ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

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LOCALIZATION OF QUASIENERGY EIGENFUNCTIONS IN ACTION SPACE

PREPRINT 85-92

НОВОСИБИРСК

LOCALIZATION OF QUASIENERGY EIGENFUNCTIONS

IN ACTION SPACE

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A b s t r a c t

It is shown that the localization length of quasienergy eigenfunctins is determined by the classical diffusion rate: $\ell = D/2$. A numerical method for the calculation of ℓ in one-dimensional systems is proposed.

A dynamical approach to the problem of the quantum limitation of classical chaos(1-3), which plays a significant role in the excitation of atoms by a strong monochromatic field (4), is proposed. This method is based on the observation that the properties of quantum quasienergy eigenfunctions can be determined by the dynamics of a classical Hamiltonian system with many degrees of freedom. We discuss here also the possibility of using such an approach for the problem of one-dimensional Anderson localization in solid state systems(5). The analogy between the problems of Anderson localization and quantum limitation of chaos was established in Ref. (6).

Let us consider the system with the Hamiltonian $H = H_0(\hat{I}) + V(\theta)\delta_T(t)$, where $\hat{I} = -i\frac{\partial}{\partial\theta}$, $\delta_T(t)$ is the periodic delta-function, θ is the phase variable, $\hbar = 1$, and H_0 is dimensionless(1-3,6). The classical equations of motinn are

$$\bar{I} = I - \partial V/\partial\theta$$
$$\bar{\theta} = \theta + T\,\partial H_0(\bar{I})/\partial\bar{I} \qquad (1)$$

Here $\bar{I}$ and $\bar{\theta}$ are the values of the variables I and θ after one period of time T. If the resonances overlap(7), then the action grows without limit according to the diffusion law: $\langle(\Delta I)^2\rangle = D\tau$, where τ is the number of periods. In the region of strong stochasticity the phases $\theta(\tau)$ are independent and random. So, the diffusion rate is equal to $D_{cl} = \int_0^{2\pi}(V')^2 d\theta/2\pi$. The quasiclassical codition has the form $D \gg 1$, $T \ll 1$(2,3).

Numerical experiments(1-3,6,8) with the quantum standard map ($H_0 = \hat{I}^2/2$, $V = k\cos\theta$) have shown that in the course of time, $\langle I^2\rangle$ stop growing. This means that the external field effectively excites only a finite number of unperturbed levels ($\Delta n =$

$=\Delta I \sim \ell$). It is natural to interpret this effect as resulting from the localization of quasienergy eigenfunctions[6,3]. The following theoretical estimate has been obtained in Refs. (2,3):

$$\ell = \alpha D \tag{2}$$

where α is an unknown numerical constant. This relation is valid when the field excites a large number of levels ($D \gg 1$). This was confirmed indirectly by numerical experiments with the quantum standard map[3] and a highly excited hydrogen atom in a monochromatic field[4] by measuring the stationary distribution $\bar{f}_n$ on the unperturbed levels.

To directly calculate ℓ from an eigenfunction, let us consider the equation for the eigenfunction with quasienergy ω [6]:

$$u_n^- = e^{i(\omega - T H_0(n))} u_n^+$$
$$u^+(\theta) = e^{-iV(\theta)} u^-(\theta) \tag{3}$$

Here $u^\mp$ are the values of the function u before and after a kick $\delta(t)$ and $u_n^\pm$ are the Fourier coefficients of $u^\pm(\theta)$. It is convenient to introduce $\bar{u} = e^{\pm iV/2} u^\pm / g$, where g is some arbitrary real function of θ . Then from (3) we obtain

$$\sum_r \bar{u}_{n+r} W_r \sin(\chi_n + \varphi_r) = 0 \tag{4}$$

Here $W(\theta) = \exp(-iV/2)g = \sum_r W_r e^{i(r\theta + \varphi_r)}$, $\chi_n = (\omega - TH_0(n))/2$ and we consider the case $W(\theta) = W(-\theta)$ only. In Ref. (6) the function $g = 1/\cos\frac{V}{2}$ was implicitly taken. Such a choice leads to a non-physical singularity which does not allow for an analysis of the wide class of potentials with $V(\theta) \geq \pi$. However, the choice of g is arbitrary and does not influence the localization in the original system (3). So, for example, in the quantum standard map it is convenient to take g =1. The formula (4) gives the relation between one-dimensional Anderson localization and localization of quasienergy eigenfunctions in an external field.

Let us assume that in (4) only W_r with $|r| \leq N$ differs from zero. Then the formula (4) determines the dynamics of some Hamiltonian system ($W(\theta) = W(-\theta)$) with N degrees of freedom in which the serial level number n plays the role of discrete time . It is well known that in the case N =1 the localization length is determined by the single positive Lyapunov exponent[5]. It appears that the calculations of ℓ for $N > 1$ have not carried out. For $N > 1$, there are N pairs Lyapunov exponents $\gamma_i^+ = -\gamma_i^- \geq 0$ [9]. The asymptotic decay rate of the quasienergy eigenfunctions is then determined by the minimal positive Lyapunov exponent $\gamma_0 = 1/\ell$ (see Fig 1). The condition for exponential localization is $\gamma_0 \neq 0$. Anumerical method for calculating all of the Lyapunov exponents is described in Ref. (9). An example of the calculation of ℓ by this method is shown in Fig.2.

To determine the value of α in (2), let us consider the exactly solvable Lloyd model (see, for example, Ref.(5)). It is obtained from (4) when $W_0 e^{i\varphi_0} = 1 - iE$, $W_{\pm 1} e^{i\varphi_{\pm 1}} = ik$ and χ_n are randomly distributed on the interval $[0, 2\pi]$ (see also Ref. (6)). Then the diffusion rate in (1) is $D = D_q \ell = 2\sqrt{4k^2 - E^2}$ (for $D \gg 1$). The comparison of D with the exact value of ℓ [5] gives α =1/2.

In the quantum standard map we have $W_r = J_r(k/2)$, $\varphi_r = -\frac{\pi}{2} r$. In this model the χ_n are not random and both D and ℓ depend on the classical parameter of stochasticity $K = kT$. A comparison between numerical data and the theory (2) gives satisfactory agreement for the value α =1/2 (see Fig.3). The parameters k and K in Fig.3 vary within the intervals

$5 \le k \le 75$ and $1.5 \le K \le 29$ and $T/4\pi$ is a typical irrational number. An example of the dependence $\ell(K)$ is shown in Fig. 4.

The resulting expression for the localization length is $\ell = D_0/2T^2$, where $D_0(K)$ is the diffusion rate for the standard map: $\bar{p} = p + K\sin\theta$, $\bar{\theta} = \theta + \bar{p}$, $\langle p^2 \rangle = D_0 \tau$, $p = T I$. The obtained average value $\langle \alpha \rangle$ =0.57, with root-mean-square deviation Δ =0.11, significantly differs from the value obtained in Ref. (3), $\langle \alpha \rangle$ =1.04, Δ =0.20. The cause of this discrepancy is apparently related to the fact that in Ref. (3) ℓ was determined from the stationary (time averaged) distribution $\bar{f}_n \propto e^{-2|n|/\ell_s}$ (here we have introduced the index s). If initially only the n =0 level were excited, then this distribution would be given by $\bar{f}_n = \sum_m |\varphi_m(0)|^2 |\varphi_m(n)|^2$, where $\varphi_m(n)$ is the eigenfunction with quasienergy ω_m. In Ref. (3) in the assumption that $|\varphi_m(n)|^2 \propto e^{-2|n-m|/\ell}$ and the fluctuations of $|\varphi_m(n)|^2$ are negligibly small it was shown that $\ell_s = \ell$. However, the influence of strong fluctuations of $|\varphi_m(n)|^2$ may be significant, that may lead to $\ell_s \neq \ell$. So, for example, in Anderson localization the fluctuations cause the difference between the rate of exponential decay of the density-density correlation function, which is analogous to $\bar{f}_n$, and the decay rate of the square of the eigenfunction[5]. A comparison of the numerical data[3] for ℓ_s with the results presented in Fig. 5 of this paper shows that $\ell_s \approx 2\ell$. The cause of difference between ℓ_s and ℓ apparently connected with the stronge fluctuations of $|\varphi_m(n)|^2$. A detailed discussion of the fluctuation properties and the localization in the region $K \lesssim 1$ will be given elsewhere.

Apparently, the analytic expression for ℓ (2), and the

numerical method of minimal Lyapunov exponent, may be used in one-dimensional solid state problems. As an example, let us consider localization in the Lloyd model with many neighbores: $W_r e^{i\varphi_r} = ik$, $W_0 e^{i\varphi_0} = 1 - iE$, $W_r = 0$ for $|r| > N$ and the χ_n are random. Then the potential is given by $V(\theta) = 2\,\mathrm{arctg}(E - 2k\sum_{r=1}^{N}\cos r\theta)$. For this model, $\ell = D_{ql}/2 \sim 2kN^2$ (for E =0) and the theory gives satisfactory agreement with the numerical data in Fig. 2 which were obtained for parameters in the intervals $0.1 \le k \le 50$, $4 \le N \le 20$. The average value of α obtained from the numerical data was $\langle \alpha \rangle$ =0.52 with Δ =0.07.

The author express his deep gratitude to B.V. Chirikov, for attention to this work and valuable comments.

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Figure captions

Fig.1. Localization of the quasienergy eigenfunctions in the quantum standard map (k =2.8, T =4.867). The points and circles represent numerical data from Ref. (6). The straight lines correspond to the value of ℓ obtained by the method of minimal Lyapunov exponent.

Fig.2. An example of a calculation of the localization length for the quantum standard map (k =40, K =10). The solid lines correspond to positive Lyapunov exponents and the dashed lines to negative. Two minimal exponents are shown.

Fig.3. The ratio $\alpha = \ell / D$ for different values of the diffusion rate D in the quantum standard map (circles) and in the Lloyd model with many neihbors (points). Here and in Fig.5 the logarithm is decimal.

Fig.4. The dependence $\ell(K)$ in the quantum standard map (crosses; k =30). The curve and circles show the theory and numerical data for the diffusion rate $D(K)$ from Ref.(10), $D_{ql} = k^2/2$.

Fig.5. The dependence of the localization length on the diffusion rate D_o of the classical standard map. The circles represent numerical data from Ref. (3) for values of ℓ_s obtained from stationary distributions. The dashed line corresponds to the average value $\langle\alpha_s\rangle$=1.04. The points show the localization lengths obtained from the quasienergy eigenfunctions by the method of minimal Lyapunov exponent. The straight line shows the theoretical localization $\ell = D/2$. In the inset the numerical data from

Ref. (3) are shown, giving the dependence of D_o on $\Delta K = K - K_{cr}$, $K_{cr} = 0.971635$.

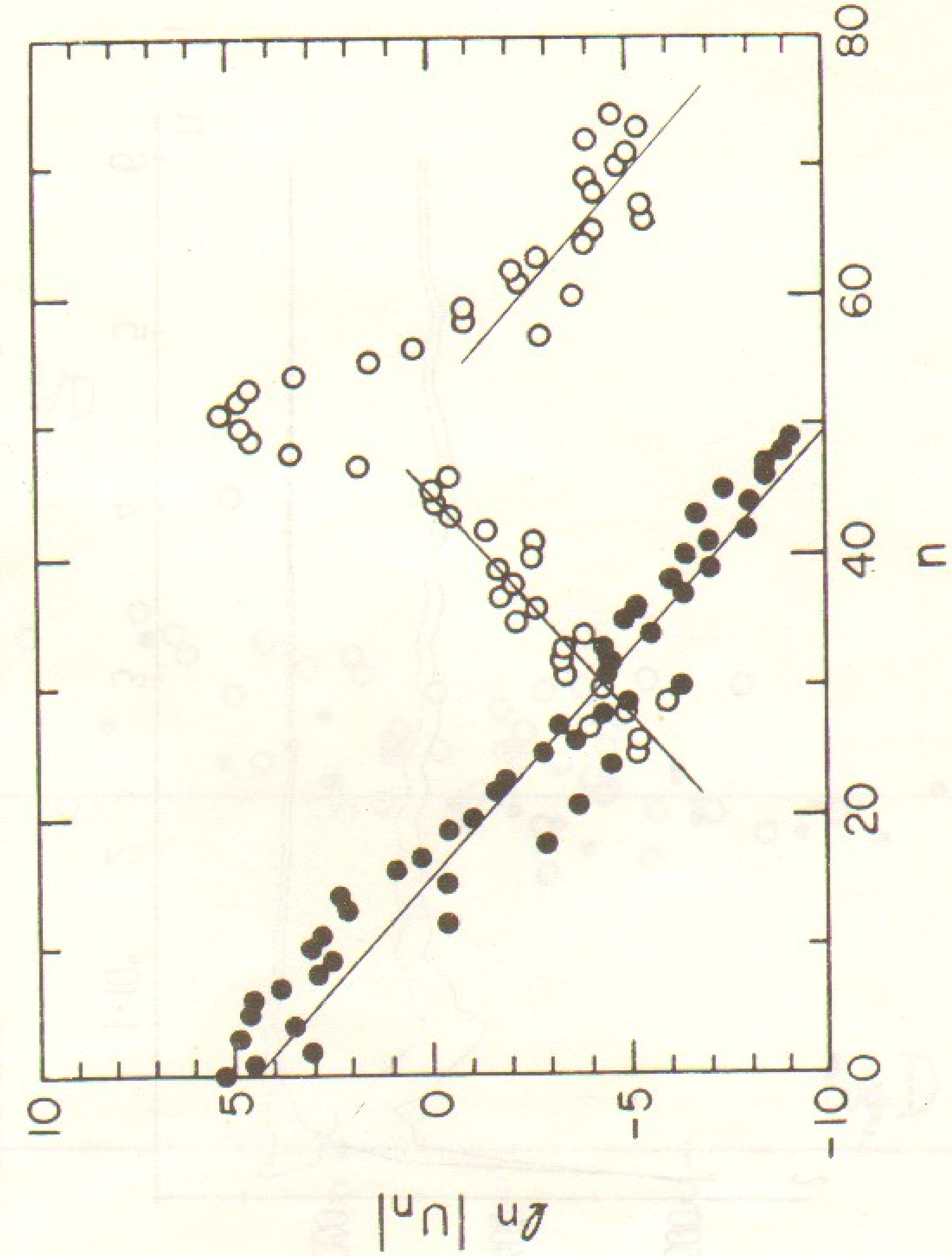


Fig.1.

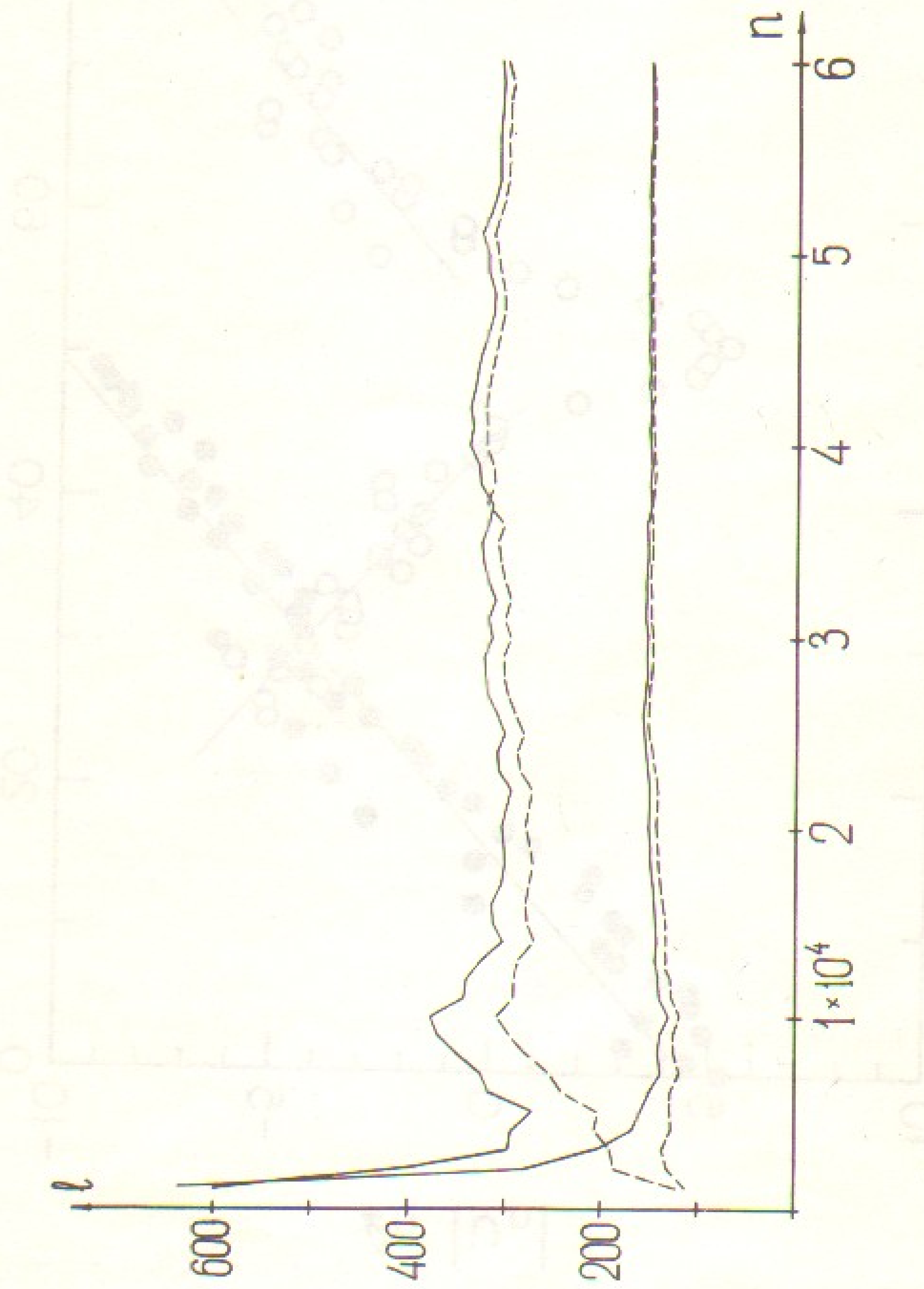


Fig.2.

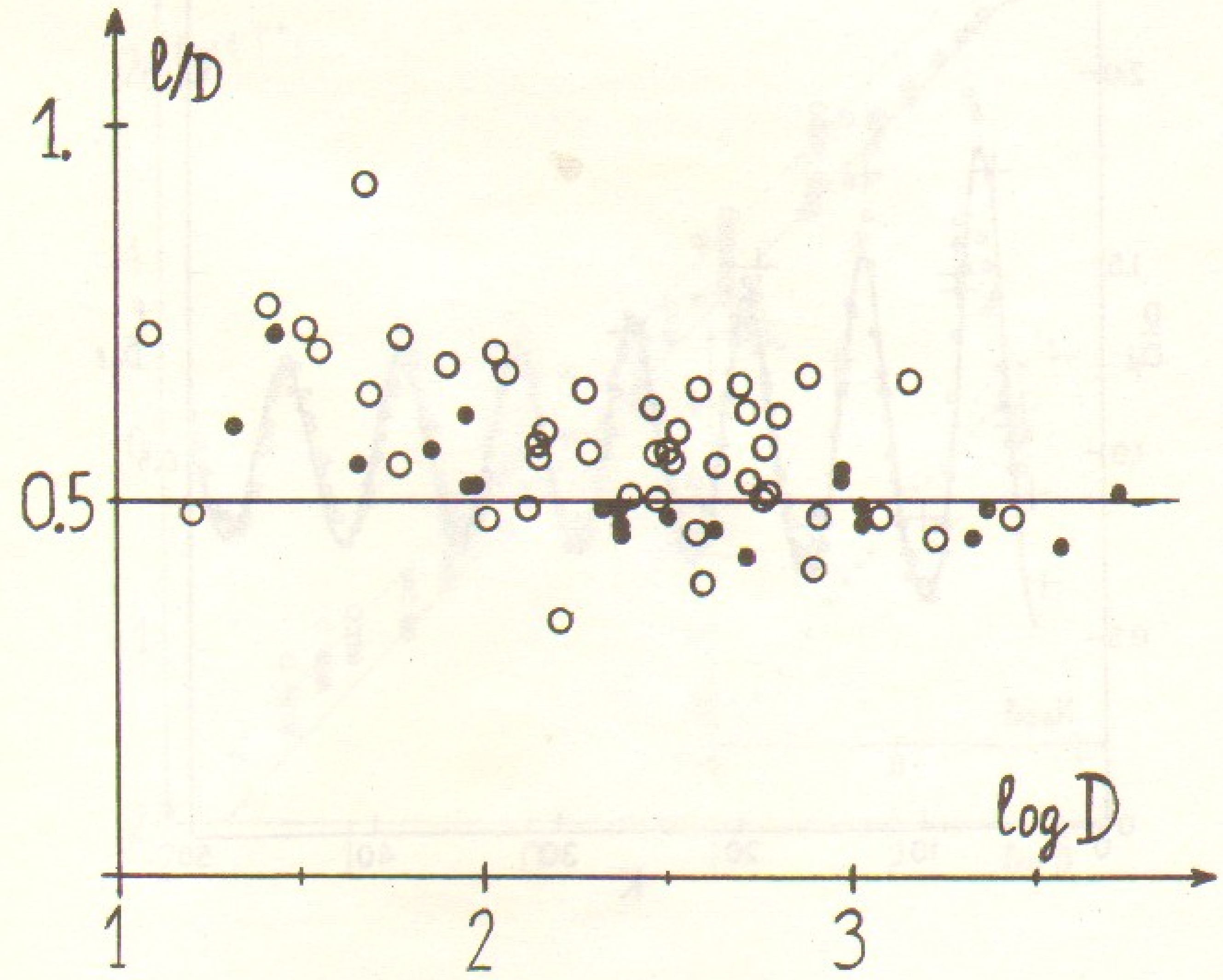
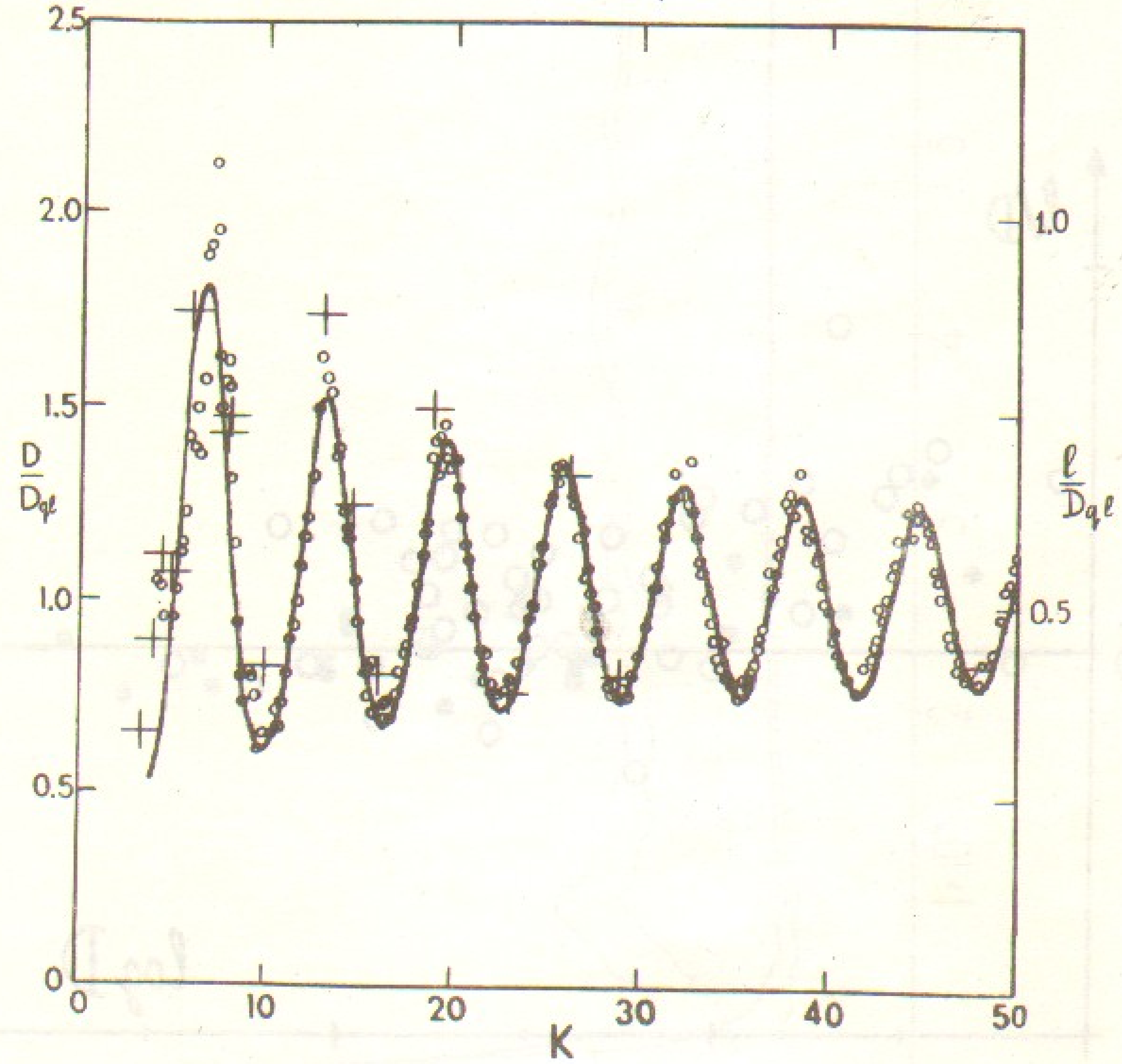
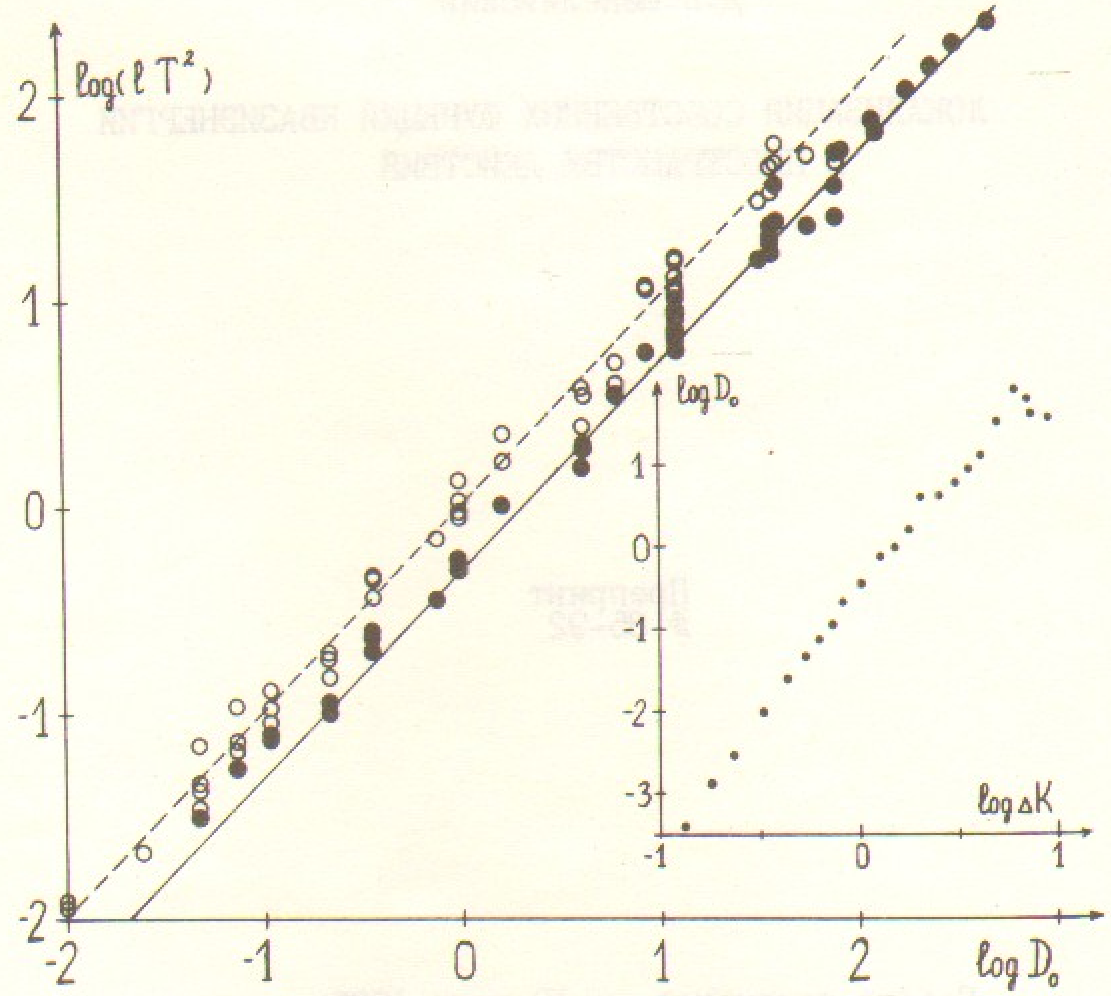


Fig.3.



2.5
2.0
1.5
1.0
0.5
0
D/D_ql
ℓ/D_qℓ
1.0
0.5
0
10
20
30
40
50
K

Fig.4.



log(ℓ T²)
2
1
0
-1
-2
-2
-1
0
1
2
log D₀
log D₀
1
0
-1
-2
-3
log ΔK
-1
0
1

Fig.5.

Д.Л.Шепелянский

ЛОКАЛИЗАЦИЯ СОБСТВЕННЫХ ФУНКЦИЙ КВАЗИЭНЕРГИИ В ПРОСТРАНСТВЕ ДЕЙСТВИЙ

Препринт
№ 85-92

Работа поступила - I0 июня I985г.

Ответственный за выпуск - С.Г.Попов
Подписано к печати 23.07.I985г. МН 05839
Формат бумаги 60x90 I/I6 Усл. 0,8 печ.л., 0,7 учетно-изд.л.
Тираж 290 экз. Бесплатно. Заказ № 92

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90