic component. It is formed by trajectories with those initial points from the square $-2\leqslant x, y\leqslant 2$ which lead to divergent trajectories in system (4). Fig. 3 shows such a stochastic component. There is seen one trajectory, more exactly, 10000 iterates of the point $(1, -0.96)$ (again $A=B=D=1$). The first iterates lie near the separatrix (cf. Fig. 1) and then they wander erratically over the whole square (torus). In case of system (4) this trajectory would escape to infinity.

The shape and position of the stable regions depend on D in an interesting way. If D decreases from $D=1$, they first shrink, then

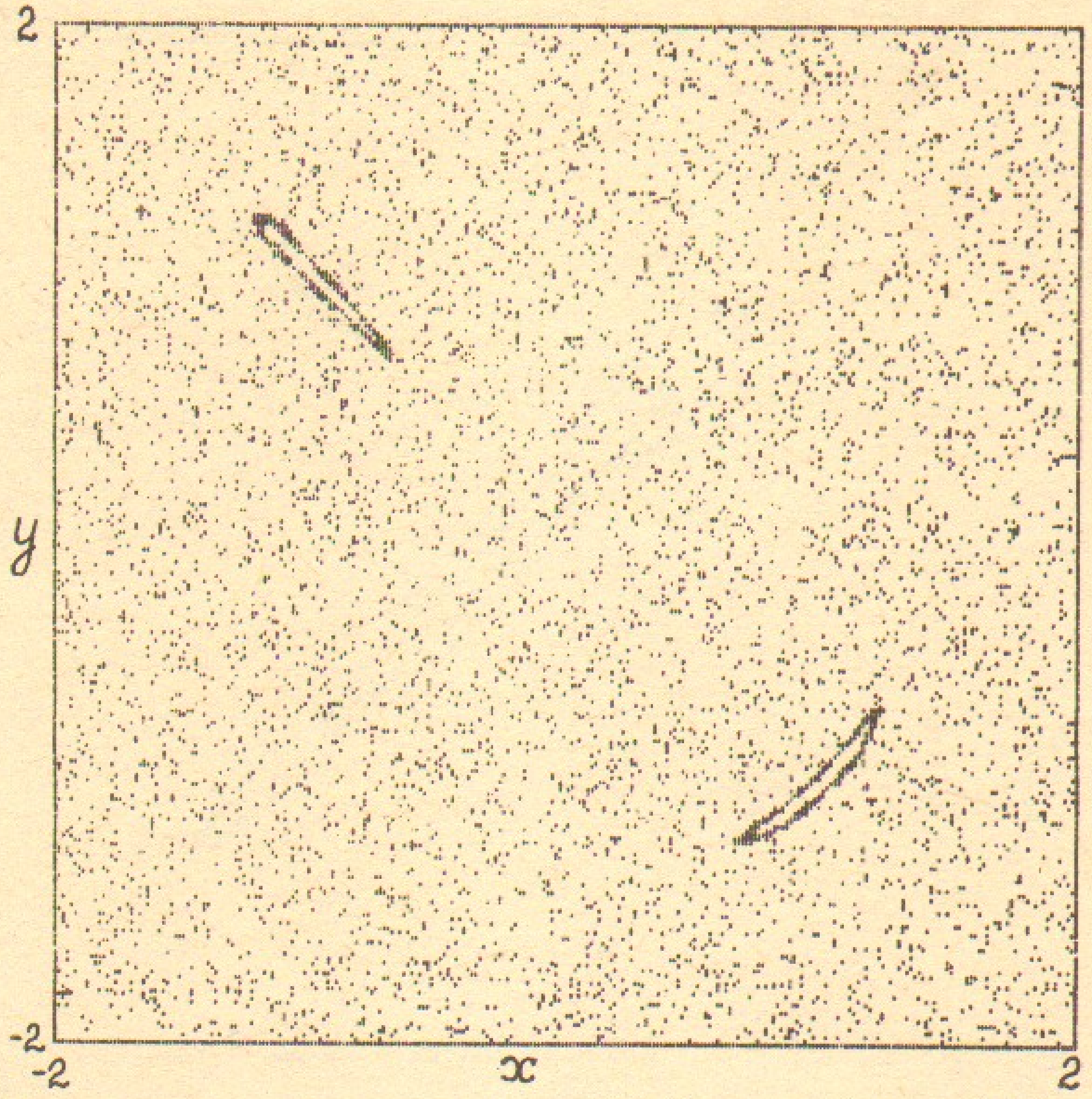

Fig. 3. System (4′), $A=B=D=1$. Stochastic component formed by 10000 iterations of the initial point (1, −0.96).

blow up, come nearer and nearer and finally fuse at (0,0). Figures 4 and 5 give impression of these features. Fig. 4 shows 10000 points of one trajectory of system (4′) for $A=B=1$, $D=0.6$. The initial point is (0.1, 0.1). The two stable islands are seen very well. Fig. 5 contains two trajectories of (4′) for $A=B=1$, $D=0.2$. The stable trajectory (the two ellipse-like figures inside the stable regions) is obtained from the initial point (0.2, −0.2) by 1000 iterations. The other trajectory is an unstable one forming a stochastic component. It is formed by 2000 iterations starting from (0.1, 0.1).

In accordance with previous observations around the boundary of the stable regions there are transient zones. Now we put in a small dissipation. Then in system (4′) the stochastic component is

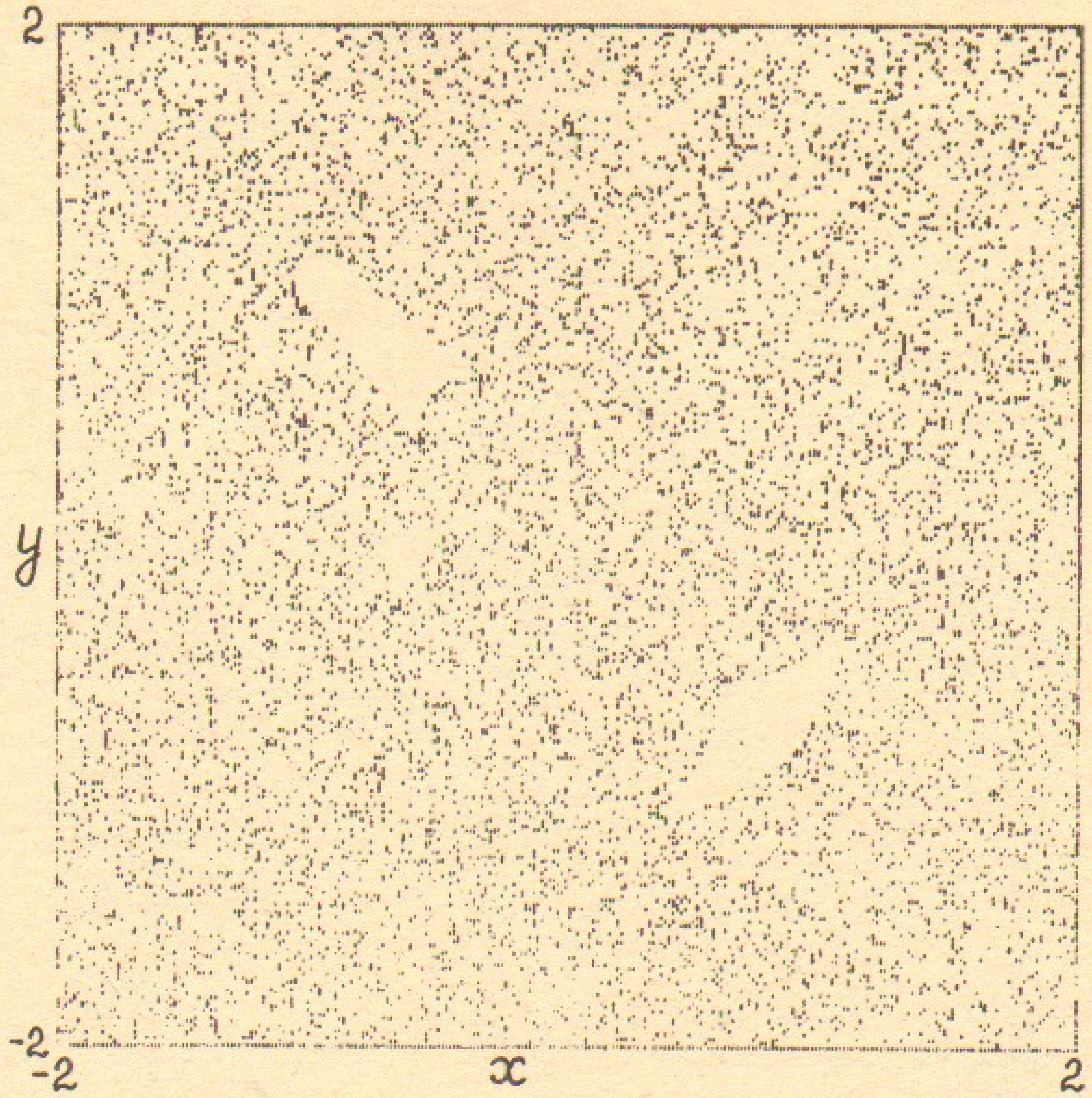

Fig. 4. System (4′), $A=B=1$, $D=0.6$. Stable islands, initial point (0.1, 0.1), 10000 iterations.

destroyed and in dependence on D different types of periodic orbits appear. Let us give some examples. First consider $D=1$. For small dissipation one finds an attracting 4-orbit which undergoes the whole period-doubling scenario with decreasing B (i. e. increasing dissipation). Finally, at $B\approx0.825$ a strange attractor with characteristic Lyapunov exponent 0.038 appears. This strange attractor consists of 4 small pieses. With increasing dissipation it develops into a large attractor. Fig. 6 shows its approximate shape for $A=D=1$, $B=0.8$. There are seen 10000 iterates of the initial point (0.1, 0.1). Thus, also for system (4′) the same transition as in [2] from Hamiltonian to dissipative chaos is observed, namely: chaotic motion (stochastic component) → periodic regime → strange attractor.

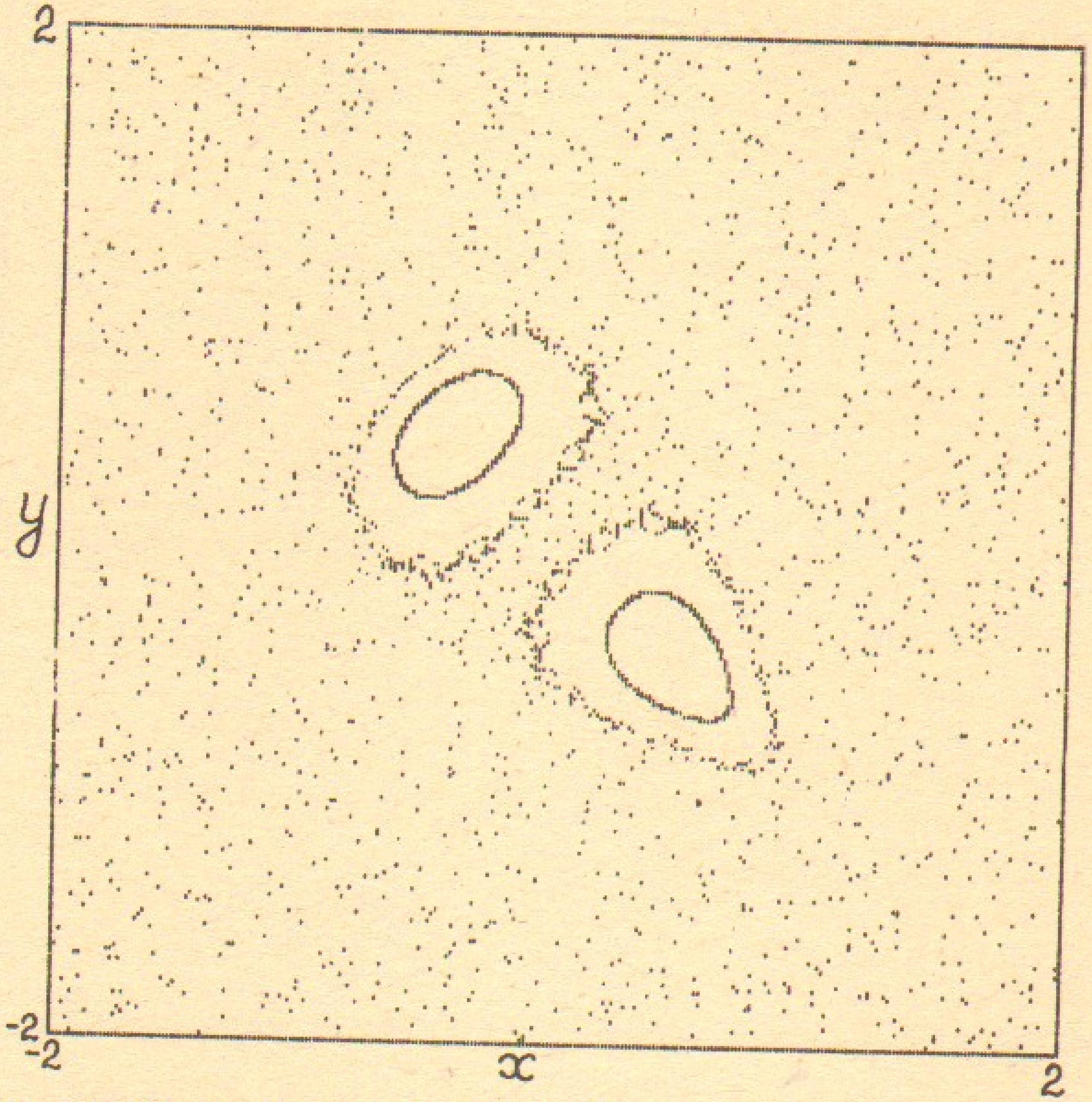

Fig. 5. System (4′), $A=B=1$, $D=0.2$. Stable islands and chaotic «sea». Inner stable trajectory obtained from (0.2, −0.2) by 1000 iterations. Initial point (0.1, 0.1) gives the stochastic component (2000 iterations).

Let us add the following remark. Since system (4) (or (4′)) has a 2-orbit for $B=1$, one would expect, that for small dissipation the periodic regime also starts with a 2-orbit (instead of the 4-orbit). However, the analysis of the stability of the 2-orbit in system (4) shows, that we are in the transient case from stable to unstable motion. More exactly, the eigenvalues of the linearized map at the 2-orbit are both equal to -1. Thus, already the weakest dissipation leads to an immediate bifurcation of this 2-orbit and the stable 4-orbit is observed.

Next we consider $D<1$. For small dissipation a 2-orbit appears.

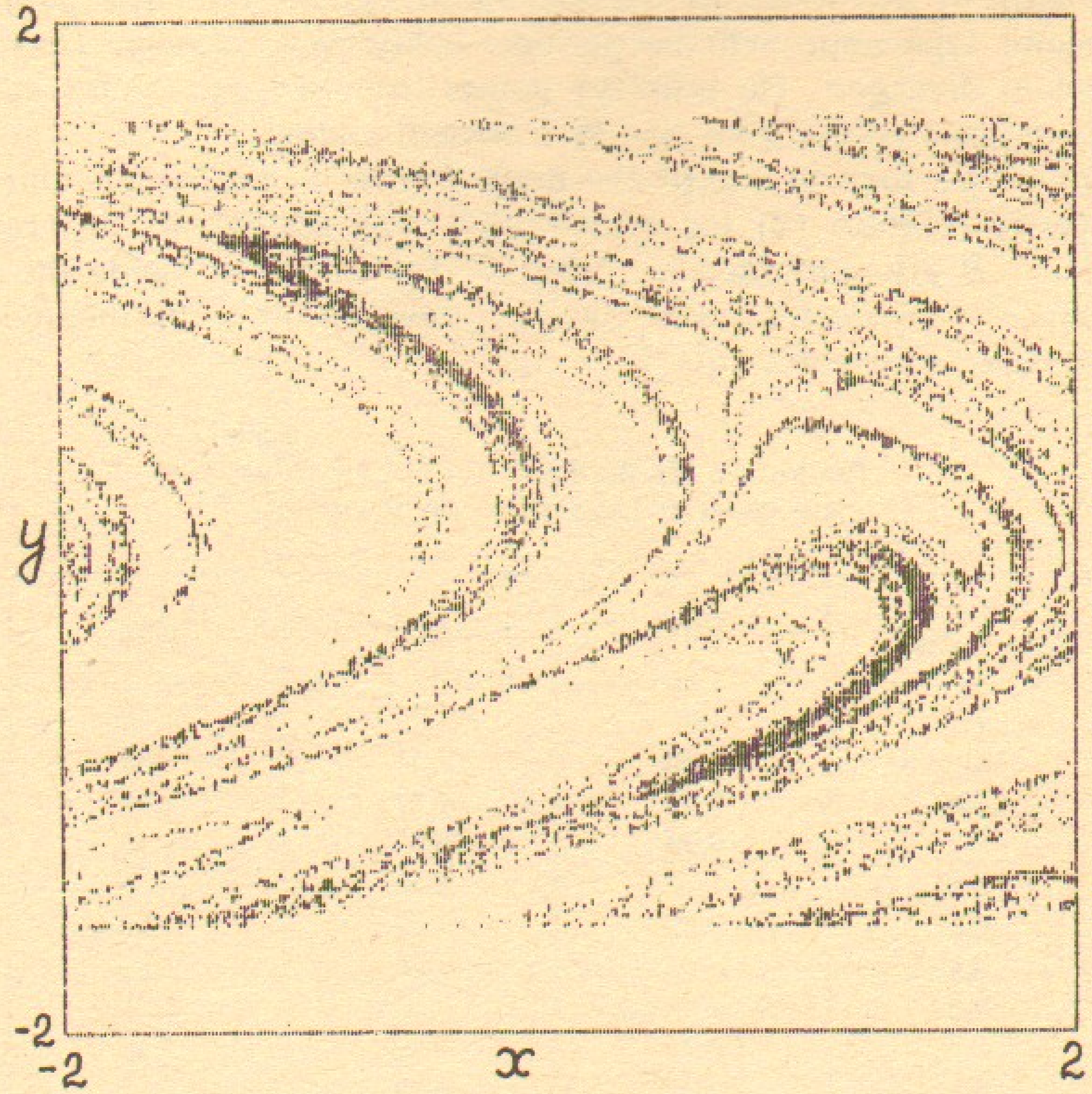

Fig. 6. System (4′), $A=D=1$, $B=0.8$. There are shown 10 000 iterates of (0.1, 0.1).

But then the behaviour under increasing dissipation depends essentially on D. Let us give some examples:

$D=0.95$: period doubling to a strange attractor at $B\approx 0.7$ with Lyapunov exponent 0.086;

$D\sim 0.8 \dots 0.9$: only one period-doubling, then again a 2-orbit appears;

$D\sim 0.6 \dots 0.7$: only the 2-orbit was observed;

$D\sim 0.1 \dots 0.5$: the 2-orbit degenerates with increasing dissipation into a fixed point.

It seems worthwhile to mention the following observation. If system (4′) becomes dissipative, there are two different ways how points approach the periodic orbits, so to say two different types of points. The points of the first type reach the orbits quite rapidly.

These are just those points of the square $-2 \leqslant x, y \leqslant 2$ which lie in the domain of attraction of the 4-orbit in system (4). The points of the second type approach the periodic orbits very slowly. They correspond to divergent (to infinity) points from the square for system (4). Just these points give rise to transient chaos.

Finally we turn to one of the main results of the paper. Our aim is to verify relation (2) for system (4′). The results for different D and $\varepsilon = 1 - B$ are contained in Table. 1. The numbers inside the Table denote the product $t_{cr} \cdot S \cdot \varepsilon$. The last row contains the means over the corresponding column.

Table 1

The Values for the Point $t_{cr} \cdot S \cdot \varepsilon$ (cf. (2)) are given in Dependence on D and ε.

D \ ε	0.01	0.02	0.05
1.0	1.01	1.02	1.12
0.9	1.06	1.41	1.33
0.8	0.47	0.72	1.09
0.7	0.99	0.91	1.46
0.6	1.11	1.08	1.56
0.5	0.92	1.34	1.67
0.4	0.86	1.26	1.81
0.3	1.04	1.06	1.76
0.2	0.75	0.95	1.40
0.1	0.92	0.88	1.22
	0.91	1.06	1.44

Let us explain a little bit more in detail how S and t_{cr} were determined. As already mentioned, in [2] S was taken to be the total area of all stable regions of the phase space in absence of dissipation. Here we consider S in dependence on ε and use for S the area of the set of non-divergent points of (4) contained in $-2 \leqslant x, y \leqslant 2$. To do this, 40000 initial points were iterated and S is proportional to the fraction of points which did not escape to infinity. This set of non-divergent points has a complicated structure. Fig. 7 shows a small part of this set, namely that contained in $0.4 \leqslant x \leqslant 0.8$; $-1.2 \leqslant y \leqslant -0.8$.

As to t_{cr}, there are no direct methods to determine the lifetime of chaos. So we used the following heuristic procedure. Take an initial

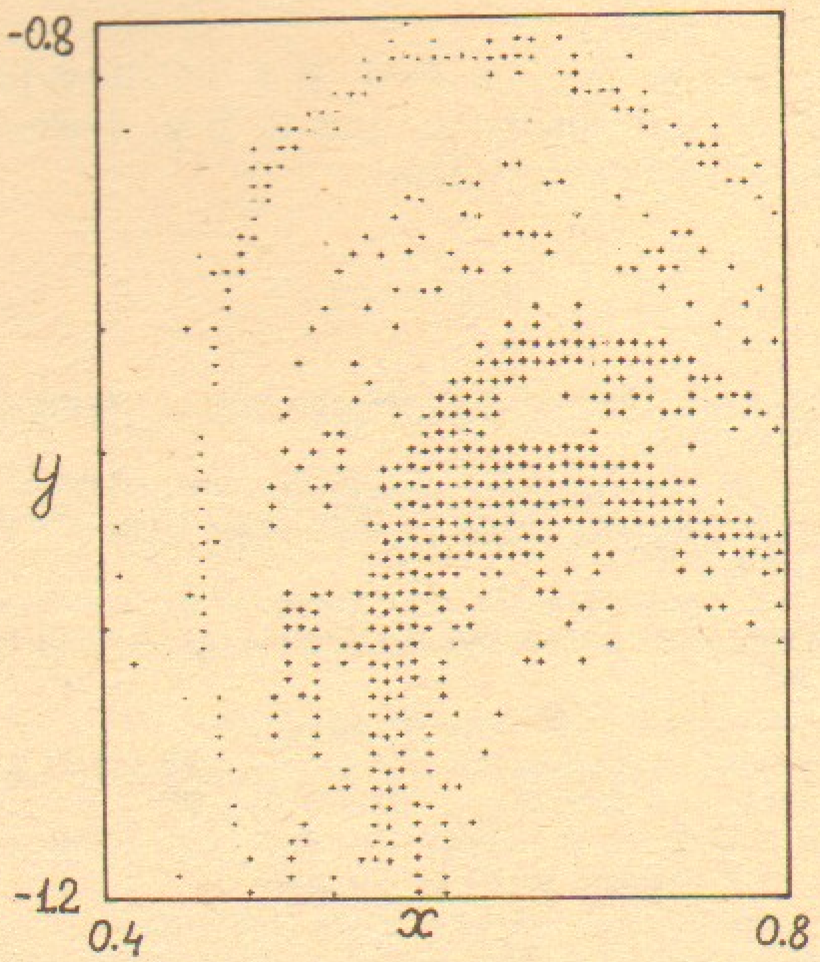

Fig. 7. System (4′), $A = D = 1$, $B = 0.825$. Set of non-divergent points contained in $0.4 \leqslant x \leqslant 0.8$; $-1.2 \leqslant y \leqslant -0.8$.

point which yields transient chaos. After each iteration it is proved whether or not it lies inside the phase square $-2 \leqslant x, y \leqslant 2$ (i. e. whether or not it is necessary to truncate the coordinates). If 100 times in succession it was not necessary to truncate, we suppose that the point had reached the periodic regime (was captured by the stable islands). And then t_{cr} for this point was taken to be the number of iterations up to this 100 times of non-truncation. To get the value t_{cr} for relation (2) we averaged over 80 initial points on the diagonal $x = y$ of the square. It appeared that this heuristic method works quite well for dissipations which are not too small. For very weak dissipations one should use the Lyapunov exponent to decide when the point comes into the periodic regime.

In conclusion one can say that as in [2] also for our model (4′) relation (2) is well satisfied for weak dissipations. Furthermore, in accordance with [2] the product $t_{cr} \cdot S \cdot \varepsilon$ increases rapidly if one approaches some critical dissipation ε_{cr}. It is a very interesting question to determine the dependence of t_{cr} in the neighbourhood of critical values of the dissipation. For some models this was done in [6, 13] and further papers.

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