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THE REFLECTION OF ALFVEN WAVES FOR SUPER-ALFVEN PLASMA FLOW TO THE WALL

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THE REFLECTION OF ALFVEN WAVES
FOR SUPER-ALFVEN PLASMA FLOW TO THE WALL

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ABSTRACT

The reflection of Alfven waves off a plasma boundary is considered for the case when the plasma is flowing towards the boundary with a velocity that exceeds the Alfven speed. The boundary conditions can be met if two whistler waves are propagated from the boundary. Two boundary conditions are considered: an insulating and a conducting endwall.

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1. Introduction

The interchange mode in a "gas dynamic trap" (GDT) [1,2] investigated in reference [3] in many respects differs from that of conventional mirror machines. One of the important differences is the large plasma flow out of the trap. If the magnetic field between the mirror throat and the absorber drops sufficiently fast, then the plasma will be flowing to the absorber with a velocity in excess of the Alfven speed. In this case, the absorber is not able to propagate Alfven waves back into the plasma, which raises the question of how the plasma maintains the proper boundary conditions at the wall. In reference [3] it is shown that the dispersion relation for flute modes is independent of whether the absorber is a conductor or an insulator if the plasma flow exceeds the Alfven speed. Thus it is not possible to stabilize the interchange mode by "line-tying" the magnetic field lines to the absorber.

Here we consider another consequence of plasma flow to the endplate that exceeds the Alfven speed. We show that low frequency MHD waves that flow through the expanded section and strike the absorber can give rise to waves that travel back from the absorber to the throat. The frequency of these reflected waves in the plasma frame is typically of the same order of magnitude as the ion cyclotron frequency. This high frequency turbulence which is propagated from the absorber to the throat may effectively scatter the ions in the region between the absorber and mirror throat.

An analogous problem also appears in astrophysics for the solar wind flowing to planetary bodies (such as the moon). Our results show that when low frequency MHD oscillations strike the moon's surface, short wavelength waves are reflected that have a large frequency in the plasma frame.

In this paper, we assume small amplitude waves, and neglect non-linear effects. The conversion of low frequency waves into high frequency waves in the plasma frame is possible because the system is not steady state in time in the rest frame of the plasma.

2. Dispersion Relation

We shall consider the case of a dense homogeneous plasma: $\omega_{pi} \gg \omega_{Bi}$, where ω_{pi} and ω_{Bi} are the plasma and cyclotron frequencies, respectively. The plasma flows along the magnetic field lines and recombines on the endplate surface which is perpendicular to the magnetic field (see Fig.1). According to the real situation [3], we are assuming the plasma to be cold. The incident Alfven wave has the frequency ω_o, $\omega_o \ll \omega_{Bi}$ and wavenumber $\vec{k}_o$ in the plasma rest frame. The frequency and wavenumber of the reflected wave in this frame will be denoted by ω and $\vec{k}$, respectively. The dependance $\sim e^{i\omega t - i\vec{k}\vec{r}}$ is assumed.

The parameters $\omega, \vec{k}$ are obtained by requiring that the reflected and incident waves have the same frequency and x-component of the wavenumber in the reference frame of the wall:

$$\omega - k_z u = \omega_o + |k_{zo}| u, \tag{1a}$$

$$k_x = k_{xo}, \tag{1b}$$

where $u > o$ is the velocity of the plasma relative to the wall (note that $k_{zo} < 0$). The frequency ω in (1a) is a function of k_x and k_z given by the cold plasma dispersion relation.

In general, it is possible for both ω and k_z to be complex. For complex k_z one should demand that the wave be attenuated as it propagates into the plasma, $\mathrm{Im}k_z < 0$. But in our case, as we shall show later, only real ω, k_z are possible. For real k_z we require:

$$\frac{\partial \omega}{\partial k_z} > u. \tag{2}$$

The intuitive explanation for (2) is that in order to propagate from the endplate the reflected wave must have group velocity larger than the flow velocity. One can also verivy this mathematically by considering the inhomogeneous form of Maxwell's equations, where the wall acts as a source term. Equation (2) follows naturally when one obtains the steady state solution by integrating around the poles of the inverse dielectric function.

In Eq.(1a) ω and k_z can have both positive and negative values, $-\infty < \omega, k_z < \infty$. Using the invariance of the cold plasma dispersion relation under sign changes in ω and k_z it is more convenient to consider only first quadrant in ω, k_z plane (i.e. ω, $k_z > 0$). In doing this we shall allow the possibility of negative ω and k_z by considering solutions of Eq.(1b) and also of the following equations:

$$-\omega + k_z u = \omega_o + |k_{zo}| u, \tag{3a}$$

$$\omega + k_z u = \omega_o + |k_{zo}| u. \tag{3b}$$

Equation (3a) is obtained from (1a) by changing the signs of both ω and k_z, and Eq. (3b) by changing the sign of k_z only. Since the sign change of k_z only changes the sign of the group velocity $\partial\omega / \partial k_z$, we require for solutions (3b)

$$-\frac{\partial \omega}{\partial k_z} > u \tag{4}$$

instead of (2). (Thus for solutions to (3b), we require that the z components of the group and phase velocities have opposite sign.) We need not consider the possibility of changing the sign of only ω in (1a) because it does not have solutions for positive ω, k_z if $\omega_o > 0$.

The lines (1a), (3a), (3b) and the cold plasma dispersion curves are shown in fig.2 for $\omega \ll \omega_{Bi}$. In this region there are two types of wave: fast magnetosonic waves with the dispersion relation $\omega^2 = (k_x^2 + k_z^2)v_A^2$, and Alfven waves for which $\omega = k_z v_A$, where v_A is the Alfven speed. There are two intersections, 1 and 2, of the line (3b) with the dispersion curves, but having positive $\partial\omega / \partial k_z$ they do not satisfy (4) (they propagate in the wrong direction). The other two intersections, 3 and 4, in figure 2, represent waves that travel in the proper direction, but have a group velocity that is less than u. Hence, by (2) and (4), none of the waves shown in figure 2 are acceptable.

In figure 3 we see the same diagram on a much larger scale. In addition to two curves whose origin is shown in fig.2, three curves appear corresponding to high frequency electron oscillations (see, for example, [4]). From figure 3, it is clear that for $u < u_k$ (one can show that u_k corresponds to the relativistic velocities which we do not consider here) a pair of roots, shown by a circle, satisfies Eq.2. According to conventional terminology, these waves are whistlers.

Counting the intersections in figures 2 and 3 we find that there are a total of 10 real roots. As shown in Appendix using the cold plasma dielectric function Eqs. (1) formally corresponds to a polynomial that has no more than 10 roots. Thus we have found all the roots, and, as stated earlier, there are no surface waves with complex ω and k_z in our problem.

Now we obtain an approximate dispersion relation for whistler waves in a high density plasma ($\omega_{pi} \gg \omega_{Bi}$). For simplicity we suppose that the whistler frequency is much smaller than the electron-cyclotron frequency ω_{Be} and the electron plasma frequency ω_{pe}. This imposes a constraint on the velocity u which will be obtained later. For frequencies $\omega \ll \omega_{Be}$, ω_{pe} we can set the electron mass equal to zero, which greatly simplifies the problem. In this zero-electron-mass approximation, the zz component of the conductivity tensor becomes infinite, which means that the parallel electric field vanishes. The matrix equation for the x and y component of the perturbed electric field in the plasma frame is:

$$\begin{bmatrix} \frac{w^2}{1-w^2} - \zeta^2 & \frac{iw^3}{1-w^2} \\ \frac{-iw^3}{1-w^2} & \frac{w^2}{1-w^2} - \zeta^2 - \xi^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0, \tag{5}$$

where $w = \omega/\omega_{Bi}$, $\zeta = ck_z/\omega_{pi}$ and $\xi = ck_x/\omega_{pi}$.
Eq.5 can be solved explicitly for $w = w(\zeta, \xi)$ or $\zeta = \zeta(w, \xi)$.

Now we find the frequency of the reflected waves as a function of plasma speed, and the Alfven wave frequency ω_0. From figure 3 we see that the whistlers will have ω and k_z much larger than the Alfven wave frequency and wavevector. Thus for oblique angle of incidence of the Alfven wave, we will have a nearly perpendicular angle of reflection: $k_z \gg k_x$. In this limit, the dispersion relation which follows from (5) takes the simple form:

$$\zeta^2 = \frac{w^2}{1 \pm w}, \tag{6}$$

where the +(-) sign refers to the whistler (Alfven) branch.

Another consequence of the inequality $\omega_0 \ll \omega_{Bi}$ is that the two lines (1a) and (3a) almost overlap in the scale of figure 3. Thus both whistlers have almost the same frequency. If we neglect the right-hand sides in (1a) and (3a) and use (6) we obtain the approximate frequency of the two whistler waves:

$$w = \mu^2 - 1, \tag{7}$$

where $\mu = u/v_A$. We can also obtain a correction to (7) which is valid to first order in ω_0/ω_{Bi}:

$$w_\pm = \mu^2 - 1 \pm \frac{2\mu^2}{\mu - 1} \frac{\omega_0}{\omega_{Bi}}, \tag{8}$$

where the sign +(-) corresponds to the upper (lower) line in fig.3. In the limit $k_x \ll k_z$, we can show from (5) (see reference [4]) that in the plasma frame both whistlers are circular polarized in the electron cyclotron direction.

Now, using (7), it is easy to show that the requirement $\omega \ll \omega_{Be}$, ω_{pe} is satisfied when (assuming for simplicity $\omega_{Be} \lesssim \omega_{pe}$)

$$\mu \ll \sqrt{\frac{m_i}{m_e}}.$$

3. Boundary Conditions

There are two boundary conditions that can be used to describe the reflection of waves off a wall. If the plasma is terminated on a conducting wall, we may set the component of the electric field, which is perpendicular to the field lines, equal to zero. For a conducting metal wall, we have the boundary condition:

$$\sum_j \vec{E}^{\,j}_{\perp w} = 0, \tag{9}$$

where the sum is over all the waves in the plasma and the subscript w shows that wave amplitudes are taken in the wall frame.

We also consider the case when the plasma is in contact with an insulator. In the limit of large density, the wavelengths are large compared with the vacuum light wavelength ($k_x \ll c/\omega\sqrt{\varepsilon}$, where ε is the insulator dielectric constant).

From Maxwell's equations, we can show that in this case the waves in the insulator are surface waves, with wavelength

$$k_z = i\sqrt{k_x^2 - \frac{\omega^2 \varepsilon}{c^2}} \; . \tag{10}$$

Analogous to the two polarizations of a propagating vacuum electromagnetic wave, we have two types of waves that satisfy (10). One wave has the polarization:

$$\vec{E} = \left(-\frac{k_z}{k_x}, 0, 1\right) \psi , \tag{11a}$$

$$\vec{B} = \left(0, -\frac{\omega \varepsilon}{c k_x}, 0\right) \psi ; \tag{11b}$$

it is nearly electrostatic (rot $\vec{E} \simeq 0$) in low frequency limit. The other surface wave in the insulator is nearly magnetostatic with electric and magnetic field:

$$\vec{E} = (0, 1, 0) \; \psi , \tag{12a}$$

$$\vec{B} = \left(-\frac{c k_z}{\omega}, 0, \frac{c k_x}{\omega}\right) \psi . \tag{12b}$$

In our limit of interest, we may approximate (10) by $k_z = i|k_x|$ and also neglect the magnetic field in (11b)(but not the electric field in (12a)).

Next, we match boundary conditions by demanding continuity of $\vec{E}_\perp$ between the plasma waves and the surfaces waves in the insulator. We can also require continuity of $\vec{B}_\perp$ because the cold plasma is not a magnetic medium(which follows from the fact the dielectric tensor does not depend on $\vec{k}$).

Since $E_z = 0$ in any reference frame for waves inside the plasma, we can relate the perpendicular components of $\vec{E}_w$ and $\vec{B}_w$ inside the plasma via Maxwell's equations:

$$\vec{B}^{\,j}_{\perp w} = \left(-\frac{c k_z^j}{\omega} E^j_{yw} \, , \; \frac{c k_z^j}{\omega} E^j_{xw} \, , \; B^j_{zw}\right) . \tag{13}$$

In order to obtain the boundary condition for a wave impinging on the plasma insulator interface, we use (11) and (12) and (13) to relate the magnetic fields in terms of electric fields in the insulator and in the plasma. From continuity of $\vec{E}_\perp$ and $\vec{B}_\perp$ we obtain:

$$\sum_j k_z^j E^j_{xw} = 0 \, , \tag{14a}$$

$$\sum_j (i k_x + k_z^j) E^j_{yw} = 0 , \tag{14b}$$

where we have made the approximations described after (12b). The corrections are easily shown to be small when $\omega \ll c k_x$, and $\omega \ll c k_z$. Using rot $\vec{B} = 4\pi \vec{j}/c$, one can show that (14a) is equivalent to the condition that the parallel current vanishes at the end. From (14b), one can show that the y component of the current vanishes at the end.

In order to match the boundary conditions (9) or (14), it is necessary to transform the wave electric fields out of plasma frame. We transform the electric field via the non-relativistic Lorentz transformation, $\vec{E}_w = \vec{E} + (\vec{u}/c) \times \vec{B}$, where $\vec{E}$ and $\vec{B}$ are the perturbed electric and magnetic fields. As an aside, we also give the transformation rule for the conductivity tensor, $\vec{j} = \hat{\sigma} \vec{E}$, where we transform the perturbed current density as $\vec{j}_w = \vec{j} + \rho \vec{u}$. If we use Maxwell's equations to relate $\vec{B}$ and $\vec{E}$, and charge continuity to relate $\vec{j}$ and ρ , we obtain the following equations for transformations involving waves with zero parallel electric field and transformations into frames moving parallel to the unperturbed magnetic field:

$$\omega_w = \omega - k_z u \, , \tag{15a}$$

$$\frac{1}{\omega_w} \vec{E}_w = \frac{1}{\omega} \vec{E} \, , \tag{15b}$$

$$\omega_w \hat{\sigma}_w = \omega \hat{\sigma} \, , \tag{15c}$$

where (15c) does not describe components of the conductivity tensor that involve z.

As it follows from (1a), (3a) and figure 3 , one reflected whistler has a phase velocity that is slightly larger than the wall velocity, and the other has a phase velocity that is slightly slower. The transformation into the wall frame (15a) changes the sign of ω for the slower wave. Because a negative ω wave corresponds to a time reversed positive ω wave, the transformation reverses also the polarization of the slower wave from electron cyclotron to ion cyclotron direction. This is convenient because in the wall frame, we obtain a left and a right circularly polarized wave, making it easy to match boundary conditions for an incident linearly polarized wave.

We now give the reflection coefficients for an Alfven wave impinging on an endplate when the plasma is flowing faster than the Alfven speed. We consider both reference frames, and either insulating or conducting endplates. We shall omit the algebra because it is straightfoward. The table gives the electric field amplitude for both reflected whistlers when the electric field amplitude of the incident Alfven wave is unity. We also give the reflection coefficients for the wave magnetic fields.

Electric Field Reflection Coefficients

	wall frame	plasma frame
conducting wall	$\frac{1}{2}$	$\frac{1}{2}(\mu^2-1)\frac{\omega_{Bi}}{\omega_0}$
insulating wall	$\frac{\mu}{2(\mu^2-1)}\frac{\omega_0}{\omega_{Bi}}$	$\frac{1}{2}\mu$

Magnetic Field Reflection Coefficients

	wall frame	plasma frame
conducting wall	$\frac{(\mu^2-1)^2(\mu+1)}{2\mu(\mu^2+1)^2}\frac{\omega_{Bi}}{\omega_0}$	$\frac{\mu^2-1}{2\mu}\frac{\omega_{Bi}}{\omega_0}$
insulating wall	$\frac{\mu(\mu^2-1)(\mu+1)}{2(\mu^2+1)^2}$	$\frac{1}{2}$

4. Conclusion

We considered the plasma flowing onto a metal or insulating surface with the velocity exceeding the Alfven velocity. We showed that the Alfven waves in the plasma reflected from the surface transform into the short wavelength whistler waves having (in the plasma frame) frequency of the order of ω_{Bi}.

These waves can cause an effective ion scattering in the plasma near the endplate. In a GDT, they can propagate back into the trap, although the frequency in the trap will be much lower than the local value of ω_{Bi} because the magnetic field in the trap is much larger than the magnetic field at the endplate.

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Appendix

Here we count the number of waves that satisfy (1a),(3a), or (3b). If we allow roots with positive or negative ω, k_z , we need only consider the following two equations for ω and k_z:

$$\omega - k_z u = \omega^*, \qquad (A^1)$$

$$D(\omega, k_z) = 0 \qquad (A^2)$$

where $\omega^* = \omega_0 + |k_{z0} u| > 0$ and $D = 0$ is the dispersion relation for the waves. For a homogeneous plasma, (A^2) takes the form:

$$\det\left[\frac{c^2k^2}{\omega^2}\left(\frac{k_ik_j}{k^2} - \delta_{ij}\right) + \varepsilon_{ij}\right] = 0 \qquad (A^3)$$

where ε_{ij} is the dielectric tensor. We shall employ the cold plasma dielectric tensor [4] that yields the five branches of the dispersion relation shown in figure 3 . Here, we shall place no restrictions on the plasma density, electron mass, or wave frequency.

It is trivial to solve (A^1) for k_z and reduce (A^2) to an equation in one unknown. Using the cold plasma conductivity tensor, we may obtain from (A^3) a polynomial in ω of order fourteen if we multiply both sides of the equation by

$$(\omega^2 - \omega_{Be}^2)^2 (\omega^2 - \omega_{Bi}^2)^2 \qquad (A^4)$$

From the polynomial of order 14, we know A(3) has at most fourteen roots. We can also show that the polynomial is zero when $\omega^2 = \omega^2_{Be}$, or when $\omega^2 = \omega^2_{Bi}$. Thus the polynomial has four roots that were introduced by multiplying by (A^4): $\omega_{Bi}, -\omega_{Bi}, \omega_{Be}, -\omega_{Be}$. We are now left with a maximum of ten roots. We can verify that for our problem, all ten roots are true roots, becouse all ten roots appear in figures (2) and (3).

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Figure captions

Fig. 1. Coordinate system (z = 0 is the endplate surface), incident wave (ω_0 , $\vec{k}_0$) and reflected wave (ω , $\vec{k}$). Magnetic field is in z -direction, plasma flows in the opposite direction.

Fig. 2. Plots of equations (1a), (3a), (3b) for $u > V_A$ (broken lines) and the dispersion curves (solid lines) for $\omega \ll \omega_{Bi}$.

Fig. 3. Figure 2 in a much larger scale. The dotted line, $\omega = u_k k_z$, touches the whistler curve.

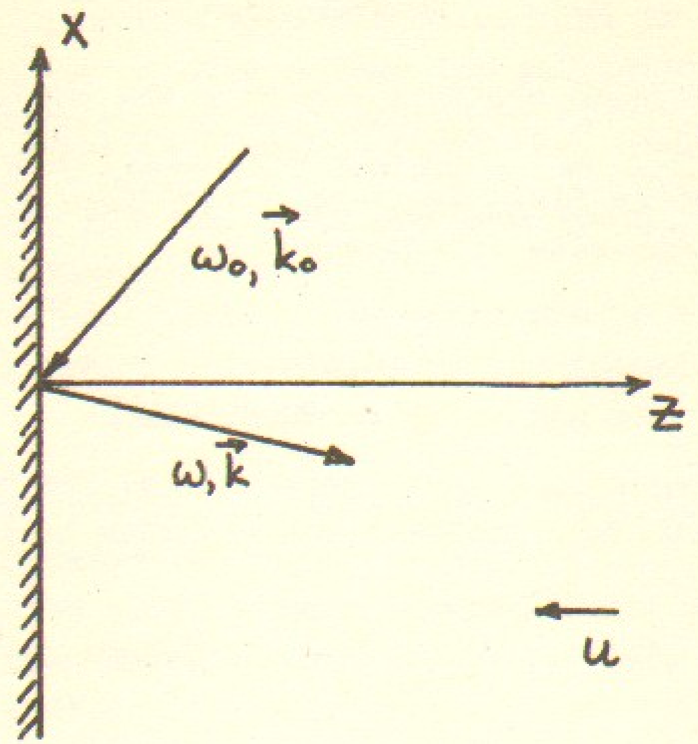

Fig. 1.

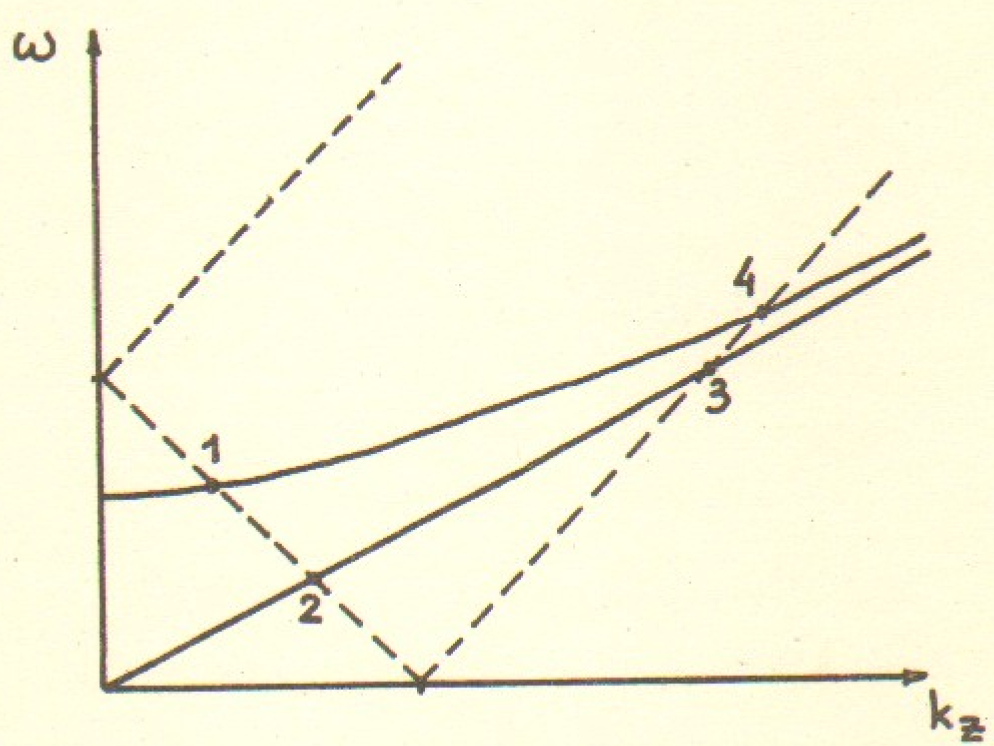

Fig. 2.

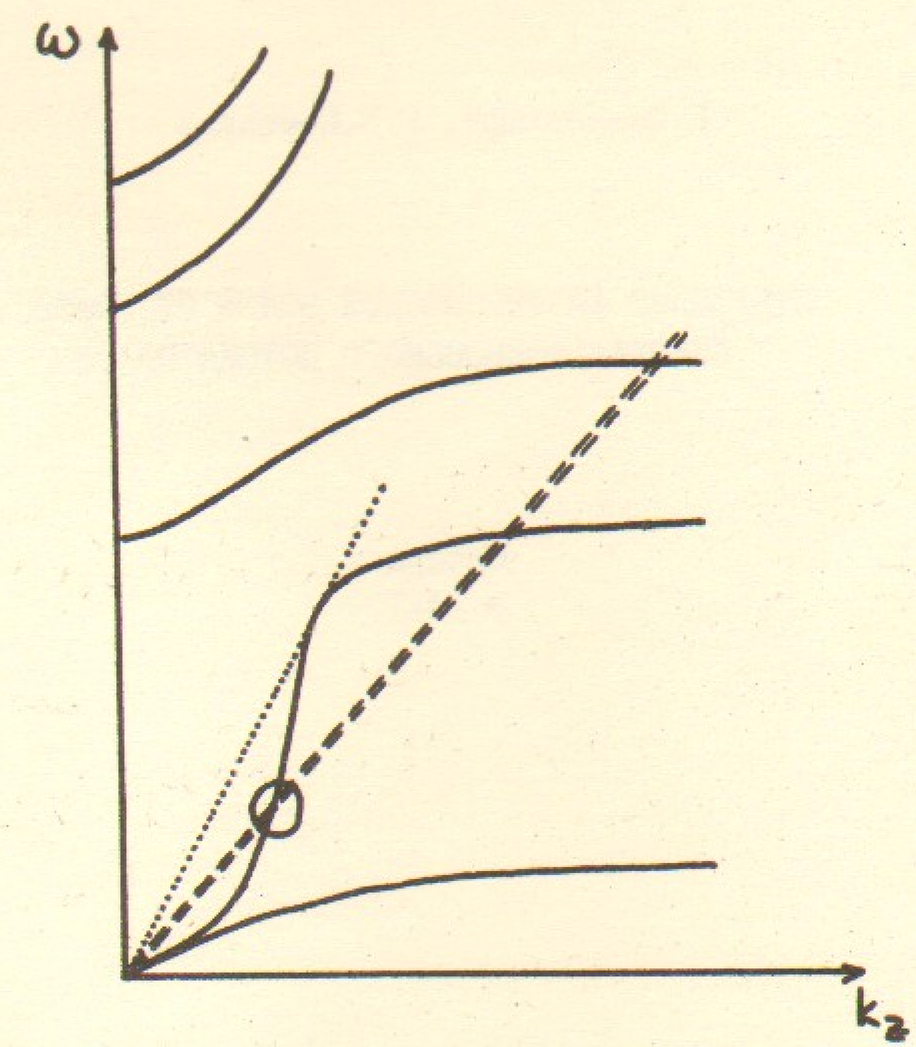

Fig. 3.

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ОТРАЖЕНИЕ АЛЬФВЕНОВСКОЙ ВОЛНЫ ОТ ГРАНИЦЫ
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