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ABSTRACT

The nonperturbative effects in QCD, which are due to the non-Abelian constant fields, are analysed. The nonrelativistic polarization operator $\Pi(E)$ is obtained in these fields. The operator expansion is not applicable if the nonperturbative corrections are larger than the relativistic corrections. At high energies, it is possible to make the analytic continuation of the derived polarization operator from the under-threshold region to the physical by means of the continuation of each operator expansion term individually. The Borel transform $\Pi(\tau)$ of the nonrelativistic polarization operator is sensitive not only to the gluon condensate value but also to the vacuum field structure. The non-Abelian constant fields don't lead to the spontaneous chiral symmetry breaking but generate the nonperturbative quark mass renormalization.

1. Introduction

One of the most important problems in QCD is the problem of the vacuum state structure. The QCD vacuum differs drastically from a perturbative one. Its complex structure is displayed in the existence of the quark /1/ and gluon /2/ condensates in it. So far the relative role of various vacuum fluctuations in the vacuum structure is, however, unknown definitively. In view of this, it is important to consider different models of the nonperturbative QCD vacuum. For this purpose, one can investigate the dependence of the vacuum correlators properties on the vacuum fluctuations form in certain models.

A lot of phenomenological features of QCD are explained by the dilute instanton gas model /3/. The consideration of the uniform field models is of interest, too. One can define these uniform fields so that all the gauge-invariant quantities are constant in space-time. Such a definition implies that the vector potential at one point is related to the vector potential at any other point by a gauge transformation /4/ :

$$B_\mu(y) = U^{-1}(x,y)\,B_\mu(x)\,U(x,y) + i\,U^{-1}(x,y)\,\partial_\mu U(x,y)\,. \tag{1}$$

Then the gauge can be chosen so that the field strength tensor $G^a_{\mu\nu}$ is constant. It is shown in /4/ that the uniform fields are generated by the vector potentials of two types only.

The first type is gauge equivalent to an Abelian potential and corresponds to the so-called covariantly constant field, which satisfies the gauge-invariant condition /5/ :

$$[\mathcal{D}_\lambda\,,\,G_{\mu\nu}] = 0\,, \tag{2}$$

where $\mathcal{D}_\lambda$ is a covariant derivative. In /5/ the vacuum polarization due to the quantum fluctuations of the gauge field in the presence of the covariantly constant field has been studied, and the presence of a chromomagnetic field has been shown to lead to the decrease of the vacuum energy. In the authors' paper /6/ the vector polarization operator and the vacuum expectation values in the covariantly constant field model have been analysed. This model is characterized by two invariants (for the gauge group $SU_c(2)$): $\mathcal{F}=\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$ and $\mathcal{G}^2=\frac{1}{16}(G^a_{\mu\nu}G^{*a}_{\mu\nu})^2$, where $G^{*a}_{\mu\nu}$ is a tensor dual to $G^a_{\mu\nu}$. The first invariant characterizes the gluon field intensity and $\mathcal{G}^2$ characterizes the topological charge density fluctuations. In /6/ it has been obtained in this model that the quark confinement at arbitrary high energies follows from the analytical properties of the polarization operator, in case of $\mathcal{G}^2\neq 0$.

The second-type vector potentials are gauge equivalent to the constant, noncommuting potentials. The vacuum polarization in the one-loop approximation for this field has been considered in /4/. Probably, the constant fields play the important role in QCD, if the master-field /7/ is reasonable.The constant field configurations were investigated in the framework of the Hamiltonian approach for arbitrary number of colours N_c in /8/.

In the present paper we consider the physical effects of the constant fields which are exemplified by the nonrelativistic polarization operator. We perform the calculations for the colour group $SU_c(2)$ in the one-loop approximation taking into account the external field exactly.The consideration for the other groups is similar.

2. Nonrelativistic polarization operator

The corresponding Green function should be known for investigation of the quantum processes with quarks in a constant field. We shall perform first the calculations in Euclidean space and pass to Minkowski space-time by means of Wick rotation. We use the metric $g_{\mu\nu}=-\delta_{\mu\nu}$ in Euclidean space. The operator $P_\mu = i\partial_\mu$ is an integral of motion in the field corresponding to the constant vector potentials. Therefore, is a c-number in the momentum space and the Green function calculation is an algebraic problem. The quark Green function, representing itself a matrix in the spinor and color spaces, is equal to

$$S(p) = \left(\mathcal{P}^2+\frac{i}{2}\sigma\cdot G-m^2\right)^{-1}\left(\hat{\mathcal{P}}+m\right) \tag{3}$$

($\hat{\mathcal{P}}=\mathcal{P}_\mu\gamma_\mu$, $\mathcal{P}_\mu=i\partial_\mu+B^a_\mu\tau^a/2$, $\sigma G=\sigma_{\mu\nu}G^a_{\mu\nu}\tau^a/2$, $\sigma_{\mu\nu}=[\gamma_\mu,\gamma_\nu]/2$, τ^a are Pauli matrices; we have absorbed the coupling constant g into the amplitudes of the fields). The field strength tensor, chromoelectric and chromomagnetic non-Abelian constant fields are equal to

$$G^a_{\mu\nu}=\epsilon^{abc}B^b_\mu B^c_\nu\,,\ \mathcal{E}^a_i=G^a_{0i}\,,\ \mathcal{H}^a_i=-\frac{1}{2}\epsilon_{ijk}G^a_{jk}\,. \tag{4}$$

Note that $G^a_{\mu\nu}G^{*a}_{\mu\nu}=0$ in this field. From (3) and (4) we obtain

$$S(p) = \frac{(H^2+h^2+\alpha)(H-h)+\beta}{H^4+\alpha H^2+\beta H+\gamma}\left(\hat{p}+\hat{B}+m\right), \tag{5}$$

where

$$H = p^2 + \varkappa/4 - m^2 ,$$
$$h = b^a \tau^a + i\sigma\cdot G/2 , \quad b^a = p_\mu B_\mu^a \tag{6}$$
$$\alpha = -2(b^2+C^2) , \quad \beta = -8 \det f ,$$
$$\gamma = (b^2+C^2)^2 - 4 b f^T f b + 2 tr (f f^T f f^T) - 2C^4 .$$

In (6) we use matrix notations; the sign "T" means a matrix transposition; $f_{ia} = (\mathcal{E}_i^a - \mathcal{H}_i^a)/2$, $C^2 = tr(f f^T)$ and $\varkappa_{\mu\nu} = B_\mu^a B_\nu^a$; $\varkappa = \varkappa_{\mu\mu}$.

The nonrelativistic approximation in Euclidean space for the Green function (5) corresponds to the analytical continuation from a real p_0 to the neighbourhood of the point $-im$; $p_0 = -im + \Delta \quad (|\Delta| \ll m)$, when the space momenta are small: $|\underline{p}| \ll m$. In this case, the expression (5) for the quark Green function becomes simpler:

$$S_{nr}(\Delta, \underline{p}) = - \frac{h_0(\Delta) + (i\varphi^a - \underline{p}\underline{B}^a/m)\tau^a/2}{h_0^2(\Delta) - (i\varphi^a - \underline{p}\underline{B}^a/m)^2/4} , \tag{7}$$

where $h_0(\Delta) = (\underline{p}^2 + \mu^2)/2m - i\Delta$, $\varphi^a = B_0^a$, $\mu^2 = \underline{B}^a \underline{B}^a/4$.

Let us consider the nonrelativistic polarization operator in the under-threshold region, where the squared total 4-momentum of a heavy quark Q and antiquark $\overline{Q}$ is $q^2 = 4m(m-E)$, $m \gg E > 0$. We have:

$$\Pi(E) = \left\langle \frac{1}{2m^2} tr \int \frac{d^3p}{(2\pi)^3} \frac{d\Delta}{2\pi} S_{nr}(iE+\Delta, \underline{p}) \overline{S}_{nr}(-\Delta, -\underline{p}) \right\rangle ; \tag{8}$$

here $\overline{S}_{nr}$ is the Green function of the antiquark $\overline{Q}$; $\overline{S}_{nr} = S_{nr}|_{B \to -B}$. In (8) the Lorentz-invariant averaging over the field B_μ^a orientation is performed after the calculation

of the polarization operator at fixed field B_μ^a . When the trace over the colour indices is taken, the gauge-invariant averaging over B_μ^a orientation in a colour space becomes trivial. In the physical region ($E < 0$) the nonrelativistic polarization operator can be obtained by the analytical continuation. Taking the integral over Δ in (8), we get

$$\Pi(E) = \left\langle \frac{1}{m^2} \int \frac{d^3p}{(2\pi)^3} \frac{H_0(H_0^2 + \varphi^2)}{\left[H_0^4 + H_0^2 \left(\varphi^2 - \frac{p_i \varkappa_{ij} p_j}{m^2} \right) - (p_i \varkappa_{i0}/m)^2 \right]} \right\rangle , \tag{9}$$

where $H_0 = (\underline{p}^2 + \mu^2)/m + E$. The expression (9) may be derived starting from the obvious relation:

$$\Pi(E) = - \frac{1}{m^2} G_S(\underline{0}, \underline{0}; E) , \tag{10}$$

where $G_S(\underline{r}, \underline{r}'; E)$ is the nonrelativistic Green function of Q and $\overline{Q}$ in the colour-singlet state.

It is necessary to renormalize the polarization operator (9). We perform it, subtracting from the integrand for $\Pi(E)$ in (9), the value of this integrand at E=0, B_μ^a=0. Moreover, it should be taken into account that $\varphi^2 + \underline{B}^2 \ll m^2$, if the nonrelativistic approximation is valid. Making the corresponding expansion and taking the integral over $\underline{p}$, for the renormalized nonrelativistic polarization operator, we obtain:

$$\Pi^R(E) = -\frac{1}{4\pi} \sqrt{\frac{E}{m}} \left\langle \left(1 + \frac{\underline{\mathcal{E}}^2}{4mE\varphi^2} \right)^{1/2} + \frac{\underline{\mathcal{E}}^2}{3mE^3\delta^4} \left[\left(2 - \sqrt{1+\delta^2}\right) \left(\frac{1+\sqrt{1+\delta^2}}{2} \right)^{1/2} - 1 \right] \right\rangle , \tag{11}$$

where $\delta = \varphi/E$, $\underline{\mathcal{E}}^2 = (G_{0i}^a)^2$ is a squared chromoelectric field, $\varphi = (\varphi^a \varphi^a)^{1/2} = (\varkappa_{00})^{1/2}$. Below we shall omit the sign "R" everywhere.

For sufficiently large E , one can neglect in (11) the effects connected with the vacuum field. This means the absence of quark confinement in the vacuum field with a constant vector potential, unlike the case of covariantly constant field /6/. The leading field corrections for $\Pi(E)$ are determined by the following formula (at $\underline{\mathcal{E}}^2 \ll mE\varphi^2$):

$$\Pi(E) = -\frac{1}{4\pi}\sqrt{\frac{E}{m}}\left[1 + \sum_{n=0}^{\infty} \frac{\langle \underline{\mathcal{E}}^2 \varphi^{2n}\rangle}{m E^{2n+3}} \cdot \frac{(-1)^n\, \Gamma(2n+5/2)}{4\pi^{1/2}\,\Gamma(2n+5)}\right]. \qquad (12)$$

It follows from the obvious relations, $\mathcal{E}^a_i = \epsilon^{abc}\varphi^b B^c_i$, $\varphi^a \mathcal{E}^a_i = 0$, that the matrix elements $\langle \underline{\mathcal{E}}^2 \varphi^{2n}\rangle$ is equal to

$$\langle \underline{\mathcal{E}}^2 \varphi^{2n}\rangle = \langle \underline{\mathcal{E}}^a (\mathcal{D}_0^{2n})^{ab} \underline{\mathcal{E}}^b \rangle . \qquad (13)$$

The nonperturbative correction due to the operators $\underline{\mathcal{E}}\mathcal{D}_0^{2n}\underline{\mathcal{E}}$ to the nonrelativistic polarization operator has been obtained in /9/. The coefficient functions for these operators are leading among all the operators in the nonrelativistic approximation. Substituting (13) into equation (12), we obtain the result in the form independent of the field shape, and this result agrees with /9/.In the case under consideration, the corresponding vacuum correlator $K_{\mathcal{E}}(\tau)$ of the chromoelectric fields /9/ is equal to

$$K_{\mathcal{E}}(\tau) = \frac{1}{6}\langle \underline{\mathcal{E}}^2 \cos(\varphi\tau)\rangle . \qquad (14)$$

In applications the Borel transform /2/ of the polarization operator (11) may be useful. We get

$$\Pi(\tau) = \frac{1}{(4\pi)^{3/2}(m\tau)^{1/2}}\left\langle \exp\left(-\frac{\underline{\mathcal{E}}^2\tau}{4m\varphi^2}\right) + \frac{\underline{\mathcal{E}}^2}{2m\tau\varphi^4}\left(1-\cos(\varphi\tau)\right)\right\rangle . \qquad (15)$$

In the nonrelativistic approximation $m\tau \gg 1$. Therefore, if $\underline{\mathcal{E}}^2 \sim \varphi^4$, then the second term in (15) is much smaller than unity.

Let us consider the averaging of the polarization operator over the vacuum fields. In case of the colour group $SU_c(2)$ it follows from Lorentz invariance that a constant vacuum field is characterized completely by the eigenvalues of the matrix $\mathcal{T}^{ab} = -B^a_\mu B^b_\mu$ (see /4/). These eigenvalues satisfy the equation

$$\lambda^3 + B^a_\mu B^a_\mu \lambda^2 + \frac{1}{2} G^a_{\mu\nu} G^a_{\mu\nu}\lambda + \frac{1}{6}\epsilon^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} = 0 . \qquad (16)$$

The coefficients of this equation are related to the eigenvalues λ_i as follows

$$\begin{aligned} B^a_\mu B^a_\mu &= -(\lambda_1+\lambda_2+\lambda_3)\,; \\ G^a_{\mu\nu} G^a_{\mu\nu} &= 2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\,, \\ \epsilon^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} &= -6\lambda_1\lambda_2\lambda_3 \,. \end{aligned} \qquad (17)$$

All the eigenvalues λ_i are positive.*) The invariant $B^a_\mu B^a_\mu$ is expressed in terms of the explicitly gauge-invariant quantities by means of the identity

$$B^a_\mu B^a_\mu G^c_{\rho\sigma} G^c_{\rho\sigma} = 2 j^a_\mu j^a_\mu - \epsilon^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \, , \qquad (18)$$

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*) Note that,in this model,the sign of $\langle \epsilon^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu}\rangle$ is opposite to that in instanton models.

where $j^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}$ is the current generated by the field.

All the quantities averaged over the vacuum field orientations depend on three invariants λ_i (i= 1, 2, 3). The direct method of averaging is the integration over B^a_μ, if the invariants (17) are fixed. However, it is convenient to make use of an approach, which is similar to that used in /6/ for averaging over the covariantly constant field orientation. Let us introduce a unit vector l_μ ($l^2 = -1$) directed at first along the zero axis: $l_\mu = g_{\mu 0}$. Then,

$$\varphi^2 = l_\mu \varkappa_{\mu\nu} l_\nu , \qquad (19)$$

$$\underline{\mathcal{E}}^2 = l_\mu G^a_{\mu\nu} G^a_{\nu\rho} l_\rho .$$

Since the result is expressed in terms of the Lorentz scalars after the averaging over B^a_μ, one can multiply both sides of (15) by $\frac{1}{\pi^2}\delta(l^2+1)$ and take the integral over l_μ. After that, the integration over B^a_μ becomes trivial. The matrix J^{ab} can be diagonalized by a global gauge transformation: $J^{ab} = -B^a_\mu B^b_\mu = \lambda_{(a)}\delta^{ab}$. Therefore, there are three orthogonal vectors B^i_μ (i= 1, 2, 3) of lengths $(\lambda_i)^{1/2}$. These vectors can be directed along the axes e^i_μ (i= 1, 2, 3). It follows from (19) that the correlator $\Pi(\tau)$ (15) averaged over B^a_μ depends on the field through the matrix $\varkappa_{\mu\nu} = B^a_\mu B^a_\nu$. Substituting the orthogonal vectors B^i_μ to (19) and (15) and integrating over l_0 and $l_1^2 + l_2^2 + l_3^2$, we get

$$\Pi(\tau) = +\frac{1}{(4\pi)^{3/2}(m\tau)^{1/2}}\int\frac{d\Omega_n}{4\pi}\left[\exp\left(-\frac{\tau\epsilon^2}{4m\omega^2}\right) + \frac{\epsilon^2}{m\tau\omega^4}\left(1 - J_0(\omega\tau)\right)\right], \qquad (20)$$

where $\epsilon^2 = \lambda_1(\lambda_2+\lambda_3)n_1^2 + \lambda_2(\lambda_1+\lambda_3)n_2^2 + \lambda_3(\lambda_1+\lambda_2)n_3^2$, $\omega^2 = \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2$, J_0 is Bessel function; in (20) the integration is performed over the direction of a unit vector $\underline{n}$ ($\underline{n}^2 = 1$). Going over to the energy representation (see, e.g.,/2/) for the polarization operator averaged over the vacuum field orientations we get

$$\Pi(E) = -\frac{1}{4\pi m^{1/2}}\left\langle \sqrt{E + \frac{\epsilon^2}{4m\omega^2}} + \frac{2}{3}\cdot\frac{\epsilon^2 E^{3/2}}{m\omega^4}\left(F\left(-\tfrac{3}{4}, -\tfrac{1}{4}; 1; -\frac{\omega^2}{E^2}\right) - 1\right)\right\rangle, \qquad (21)$$

where F is a hypergeometric function, which can be expressed via the Legendre functions. In (21) and in the following formulas the averaging means the integration over the vector $\underline{n}$ orientation ($\int d\Omega_{\underline{n}}/4\pi \ldots$), unlike (8), (9) and (11)-(15).

Let us note that the same averaging, applied to the correlator $K_\xi(\tau)$ (14), gives

$$K_\xi(\tau) = \left\langle \frac{\epsilon^2}{24}\left(3J_0(\omega\tau) - 4J_2(\omega\tau) + J_4(\omega\tau)\right)\right\rangle ; \qquad (22)$$

the notations are introduced after (20).

3. Properties of the vacuum expectation values

The nonrelativistic polarization operator $\Pi(E)$ (21) represents itself one of a few examples of exact taking into account the vacuum field in the vacuum correlators. Other known examples are the polarization operator of massless quarks in the one-instanton field /10/ and the polarization operator in the covariantly constant field /6/.

Starting from the expression for the vacuum correlator, in which the vacuum field is taken into account exactly, one can study the important question on the operator expansion.

Let us consider first the correlators in the under-threshold region, where $Q^2 = 4m^2 - q^2 > 0$. In this case, the polarization operator of massless quarks in the one-instanton field contains, as it is known, an operator expansion term, proportional to Q^{-4}, and a term, exponentially small as $e^{-Q\rho}$ (at $Q \to \infty$), and not expanded over the inverse powers of Q^2 /10/. The latter term doesn't destroy the applicability of the operator expansion owing to its exponential smallness at $Q \to \infty$. However, this term can significantly affect the polarization operator properties at moderate Q^2 .

As it has been shown in /6/, the operator expansion for the polarization operator in the under-threshold region is valid for the case of a covariantly constant field. There are all the powers of $1/Q^4$ in the operator expansion. However, all the matrix elements containing covariant derivatives $\mathcal{D}_\mu$ are zero. There are also the terms exponentially small as $e^{-Q^2/\mathcal{H}}$, $e^{-Q^2/\mathcal{E}}$ ($\mathcal{H}^2, \mathcal{E}^2 = \mathcal{F} \pm (\mathcal{F}^2 + \mathcal{G}^2)^{1/2}$) at $Q^2 \to \infty$. These terms are very important for a behaviour of the polarization operator at moderate Q^2 and also in the physical region.

The nonrelativistic polarization operator properties in a constant field is of interest, too. Using (21), one can verify easily that the operator expansion takes place in the under-threshold region (compare with (12)):

$$\Pi(E) = -\frac{1}{4\pi}\sqrt{\frac{E}{m}}\left\langle 1 + \frac{\epsilon^2}{m 2^5 E^3}\sum_{n=0}^{\infty}\frac{(-1)^n \Gamma(2n+5/2)}{\sqrt{\pi}\, 2^{2n}\Gamma^2(n+3)}\left(\frac{\omega}{E}\right)^{2n} + O\left(\frac{1}{m^2}\right)\right\rangle . \quad (23)$$

Formula (23) is valid at $E \gg \epsilon^2/m\omega^2$. The corrections incorporated in (23) correspond to the operators $\underline{\mathcal{E}}\,\mathcal{D}_0^{2n}\,\underline{\mathcal{E}}$,

whereas the neglected terms $O(1/m^2)$ correspond to the operators including the chromomagnetic field $\mathcal{H}$, covariant derivatives $\underline{\mathcal{D}}$, or high powers of $\underline{\mathcal{E}}$. Note that the exponentially small terms are absent in $\Pi(E)$ (21). This point are displayed by the analytical properties of $\Pi(E)$.

One can operate in terms of the parameters ϵ and ω because the integration region over the vector $\underline{n}$ direction is finite. The series (23) (the contribution of the operators $\underline{\mathcal{E}}\,\mathcal{D}_0^{2n}\,\underline{\mathcal{E}}$) is reduced to $\epsilon^2/8mE\omega^2$ at low energies, $E \ll \omega$. Therefore, in this case equation (23) represents itself an expansion of the square root $(1 + \epsilon^2/4mE\omega^2)^{1/2}$:

$$\Pi(E) = -\frac{1}{4\pi}\sqrt{\frac{E}{m}}\left\langle 1 + \frac{\epsilon^2}{8mE\omega^2} + \ldots\right\rangle .$$

If all the invariants λ_i are of the same order (let us refer to this case as general) then $\epsilon \sim \omega^2$, and the value of correction in (23) is of the order of $\epsilon^2/mE^3 \sim$ $\sim (E/m)(\omega/E)^4 \lesssim E/m$, at relatively high energies $E \gtrsim \omega$. The nonperturbative effects in this field become compared with the perturbative effects not at $E \sim \omega(\omega/m)^{1/3}$, when the operator $G^2 = G^a_{\mu\nu}G^a_{\mu\nu}$ gives the contribution of order of unity, but at lower energies $E \sim \omega^2/m$ when $\epsilon^2/mE\omega^2 \sim 1$. At $E \gtrsim \omega$ the nonperturbative effects are compared with the relativistic corrections, or smaller. At lower energies $\omega^2/m \ll E \ll \omega$ the sum of the contributions of the operators $\underline{\mathcal{E}}\,\mathcal{D}_0^{2n}\,\underline{\mathcal{E}}$ (n= 0, 1, ...) should be taken into account. Consequently, we cannot restrict ourselves to a contribution of the operator G^2 to $\Pi(E)$ provided the nonperturbative effects are larger than the relativistic corrections. This means that the operator expansion is not applicable in this situation.

It has been shown in /9/ that it is necessary to take into account the contributions of all operators $\mathcal{E}\mathcal{D}_c^{2n}\mathcal{E}$ to the nonrelativistic polarization operator in the region of exceeding of the nonperturbative effects under the relativistic corrections in case of the dilute instanton gas. This is similar to the conclusion derived above on the operator expansion applicability.

Thus, taking the leading contribution to $\Pi(E)$, we get

$$\Pi(E) = -\frac{1}{4\pi m^{1/2}}\left\langle\left(E+\frac{\epsilon^2}{4m\omega^2}\right)^{1/2}\right\rangle, \tag{24}$$

and, correspondingly,

$$\Pi(\tau) = (4\pi)^{-3/2}(m\tau)^{-1/2}\left\langle \exp\left(-\tau\epsilon^2/4m\omega^2\right)\right\rangle. \tag{25}$$

Note that, with the relativistic correction E/m taken into account, we have the following region of the operator expansion applicability: $1 \ll E/\omega \ll (m/\omega)^{1/5}$. Emphasize that the operator expansion is not applicable in all the other cases of the relations between λ_i, with neglect of the relativistic corrections (see below).

Let us consider now a question on the sensitivity of $\Pi(\tau)$ (20) to the vacuum field structure in this model, i.e. to the invariants λ_i (the gluon condensate value $\langle\frac{\alpha_s}{\pi}G^2\rangle$ is fixed). We introduce the dimensionless variable $T = \tau C^2/4m$ and the parameter $\eta = C/m$ ($C = (\langle G^2\rangle/2)^{1/4}$). The parameter η should be small for an applicability of the nonrelativistic approximation, $\eta \ll 1$. Then, we change over to the variables $\Lambda_i = \lambda_i C^{-2}$, which satisfy the rela-

tion :

$$\Lambda_1\Lambda_2 + \Lambda_1\Lambda_3 + \Lambda_2\Lambda_3 = 1 , \tag{26}$$

We introduce the function

$$F(T) = \frac{\Pi(\tau)}{\Pi_0(\tau)} = F_1(T) + F_2(T) = \left\langle e^{-f^2T} + \frac{f^2\eta^2}{4\Omega^2T}\left(1 - J_0\left(\frac{4\Omega T}{\eta}\right)\right)\right\rangle, \tag{27}$$

where $\Pi_0(\tau) = (4\pi)^{-3/2}(m\tau)^{-1/2}$ is the perturbative correlator: $\Omega = \omega/C$ and $f = \epsilon/C\omega$. There are four types of the relations between the invariants Λ_i. Let us consider the characteristic cases.

a) $\Lambda_1 = \Lambda_2 = \Lambda_3 \equiv \Lambda$. Then $\Omega^2 = \Lambda$, $f^2 = 2\Lambda$, and from (26) we have $\Lambda = 1/\sqrt{3}$. The integral over the vector $\underline{n}$ orientations is trivial in this case. We get

$$F(T) = \exp(-2T/\sqrt{3}) , \tag{28}$$

(the term with the Bessel function is omitted according to the arguments mentioned above).

b) $\Lambda_1 = \Lambda_2 \equiv \lambda \ll \Lambda_3 \equiv \Lambda$. It follows from (26) that $\lambda \ll 1$ and $\Lambda = (1-\lambda^2)/2\lambda \approx 1/2\lambda$. Therefore, at $T \gg \lambda$ we have

$$F(T) = \exp(-2T\lambda) - \frac{\sqrt{\pi}}{2}\exp(-T\lambda)\sqrt{T\lambda}\,\mathrm{erfc}(\sqrt{T\lambda}) + \frac{\pi\eta^2}{16\sqrt{2}\lambda T}, \tag{29}$$

$$\mathrm{erfc}\,x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt ,$$

i.e. the operator expansion is not applicable under this condition. It is not hard to see that the function $F_1(T)$ can be expanded in a series of T , if $T \ll \lambda$. Then, the operator expansion is not applicable in the region $T \sim \lambda$, too. The function $F_2(T)$ can be expanded in a series of T if $T \ll \eta/\sqrt{\Lambda}$. In this case the non-leading terms of $F_1(T)$ expansion (of the order of T^2 and higher) contain the powers of $1/\lambda$. In order to neglect these terms, it is necessary for them to be much smaller than the contribution of the operator G^2 . This condition restricts the region of the applicability of (20) and (27): $h \ll \lambda^{3/2}$ and $T \gg h^2/\lambda$. Then the terms of the expansion in (27), proportional to the variable T , cancel out. The following term of $F_2(T)$, proportional to T^3, corresponds to the contribution of the operator G^2 . This contribution is larger than the relativistic correction of order of $(m\tau)^{-1}$ if $T \gg h$. So, the operator expansion exists in the region

$$h^2/\lambda \ll T \ll h\sqrt{\lambda} , \qquad (30)$$

but is not applicable without taking into account of the relativistic corrections in this case, too. For this relation between the invariants Λ_i we have (see (17) and (18)):

$$-j^2 \gg (G^2)^{3/2} \gg -G^3$$

c) $\Lambda_1 = \Lambda_2 \equiv \Lambda \gg \Lambda_3 \equiv \lambda$. It follows from (26) that $\Lambda \approx 1$, $\lambda = (1-\Lambda^2)/2\Lambda \ll 1$. Then

$$F(T) = e^{-T} + \frac{h^2}{4T}\left(\gamma + \ln\frac{4T}{h} - Ci\left(\frac{4T}{h}\right)\right) , \qquad (31)$$

$Ci(x)$ is a cosine-integral function, γ = 0.577...

d) $\Lambda_1 \ll \Lambda_2 \ll \Lambda_3$. In this case, for $T \gg \Lambda_2 \equiv \lambda$ we have:

$$F(T) = \frac{\sqrt{\pi}}{2}\, erfc(\sqrt{T\lambda}) + \frac{h^2}{8T\lambda}\ln\left(\frac{T\sqrt{\lambda}}{h}\right) . \qquad (32)$$

It is interesting that a constant field strength tensor $G^a_{\mu\nu}$ doesn't determine the quark-antiquark dynamics. The Wu-Yang ambiguity takes place for the uniform fields /4/. If one of the invariants λ_i is zero (if two λ_i are zero,

then $G^a_{\mu\nu} = 0$), then the corresponding constant tensor can be also generated by an Abelian vector potential $B^a_\mu = -n^a F_{\mu\nu} x^\nu/2$. It is necessary for the existence of this ambiguity that the equation $G^a_{\mu\nu} \tilde{G}^{*a}_{\mu\nu} = 0$ be fulfilled (for non-Abelian constant field this equality is an identity). In this case, it follows from (13) of /6/ that

$$F_A(T) = (1 - e^{-t})/t , \quad t = 4T^3/3h^2 \qquad (33)$$

Evidently, the form of the function $F_A(T)$ substantially differs from the corresponding function, obtained from (27) at $\lambda_3 = 0$. If λ_1 and λ_2 are of the same order, then

$$F_{nA}(T) = \frac{2}{\pi}\int_0^{\pi/2} d\varphi \exp\left(-\frac{T}{\Lambda_1\cos^2\varphi + \Lambda_2\sin^2\varphi}\right) , \qquad (34)$$

where $\Lambda_1 = \Lambda_2^{-1} \equiv \Lambda \leq 1$. Note that $F_{n-A} \sim T^{-1/2} e^{-\Lambda T}$ at large T ($\Lambda \neq 1$).

The examples discussed above demonstrate explicitly that the correlator $\Pi(\tau)$ depends strongly on the relations between Λ_i , i.e. on the vacuum field structure. This rejects the related conclusions in /11/.

A study of the properties of the polarization operator in a constant field, in the physical region, is also of interest in view of the question on the vacuum structure. The analytical continuation to the physical region of the polarization operator of light and heavy quarks in the instanton field was considered in /9,12/. The question on the operator expansion in this region was considered as well. The oscillations of the imaginary part of the polarization operator was found in /9/.

These oscillations in the physical region are connected with the existence of the exponentially small terms in the under-threshold region (at $Q^2 \to \infty$). Evidently, the oscillations violate the applicability of the operator expansion. It was shown in /6/ that the analytical continuation of the polarization operator $\Pi(E)$, obtained in the one-loop approximation in a covariantly constant field, from the under-threshold region to the physical leads to the absence of the imaginary part of $\Pi(E)$. This means the quark confinement in these fields. Then, the usual dispersion relations are violated.

It is convenient to make the analytical continuation, starting from (21) for $\Pi(E)$. Since the region of integration in (21) is finite, the value of ω varies in a finite region, too. Due to the convergence of a hypergeometric series $F(a,b;c;z)$ inside the circle, $|z|<1$, one can use the expansion of a hypergeometric function at $|E| > max(\omega)$. then we obtain that the operator expansion is valid at $|E|>max(\omega)$ (in this case we have $|E| > max(\epsilon^2/m\omega^2)$ if all λ_i are of the same order). That is why one can use, in this field, a procedure of the naive analytical continuation of each operator expansion term individually. The conditions of applicability of the operator expansion in the physical region are the same as in the under-threshold region. Making the analytical continuation of $\Pi(E)$ (21) to the physical region ($E \to E e^{-i\pi}$), we get:

$$\Pi(E) = -\frac{1}{4\pi m^{1/2}} \Big\langle \Big(\frac{\epsilon^2}{4m\omega^2} - E \Big)^{1/2} \theta\Big(\frac{\epsilon^2}{4m\omega^2} - E \Big) - i \Big(E - \frac{\epsilon^2}{4m\omega^2} \Big)^{1/2} \cdot \tag{35}$$

$$\cdot \theta\Big(E - \frac{\epsilon^2}{4m\omega^2} \Big) + \frac{2}{3} i \frac{\epsilon^2 E^{3/2}}{m\omega^4} \Big[\theta(E-\omega) F\Big(-\frac{3}{4}, -\frac{1}{4}; 1; -\frac{\omega^2}{E^2}\Big) - 1 +$$

$$i\theta(\omega - E) \Big\{ \frac{\Gamma^2(1/4)}{6\pi^{3/2}} \Big(\frac{\omega}{E}\Big)^{\frac{3}{2}} F\Big(-\frac{3}{4}, -\frac{3}{4}; \frac{1}{2}; -\frac{E^2}{\omega^2}\Big) + \frac{6\pi^{1/2}}{\Gamma^2(1/4)} \Big(\frac{\omega}{E}\Big)^{1/2} F\Big(\frac{1}{4}, -\frac{1}{4}; \frac{3}{2}; -\frac{E^2}{\omega^2}\Big) \Big\} \Big] \Big\rangle$$

In (35) we have used the formula of the analytical continuation of a hypergeometric function $F(a,b;c;z)$ to the domain $|z|>1$. Then we obtain

$$\mathrm{Im}\, \Pi(E) = \frac{1}{4\pi} \sqrt{\frac{E}{m}} \Big\langle \Big(1 - \frac{\epsilon^2}{4mE\omega^2} \Big)^{1/2} \theta\Big(E - \frac{\epsilon^2}{4m\omega^2} \Big) + \frac{2}{3} \frac{E}{m} \frac{\epsilon^2}{\omega^4} \Big[1 - \theta(E-\omega) F\Big(-\frac{3}{4}, -\frac{1}{4}; 1; -\frac{\omega^2}{E^2}\Big) \Big] \Big\rangle \tag{36}$$

It is seen from (36) that a constant vacuum field modifies the cross section of the e^+e^- annihilation into a heavy quark and antiquark. In the general case of the relations between λ_i , one can neglect the second term in (36) because it is compared with the relativistic correction. In case of a strong inequality between λ_i , the second term is small, too. We can neglect this term in the region where the nonperturbative effects are significant. Then, one can verify, with the use of (21) and (36), that the usual dispersion relation is valid.

It is seen from (36) that a cut of the polarization operator in a constant field is shifted in comparison with the case when the field is absent; i.e. a constant field generates a mass gap. However, the nonperturbative effects due to the field are not reduced only to the quark mass renormalization Δm , because of the averaging over the vector n orientations. One can interpret this mass renormalization as the going over from the current quarks to the constituent ones. From (25) at $\tau \to \infty$ we get:

$$\Delta m = (\lambda_1 + \lambda_2 + \lambda_3 - max\, \lambda_i)/4m \tag{37}$$

It follows from (36) that a constant field doesn't generate any resonances. Previously /6/, we have shown that the bound states are absent in a covariantly constant field (in the one-loop approximation). Thus, we conclude that the field inhomogeneity is very important for a resonance formation.

One of the fundamental problems in QCD is a clarification of the spontaneous chiral symmetry breaking mechanism. One of the way to resolve this problem is to seach for the field configurations providing such a breaking. Starting from the definition of the chiral parameter of the order $\langle\bar{\Psi}\Psi\rangle$, in the case under consideration (p_μ is an integral of motion) we have (see notations after (3)):

$$\langle\bar{\Psi}\Psi\rangle = \lim_{x\to 0} m \left\langle \mathrm{Tr} \int \frac{d^4p}{(2\pi)^4} e^{ipx} \left[\frac{1}{\hat{\mathfrak{p}}^2 - m^2} e^{i\tau^a B^a_\mu x_\mu/2} - \frac{1}{\hat{p}^2 - m^2} \right] \right\rangle, \quad (38)$$

here the trace is taken over both the Lorentz and colour indices; the averaging corresponds to the integration over B^a_μ orientations. Using (5) and (6) we get:

$$\langle\bar{\Psi}\Psi\rangle = 8m \int \frac{d^4p}{(2\pi)^4} \left[\frac{H^3 + \alpha H/2 + \beta/4}{H^4 + \alpha H^2 + \beta H + \gamma} - \frac{1}{p^2 - m^2} - \frac{m^2 \varkappa}{4(p^2 - m^2)^3} \right] \quad (39)$$

It follows from (39) that $\langle\bar{\Psi}\Psi\rangle \sim m \ln m$ in the chiral limit, i.e. the spontaneous chiral symmetry breaking is absent in a constant field. For large m , the quark condensate generated by the field is equal to

$$\langle\bar{\Psi}\Psi\rangle = -\frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{24\pi^2 m} + \frac{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3) + 7\lambda_1\lambda_2\lambda_3/2}{120\pi^2 m^3} \quad (40)$$

where λ_i are the eigenvalues of the matrix J^{ab} . Using (17), we can represent $\langle\bar{\Psi}\Psi\rangle$ in the form

$$\langle\bar{\Psi}\Psi\rangle = -\frac{\langle G^a_{\mu\nu} G^a_{\mu\nu}\rangle}{48\pi^2 m} - \frac{1}{120\pi^2 m^3}\left[\langle j^a_\mu j^a_\mu\rangle + \frac{1}{12}\langle \epsilon^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu}\rangle\right]. \quad (41)$$

In /13/ the contribution of heavy quarks to the effective Lagrangian $\mathcal{L}(B_\mu, m)$ was obtained in the one-loop approximation with taking into account the gluon operators of dimensions 4, 6, 8. Since $\langle\bar{\Psi}\Psi\rangle = -\frac{\partial}{\partial m}\mathcal{L}(B_\mu, m)\rangle$, it was obtained also the corresponding expansion of $\langle\bar{\Psi}\Psi\rangle$. However, in /13/ the operators containing the current $j^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}$, were not considered. From (41) the corresponding contribution to the gauge field Lagrangian is found:

$$\mathcal{L}_{eff} = -\frac{1}{48\pi^2} \ln\frac{M}{m} \cdot G^a_{\mu\nu} G^a_{\mu\nu} - \frac{1}{240\pi^2 m^2}\left[(j^a_\mu j^a_\mu) + \frac{1}{12}(\epsilon^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu})\right], \quad (42)$$

here M is a regulator mass. If we put $j^a_\mu = 0$ in (42), then this formula agrees with the result of /13/.

4. Conclusions

The analytical calculation of the nonrelativistic polarization operator $\Pi(E)$ permits one to investigate the properties of $\Pi(E)$ in the physical region as well as in the under-threshold region. The impossibility to use the operator expansion for the nonrelativistic systems in the region, where the nonperturbative corrections are larger than the relativistic corrections, is most likely to hold for the wide class of the vacuum field configurations. The possibility of the analytical continuation of the operator expansion series to the physical region at high enough energies is connected with the absence of the exponentially small terms in the under-threshold region, in the field under consideration. The correlator $\Pi(\tau)$ depends strongly on the vacuum field structure as well as on the gluon condensate value. The example of the Wu-Yang ambiguity shows explicitly that the behaviour of the polarization operator in the vacuum fields with the different structures may be so that the effect of the field of one type cannot be imitated by the effect of another field of any gluon condensate magnitude.

The consideration of the vacuum expectation values in the fields of certain types may be useful to obtain the model-independent relations. We have demonstrated this point by the calculation of the heavy quark condensate value.

The analytical investigation of the correlators in certain vacuum fields may be very important to clarify some theoretical questions, especially, if the effective field saturating the vacuum expectation values in the low-energy region, exists in QCD.

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