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НОВОСИБИРСК

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ABSTRACT

The propagation of long-wave magnetosonic oscillations of finite amplitude in a plasma with random inhomogeneities of density, temperature and magnetic field is studied. The equations describing the evolution of averaged characteristics of the medium are derived. It is shown that the initial perturbation splits into two "simple waves" propagating in the opposite directions. Each of the "simple waves" has a tendency to the steepening and consequent overturning which results in formation of shocks.

For understanding the various processes in Solar atmosphere the problem of propagation of MHD-waves in a plasma with random inhomogeneities of density, temperature and magnetic field is of the substantial interest. This problem, being of general interest - in laboratory plasmas the situation when all parameters of medium are random functions of coordinates can offen be met - is of the particular importance for the physics of the Solar atmosphere, where the large-scale MHD-waves are one of the most important agents contributing to the energy balance in upper chromosphere and lower Corona. Previously the different sides of this problem were discussed in the framework of the linear MHD-waves [1-5].

In the present paper we consider the influence of the inhomogeneities on the propagation of the magnetosonic waves of an arbitrary amplitude, restricting ourselves with one-dimensional problem and with the assumption that the characteristic size of the inhomogeneities a is small as compared with the length of the magnetosonic waves λ :

$$a \ll \lambda$$

At the same time, we do not assume that the amplitude of inhomogeneities is small.

We derive the equations that describe the evolution of all the averaged (over inhomogeneities) characteristics of the medium. We show, that in their structure, these equations are analogous to the equations for homogeneous medium. In particular, it remains valid the conclusion that the perturbation of not too large amplitude splits into two "simple waves", propagating in the opposite directions. Each of the "simple waves" has a tendency to the steepening and consequent overturning. In the unperturbed state all the plasma parameters - density ρ_0 , pressure P_0 , temperature T_0 and the magnetic field B_0 (which is parallel to z - direction) - depend only on the coordinate x : $\rho_0 = \rho_0(x)$, $P_0 = P_0(x)$, $T_0 = T_0(x)$, $B_0 = B_0(x)$. From the mechanical equilibrium condition we have:

$$P_0(x) + \frac{B_0^2(x)}{8\pi} = \mathcal{P}_0 = const \qquad (1)$$

The pressure P_0 and the density ρ_0 are considered to be the random functions of x. For what follows, it is convenient to introduce the distribution function $f(P_0, \rho_0)$ of the random quantities P_0 and ρ_0, which is defined as follows: the fraction of those segments of the axis x, where P_0 and ρ_0 take the values in the intervals $(P_0, P_0 + dP_0)$, $(\rho_0, \rho_0 + d\rho_0)$ is proportional to $f(P_0, \rho_0)\, dP_0\, d\rho_0$:

$$\text{mes}\, x \sim f(P_0, \rho_0)\, dP_0\, d\rho_0$$

$$dP_0\, d\rho_0$$

To describe the magnetosonic waves we use the ideal MHD-equations:

$$\rho \frac{dv}{dt} = -\frac{\partial \mathcal{P}}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho v = 0 \qquad (2)$$

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x} B v = 0$$

$$\frac{d}{dt}\left(\rho^{-\gamma} P\right) = 0$$

where γ is the specific heat ratio, and $\frac{d}{dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$. From the second and the third equations of the system (2) there follows the line-tying condition:

$$\frac{B}{\rho} = \frac{B_0}{\rho_0} \qquad (3)$$

where B_0 and ρ_0 are the values of B and ρ in the point where the given element of the medium was located at the initial instant of time (t = 0). Similarly, from the equation the entropy conservation we have

$$P\rho^{-\gamma} = P_0 \rho_0^{-\gamma} \qquad (4)$$

Let's average the first and the second equations of the system (2) over the scale that is much larger than the size of inhomogeneities a, but much smaller than the length of magnetosonic wave λ. Denoting this averaging by angular brackets, we find:

$$\left\langle \rho \frac{dv}{dt} \right\rangle = -\frac{\partial}{\partial x}\langle \mathcal{P} \rangle \qquad (5)$$

$$\frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial}{\partial x}\langle \rho v \rangle = 0 \qquad (6)$$

In order to simplify these equations we use considerations similar to those of paper [1], devoted to linear waves. Let's now write an exact equation:

$$\frac{\partial v}{\partial x} = -\frac{1}{\gamma + 1}\frac{1}{P}\frac{dP}{dt} \qquad (7)$$

which follows from continuity equation and constancy of the enthropy. Since we are considering the motions with the size $\lambda >> a$, the logarithmic derivative $d\ln P/dt$, which can be estimated as v/λ, is small with respect to v/a. So that we find from (7) that

$$\frac{\partial v}{\partial x} \sim \frac{v}{\lambda} \ll \frac{v}{a} \qquad (8)$$

This means that despite the presence of the inhomogeneities of density, pressure and magnetic field which have the scale a, the velocity v is a "smooth" function, changing only at a scale $\lambda >> a$. This circumstance allows us to write the following relations:

$$\left\langle \rho \frac{dv}{dt} \right\rangle \simeq \langle \rho \rangle \frac{d\langle v \rangle}{dt}$$

$$\langle \rho v \rangle \simeq \langle \rho \rangle \langle v \rangle$$

These relations are valid with the accuracy of the order of $(a/\lambda) \ll 1$. We remind that the scale over which the averaging is made is small as compared to λ and large as compared to a. As a result we obtain instead of equations (5) and (6) the following equations

$$\langle \rho \rangle \frac{d\langle v \rangle}{dt} = -\frac{\partial}{\partial x}\langle \mathcal{P} \rangle \tag{9}$$

$$\frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial}{\partial x}\langle \rho \rangle \langle v \rangle = 0 \tag{10}$$

The form of equations (9) and (10) is similar to that of the equations for 1D gas dynamics. The analogy would become complete if we could find the "closing" relationship between $\langle \rho \rangle$ and $\langle \mathcal{P} \rangle$. Now we proceed to this part of the problem. First of all, we note that the density ρ of each plasma element can be expressed in terms of its initial density ρ_0, pressure $\mathcal{P}$ and full pressure   at a given point. By using the definition of $\mathcal{P}$ ($\mathcal{P} = p + B^2/8\pi$) and the relationships (1), (3) and (4), we obtain:

$$\mathcal{P} = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma} + \frac{\mathcal{P}_0 - p_0}{\rho_0^2}\rho^2 \tag{11}$$

This relationship determines an unexplicit dependence of ρ on $\mathcal{P}$, ρ_0 and p_0 ($\mathcal{P}_0$ is assumed to be known):

$$\rho = \rho(\mathcal{P}, p_0, \rho_0) \tag{12}$$

Now let's consider a plasma between two planes which are sticked to plasma particles. We assume that the distance ℓ between these planes is much larger than a and much smaller than λ. When the full pressure $\mathcal{P}$ is changing the distance between the planes is changing too due to the finite compressibility of plasma. Since the distance ℓ is small compared with λ, the changing of the full pressure is small and we can put $\mathcal{P}$ equal to $\langle \mathcal{P} \rangle$ in a corresponding district. Thus we can find the density of each element of plasma with the help

of equation (12) where $\mathcal{P}$ is substituted by $\langle \mathcal{P} \rangle$. The segment dx of the whole district, corresponding to these elements is connected with the segment dx_0 which these element occupied in the initial state by the evident relation

$$dx = dx_0 \frac{\rho_0}{\rho(\langle \mathcal{P} \rangle, p_0, \rho_0)} \tag{13}$$

Taking into account that $dx_0 \sim \ell_0 f(p_0, \rho_0)\, dp_0 d\rho_0$, from this relation we obtain that

$$\ell = \ell_0 \frac{\int f(p_0,\rho_0) \frac{\rho_0}{\rho(\langle \mathcal{P} \rangle, p_0, \rho_0)} dp_0 d\rho_0}{\int f(p_0,\rho_0)\, dp_0 d\rho_0} \tag{14}$$

The whole mass of the substance between the planes is obviously the following

$$\ell_0 \frac{\int \rho_0 f(p_0,\rho_0)\, dp_0 d\rho_0}{\int f(p_0,\rho_0)\, dp_0 d\rho_0}$$

Dividing this mass by the distance defined by (14) we obtain the expression for the average density:

$$\langle \rho \rangle = \frac{\int \rho_0 f(p_0,\rho_0)\, dp_0 d\rho_0}{\int f(p_0,\rho_0) \frac{\rho_0}{\rho(\langle \mathcal{P} \rangle, p_0, \rho_0)} dp_0 d\rho_0} \equiv R(\langle \mathcal{P} \rangle) \tag{15}$$

So, in principle, one can find the connection between $\langle \rho \rangle$ and $\langle \mathcal{P} \rangle$ for any distribution function $f(p_0, \rho_0)$. Thus, the relationship (15) together with equations (9) and (10) forms a closed system, describing self-consistently the propagation of longwave magnetosonic oscillations of finite amplitude in an inhomogeneous plasma. As a result the problem under consideration becomes quite analogous to the problem of onedimensional compressible gas. In the ordinary gas dynamics the initial perturbation of finite amplitude splits into two "simple waves", propagating in left and right directions. Each of the simple waves is gradually steepening and finely overturning. That leads to the formation of shocks. The condition for overturning reads (see [6], §94):

$$\frac{du}{dp} > 0 \qquad (16)$$

where $u = v + c$ and $v = \int \frac{c}{\rho} d\rho$, $c^2 = dp/d\rho$. The condition (16) can be represented in more general form through the relationship between density and pressure. Actually, using the expression for V the condition (16) can be written as follows:

$$\frac{c}{\rho} + \frac{dc}{d\rho} = \frac{c}{\rho} + \frac{dc}{dp}\frac{dp}{d\rho} > 0$$

or as follows:

$$\frac{c}{\rho} + \frac{1}{2}\frac{dc^2}{dp} > 0 \qquad (17)$$

substituting here $c^2 = \frac{dp}{d\rho}$ we obtain

$$\frac{\rho}{2}\frac{d}{dp}\frac{1}{\frac{d\rho}{dp}} > -1$$

Performing the derivation instead of condition (16) we have

$$\rho\frac{d^2\rho}{dp^2} < 2\cdot\left(\frac{d\rho}{dp}\right)^2 \qquad (18)$$

In general the conclusion about overturning is connected with the dependence of ρ on $\langle\mathcal{P}\rangle$ that is with the concrete form of a function $\rho = R(\langle\mathcal{P}\rangle)$: in accordance with (18) the overturning takes place if it is satisfied the condition

$$R \cdot R'' < 2R'^2 \qquad (19)$$

here the stroke means the derivative $d/d\langle\mathcal{P}\rangle$ taken in the point where $\langle\mathcal{P}\rangle = \mathcal{P}_0$. For the ideal homogeneous gas this condition is automatically satisfied because of $R \sim \langle\mathcal{P}\rangle^{1/\gamma}$, $\gamma > 1$. Let's elucidate now if the condition (19) is satisfied in our case when the function $R(\langle\mathcal{P}\rangle)$ is defined by the expression (15). The first and second derivatives of this function have a form:

$$\frac{\partial R}{\partial\langle\mathcal{P}\rangle} = \int \rho_0 f(P_0,\rho_0)\,dP_0 d\rho_0 \frac{\int f(P_0,\rho_0)\frac{\rho_0}{\rho^2}\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}dP_0 d\rho_0}{\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho}dP_0 d\rho_0\right]^2} \qquad (20)$$

$$\frac{\partial^2 R}{\partial\langle\mathcal{P}\rangle^2} = \int \rho_0 f(P_0,\rho_0)\,dP_0 d\rho_0 \cdot \left\{ \frac{2\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho^2}\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}dP_0 d\rho_0\right]^2}{\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho}dP_0 d\rho_0\right]^3} + \right. \qquad (21)$$

$$\left. + \frac{\int f(P_0,\rho_0)\left[\frac{\rho_0}{\rho}\frac{\partial^2\rho}{\partial\langle\mathcal{P}\rangle^2} - 2\frac{\rho_0}{\rho^3}\left(\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}\right)^2\right]dP_0 d\rho_0}{\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho}dP_0 d\rho_0\right]^2} \right\}$$

Compose now the condition (19) with the help of expressions (15), (20) and (21):

$$\left[\int \rho_0 f(P_0,\rho_0)\,dP_0 d\rho_0\right]^2 \cdot \left\{ \frac{2\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho^2}\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}dP_0 d\rho_0\right]^2}{\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho}dP_0 d\rho_0\right]^4} + \right.$$

$$\left. + \frac{\int f(P_0,\rho_0)\left[-2\frac{\rho_0}{\rho^3}\left(\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}\right)^2 + \frac{\rho_0}{\rho^2}\frac{\partial^2\rho}{\partial\langle\mathcal{P}\rangle^2}\right]dP_0 d\rho_0}{\left[\int f(P_0,\rho_0)\frac{\rho_0}{\rho}dP_0 d\rho_0\right]^3} \right\} <$$

$$< 2\left[\int \rho_0 f(p_0,\rho_0)\, dp_0\, d\rho_0\right]^2 \frac{\left[\int f(p_0,\rho_0)\frac{\rho_0}{\rho^2}\frac{\partial \rho}{\partial\langle\mathcal{P}\rangle}\, dp_0\, d\rho_0\right]^2}{\left[\int f(p_0,\rho_0)\frac{\rho_0}{\rho}\, dp_0\, d\rho_0\right]^4}$$

In this inequality the first term in left hand side is reduced with the right hand side and the condition (19) takes a form

$$\frac{\left[\int \rho_0 f(p_0,\rho_0)\, dp_0\, d\rho_0\right]^2}{\left[\int f(p_0,\rho_0)\frac{\rho_0}{\rho}\, dp_0\, d\rho_0\right]^3}\cdot \int f(p_0,\rho_0)\left[-2\frac{\rho_0}{\rho^3}\left(\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}\right)^2 + \frac{\rho_0}{\rho^2}\frac{\partial^2\rho}{\partial\langle\mathcal{P}\rangle^2}\right] dp_0\, d\rho_0 < 0$$

It is obvious beforehand that

$$\int \frac{\rho_0}{\rho} f(p_0,\rho_0)\, dp_0\, d\rho_0 > 0$$

and the condition (19) is reduced to the form

$$\int f(p_0,\rho_0)\left[-2\frac{\rho_0}{\rho^3}\left(\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}\right)^2 + \frac{\rho_0}{\rho^2}\frac{\partial^2\rho}{\partial\langle\mathcal{P}\rangle^2}\right] dp_0\, d\rho_0 < 0 \qquad (22)$$

Now let's find the expressions for $\partial\rho/\partial\langle\mathcal{P}\rangle$ and $\partial^2\rho/\partial\langle\mathcal{P}\rangle^2$. Derivating the equation (11) over $\langle\mathcal{P}\rangle$ we can find the expression for $\partial\rho/\partial\langle\mathcal{P}\rangle$:

$$\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle} = \frac{\rho}{\gamma\langle\mathcal{P}\rangle + (2-\gamma)\frac{\mathcal{P}_0 - p_0}{\rho_0^2}\rho^2}$$

After simple calculations and transformation for the second derivative one can obtain the following expression:

$$\frac{\partial^2\rho}{\partial\langle\mathcal{P}\rangle^2} = \frac{\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}\left[\gamma\langle\mathcal{P}\rangle - (2-\gamma)\frac{\mathcal{P}_0 - p_0}{\rho_0^2}\rho^2\right] - \rho\gamma}{\left[\gamma\langle\mathcal{P}\rangle + (2-\gamma)\frac{\mathcal{P}_0 - p_0}{\rho_0^2}\rho^2\right]^2}$$

These derivatives taken in the point $\langle\mathcal{P}\rangle = \mathcal{P}_0$ become as follows:

$$\left.\frac{\partial\rho}{\partial\langle\mathcal{P}\rangle}\right|_{\mathcal{P}_0,\rho_0} = \frac{\rho_0}{\gamma p_0 + 2(\mathcal{P}_0 - p_0)}$$

$$\left.\frac{\partial^2\rho}{\partial\langle\mathcal{P}\rangle^2}\right|_{\mathcal{P}_0,\rho_0} = \frac{\rho\left[\gamma p_0 - \gamma^2 p_0 - 2(\mathcal{P}_0 - p_0)\right]}{\left[\gamma p_0 + 2(\mathcal{P}_0 - p_0)\right]^3}$$

Substituting these expressions into the condition (22) we obtain:

$$-\int f(p_0,\rho_0)\frac{\gamma(\gamma+1)p_0 + 6(\mathcal{P}_0 - p_0)}{\left[\gamma p_0 + 2(\mathcal{P}_0 - p_0)\right]^3}\, dp_0\, d\rho_0 < 0 \qquad (23)$$

In accordance with the mechanical equilibrium condition (1) the magnitude $(\mathcal{P}_0 - p_0)$ is always positive:

$$\mathcal{P}_0 - p_0 = \frac{B_0^2}{8\pi} > 0$$

That means that integrand in the expression (23) is definitely positive. Thus the condition (23) as well as the condition (19) is satisfied for any distribution function $f(p_0,\rho_0)$ and it is valid the conclusion that the magnetosonic wave of finite amplitude propagating in a plasma with random inhomogeneities splits into two simple waves with consequent steepening and overturning. Note that when the width of the wave-

-front becomes comparable with the characteristic scale of inhomogeneities our assumption may not be valid. In this case the dispersion effects play an essential role, since at $\lambda \sim a$ the dispersion of magnetosonic waves becomes nonlinear and steepening of wavefront ceases. Corresponding effects need the separate investigation.

Thus, we have shown the way to get the equations describing the evolution of longwave magnetosonic oscillations of finite amplitude propagating in a plasma with random inhomogeneities of density, temperature and magnetic field. It is shown that the influence of these inhomogeneities (which are not assumed to be small) is such that the initial perturbation splits into two simple waves resulting in steepening effect and formation of shocks. The problem under consideration is of particular interest for heating processes in active regions of Solar atmosphere which are seats of strong magnetic fields concentrated in tightly settled flux tubes.

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НЕЛИНЕЙНЫЕ МАГНИТО-ЗВУКОВЫЕ ВОЛНЫ В НЕОДНОРОДНОЙ ПЛАЗМЕ

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