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НОВОСИБИРСК

GENERAL STRUCTURE OF NONLINEAR EQUATIONS IN
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ABSTRACT

General form of the nonlinear evolution equations in the two-dimensional space integrable by the generalized two-dimensional Zakharov-Shabat-Gelfand-Dikij spectral problem is found. The applicability of AKNS method to this problem is demonstrated.

I. Introduction

The inverse scattering transform (IST) method is a powerful tool for the investigation of nonlinear evolution equations (see e.g. [1-3]). A number of nonlinear evolution equations have been integrated by this method.

One of the most important problems of IST method is the problem of effective description of nonlinear equations to which this method is applicable. There exist different approaches to this problem. A very convenient and simple method of description of the nonlinear equation integrable by second order matrix spectral problem was proposed by Ablowitz, Kaup, Newell and Segur in [4] . The method suggested in [4] (AKNS--method) has been generalized to different spectral problems (see e.g. [5-18]).

In the present paper we consider general two-dimensional scalar spectral problem of the form

$$\left(\partial_x^N + V_{N-2}(x,y,t)\partial_x^{N-2} + \ldots + V_1(x,y,t)\partial_x + V_0(x,y,t) + \rho^N(x,y,t)\partial_y\right)\psi = 0 \qquad (1.1)$$

where $\partial_x \overset{def}{=} \partial/\partial x$, $\partial_y \overset{def}{=} \partial/\partial y$ and $V_0(x,y,t), \ldots, V_{N-2}(x,y,t)$, $\rho^N(x,y,t)$ are scalar functions such that

$$V_k(x,y,t) \xrightarrow[\sqrt{x^2+y^2}\to\infty]{} 0 \quad (k=0,1,\ldots,N-2)$$

and

$$\rho^N(x,y,t) - 1 \xrightarrow[\sqrt{x^2+y^2}\to\infty]{} 0$$

fastly enough. Problem (1.1) is a generalization of the well known two dimensional Zakharov-Shabat-Gelfand-Dikij spectral problem for which $\rho^N(x,y,t) \equiv 1$. Problem (1.1) is also a two-dimensional generalization for arbitrary N of the problem $(\partial_x^2 + V_0(x,t))\psi = \rho^2(x,t)\lambda^2\psi$ which was discussed in [3]. For $\rho^N \equiv 1$ the problem (1.1) has been analyzed in [16, 18].

Here we extend the AKNS method to the problem (1.1) and find the general form of nonlinear evolution equations integrable by (1.1). These equations are of the form

$$\frac{\partial V}{\partial t} - \hat{L}_1^+ \frac{\partial \rho^N}{\partial t} U_{(0)} - \sum_{k=0}^{N-1}\sum_{n=0}^{\infty} \omega_{kn}(t)\left[\mathcal{L}^+_{(k,n)} V - \mathcal{M}^+_{(k,n)}(\rho^N-1)U_{(0)} - \mathcal{N}^+_{(k,n)}(\rho^N-1)U_{(k)}\right] = 0 \qquad (1.2)$$

where $V \overset{def}{=} (V_0,\dots,V_{N-2},0)^T$, $U_{(k)} \overset{def}{=} (\underbrace{0,\dots,0}_{k},1,0,\dots,0)^T$ $(k=0,1,\dots,N-1)$ and $\hat{L}_1^+, \mathcal{L}^+_{(k,n)}, \mathcal{M}^+_{(k,n)}, \mathcal{N}^+_{(k,n)}$ are certain matrix operators and $\omega_{kn}(t)$ are arbitrary functions on t.

The system (1.2) is the system of N-1 equations for N functions ($V_0, V_1,\dots,V_{N-2}, \rho^N$). Therefore, these system admits certain reductions. For N = 2 the system (1.2) is the one equation for V_0 and ρ. This equation admits the reduction to one function. In particular, at $V_0 = 0$ we have the two-dimensional generalization of Harry Dym's family of equations [3]. For N = 3 the system (1.2) for V_0, V_1, ρ admit (at $\omega_{21}\neq 0$ and other ω_{kn} = 0) the reduction $V_0 = V_1 = 0$. As a result we obtain the equation for one field $\rho(x,y,t)$. This equation of the fifth order does not contained in the Harry Dym's family and is a new one. In the one-dimensional limit $\partial\rho/\partial y = 0$ this equation is of the form $\varphi_t = \varphi^{5/2}\varphi_{xxxxx}$ or $\eta_t = -\frac{3}{2}(\eta^{-\frac{2}{3}})_{xxxxx}$.

The paper is organized as follows. In section 2 we consider the direct problem for (1.1) and obtain some important relations. In section 3 we calculate recursion operators. The general form of the integrable equations is found in section 4. In the calculation of the sections 3 and 4 we use the bilocal approach proposed in [18]. The examples of equations for N = 2 and 3 are considered in sections 5 and 6.

II. Some important relations

Firstly we represent the problem (1.1) in the matrix form

$$\frac{\partial \hat{F}}{\partial x} + A\rho^N(x,y,t)\frac{\partial \hat{F}}{\partial y} + P(x,y,t)\hat{F} = 0 \qquad (2.1)$$

where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0, & -1, & 0, & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & -1 \\ V_0 & V_1 & V_2 & \dots & V_{N-2} & 0 \end{pmatrix}. \qquad (2.2)$$

The adjoint problem is

$$\frac{\partial \check{F}}{\partial x} + \frac{\partial(\rho^N(x,y,t)\check{F})}{\partial y}A - \check{F}P(x,y,t) = 0. \qquad (2.3)$$

Introduce in the standard way matrices-solutions $\hat{F}^{\pm}_{\lambda}(x,y,t)$ and $\check{F}^{\pm}_{\lambda}(x,y,t)$ of the problems (2.2) and (2.3) given by their asymptotic behaviour

$$\hat{F}^{\pm}_{\lambda}(x,y,t) \xrightarrow[x\to\pm\infty]{} (2\pi i)^{-\frac{1}{2}}\lambda^{\frac{N-1}{2}} D(\lambda)\exp(-\lambda^N y + \bar{A}x),$$

$$\check{F}^{\pm}_{\lambda}(x,y,t) \xrightarrow[x\to\pm\infty]{} (2\pi i)^{-\frac{1}{2}}\lambda^{\frac{N-1}{2}} \exp(\lambda^N y - \bar{A}x)D^{-1}(\lambda) \qquad (2.4)$$

where $\lambda > 0$, $\bar{A}_{ik} = \lambda q^{i-1}\delta_{ik}$ $\left(q = \exp\frac{2\pi i}{N}\right)$,

$$\delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}, \quad D_{ik} = \frac{1}{\sqrt{N}}(\lambda q^{k-1})^{i-1},$$

$$i,k = 1,2,\dots,N.$$

Quantities λq^{i-1} are the eigenvalues of the matrix $\tilde{A}$ = $= \lambda^N A + P_\infty$ where $P_\infty = -\lim_{\sqrt{x^2+y^2}\to\infty} P(x,y,t)$ and $\tilde{A} = D(\lambda)\bar{A}D^{-1}(\lambda)$.

Scattering matrices for problems (2.1), (2.3) are introduced as follows [15, 16]

$$\hat{F}^{+}_{\lambda}(x,y,t) = \int_0^\infty d\tilde{\lambda}\, \hat{F}^{-}_{\tilde{\lambda}}(x,y,t)\hat{S}(\tilde{\lambda},\lambda,t),$$

$$\check{F}^{+}_{\lambda}(x,y,t) = \int_0^\infty d\tilde{\lambda}\, \check{S}(\lambda,\tilde{\lambda},t)\check{F}^{-}_{\tilde{\lambda}}(x,y,t). \qquad (2.5)$$

Using (2.4) and (2.5) one can prove that

$$\int_{-\infty}^{+\infty} dy\, \check F^{\pm}_{\bar\lambda}(x,y,t)\, \hat F^{\pm}_{\lambda}(x,y,t) = \delta(\bar\lambda-\lambda),$$
$$\int_{0}^{\infty} d\mu\, \check S(\bar\lambda,\mu,t)\, \hat S(\mu,\lambda,t) = \delta(\bar\lambda-\lambda). \tag{2.6}$$

Let us consider now two sets of potentials P, ρ^N and P', ρ'^N. Let $\hat F^+$, $\check F^+$, $\hat S$ and $\hat F^{+'}$, $\check F^{+'}$, $\hat S'$ are corresponding solutions and scattering matrices for (2.1) and (2.3). Analogously to [16, 18] one can obtain the following important relation

$$\hat S'(\bar\lambda,\lambda,t) - \hat S(\bar\lambda,\lambda,t) = \int_0^{+\infty} d\mu\, \hat S(\bar\lambda,\mu,t) \int_{-\infty}^{+\infty} dx\,dy\, \check F^+_{\mu}(x,y,t)\cdot \tag{2.7}$$
$$\cdot[A(\rho'^N(x,y,t)-\rho^N(x,y,t))\partial_y + P'(x,y,t) - P(x,y,t)]\hat F^{+'}_{\lambda}(x,y,t).$$

To prove (2.7) we use (2.4), (2.6) and the relation $\int_{-\infty}^{+\infty} dy \frac{\partial}{\partial y}(\dots) = 0$ which is valid if the potentials P, P', ρ^N-1, ρ'^N-1 decrease at the infinity fast enough.

Using (2.6), from (2.7) we obtain

$$\int_0^{\infty} d\mu\, \check S(\bar\lambda,\mu,t)\frac{d\hat S(\mu,\lambda,t)}{dt} = \tag{2.8}$$
$$= \int_{-\infty}^{+\infty} dx\,dy\, \check F^{+}_{\bar\lambda}(x,y,t)\Big[A\frac{\partial \rho^N(x,y,t)}{\partial t}\partial_y + \frac{\partial P(x,y,t)}{\partial t}\Big]\hat F^{+}_{\lambda}(x,y,t).$$

Further taking into account (2.1), (2.3)-(2.6) one can show that the following identity holds

$$\int_{-\infty}^{+\infty} dy\, \check F^{+}_{\bar\lambda}(x,y,t)\tilde Y(-\partial_y,t)\hat F^{+}_{\lambda}(x,y,t)\Big|_{x=-\infty}^{x=+\infty} =$$
$$= \int_0^{+\infty} d\mu\, \check S(\bar\lambda,\mu,t)[\hat S(\mu,\lambda,t)\bar Y(\lambda,t) - \bar Y(\mu,t)\hat S(\mu,\lambda,t)] = \tag{2.9}$$
$$= -\int_{-\infty}^{+\infty} dx\,dy\, \check F^{+}_{\bar\lambda}(x,y,t)[\tilde Y(-\partial_y,t),(\rho^N-1)A\partial_y + \tilde P]\hat F^{+}_{\lambda}(x,y,t)$$

where $\tilde P = P + P_\infty$ and $\bar Y(\lambda,t)$ is an arbitrary diagonal matrix and $\tilde Y(\lambda,t) = D(\lambda)\bar Y(\lambda,t)D^{-1}(\lambda)$. Since $D(\lambda)\bar A D^{-1}(\lambda) = \tilde A$ then $\tilde Y$ can be arbitrary matrix which commutes with $\tilde A$. Since all eigenvalues of $\tilde A = \lambda^N A + P_\infty$ are different then arbitrary $\tilde Y(\lambda,t)$ can be represented in the form $\tilde Y(\lambda,t) = \sum_{k=0}^{N-1}\Omega_k(\lambda^N,t)\tilde A^k$ where $\Omega_k(\lambda^N,t)$ are arbitrary scalar functions.

Combining (2.8) and (2.9) we get

$$\int_0^{+\infty} d\mu\, \check S(\bar\lambda,\mu,t)\Big[\frac{d\hat S(\mu,\lambda,t)}{dt} - (\bar Y(\mu,t)\hat S(\mu,\lambda,t) - \hat S(\mu,\lambda,t)\bar Y(\lambda,t))\Big] =$$
$$= \int_{-\infty}^{+\infty} dx\,dy\, \check F^{+}_{\bar\lambda}(x,y,t)\Big\{A\frac{\partial\rho^N}{\partial t}\partial_y + \frac{\partial P}{\partial t} - [\tilde Y(-\partial_y,t),(\rho^N-1)A\partial_y + \tilde P]\Big\}\hat F^{+}_{\lambda}(x,y,t). \tag{2.10}$$

Let us consider now the linear evolution law of the scattering matrix

$$\frac{d\hat S(\mu,\lambda,t)}{dt} = \bar Y(\mu,t)\hat S(\mu,\lambda,t) - \hat S(\mu,\lambda,t)\bar Y(\lambda,t). \tag{2.11}$$

If the scattering matrix $\hat S$ envolves according to equation (2.11) then by virtue of (2.10) the potentials P, ρ^N satisfy the following fundamental relation

$$\int_{-\infty}^{+\infty} dx\,dy\, \check F^{+}_{\bar\lambda}(x,y,t)\Big\{A\frac{\partial\rho^N}{\partial t}\partial_y + \frac{\partial P}{\partial t} - [\tilde Y(-\partial_y,t),(\rho^N-1)A\partial_y + \tilde P]\Big\}\hat F^{+}_{\lambda}(x,y,t) = 0. \tag{2.12}$$

The relation (2.12) is the starting point for the construction of the integrable equations connected with the problem (1.1).

For further transformation of the relation (9.12) it is convenient to introduce the bilocal product

$$\overset{++in}{\varphi}_{ke}(x,\tilde y,y,t) \overset{def}{=} \hat F^{+}_{kn}(x,\tilde y,t)\check F^{+}_{ie}(x,y,t) \tag{2.13}$$
$$(i,k,e,n = 1,\dots,N).$$

The quantity $\overset{++in}{\varphi}_{ke}(x,\tilde y,y,t)$ transforms under the adjoint representation of the group which is determined by (1.1) [17]. The use of the bilocal quantity $\overset{++}{\varphi}(x,\tilde y,y,t)$ essentially simplifies all calculations in comparison with the calculations which use the local quantity $\overset{++in}{\varphi}_{ke}(x,y,t) = \overset{++in}{\varphi}_{ke}(x,\tilde y,y,t)|_{\tilde y=y}$ (for $\rho^N = 1$ see [18]).

As it follows from (2.1) and (2.3) the bilocal quantity $\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t)$ satisfies the equation

$$\frac{\partial \overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t)}{\partial x} + A\rho^N(x,\tilde{y},t)\frac{\partial \overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t)}{\partial \tilde{y}} + \tag{2.14}$$
$$+ \frac{\partial(\rho^N(x,y,t)\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t))}{\partial y}A + P(x,\tilde{y},t)\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t) - \overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t)P(x,y,t) = 0$$

Use of $\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t)$ given by (2.13) and introduction of the additional integration over $\tilde{y}$, allow us to rewrite (2.12) in the form

$$\int_{-\infty}^{+\infty} dx\,dy\,d\tilde{y}\,\delta(y-\tilde{y})\, tr\Big[\Big(A\frac{\partial\rho^N(x,y,t)}{\partial t}\partial_{\tilde{y}} + \frac{\partial P(x,y,t)}{\partial t}\Big)\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t) -$$
$$- \tilde{Y}(\partial_y,t)\tilde{P}(x,\tilde{y},t)\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t) + \tilde{P}(x,y,t)\tilde{Y}(-\partial_{\tilde{y}},t)\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t) - \tag{2.15}$$
$$- \tilde{Y}(\partial_y,t)(\rho^N(x,y,t)-1)A\partial_{\tilde{y}}\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t) + (\rho^N(x,y,t)-1)A\tilde{Y}(-\partial_{\tilde{y}},t)\partial_{\tilde{y}}\overset{++}{\varphi}{}^{in}(x,\tilde{y},y)\Big] = 0$$

where tr denotes a usual matrix trace over the low indices. To change $\tilde{Y}(-\partial_{\tilde{y}})$ on $\tilde{Y}(\partial_y)$ we use the equality

$$\int_{-\infty}^{+\infty} dy\,d\tilde{y}\,\delta(y-\tilde{y})(\partial_y + \partial_{\tilde{y}})(\ldots) = \int_{-\infty}^{+\infty} dy\,\partial_y(\ldots) = 0.$$

We will consider here only the functions $\Omega_k(\lambda^N,t)$ entire on λ^N. So

$$\tilde{Y}(-\partial_y,t) = \sum_{k=0}^{N-1}\Omega_k(-\partial_y,t)(-A\partial_y + P_\infty)^k = \tag{2.16}$$
$$= \sum_{k=0}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)(-\partial_y)^n\big(-(P_\infty^T)^{N-k}\partial_y + P_\infty^k\big).$$

Substituting (2.16) into (2.15), we finally have

$$\int_{-\infty}^{+\infty} dx\,dy\,d\tilde{y}\,\delta(y-\tilde{y})\, tr\Big\{\Big(A\frac{\partial\rho^N(x,y,t)}{\partial t}\partial_{\tilde{y}} + \frac{\partial P(x,y,t)}{\partial t}\Big)\overset{++}{\varphi}{}^{in}(x,\tilde{y},y,t) - \tag{2.17}$$

$$-\sum_{k=0}^{N-1}\sum_{m=0}^{\infty}\omega_{km}(t)\Big[\partial_y^m\,\overset{++}{\varphi}{}^{in}_{\Delta_{N-k}}(x,\tilde{y},y,t)P_\infty^k\tilde{P}(x,\tilde{y},t) -$$
$$-(-1)^{m+1}(P_\infty^T)^{N-k}\partial_{\tilde{y}}^{m+1}\overset{++}{\varphi}{}^{in}_{\Delta_N}(x,\tilde{y},y,t)\tilde{P}(x,y,t) - (-1)^m P_\infty^k\,\partial_{\tilde{y}}^m\overset{++}{\varphi}{}^{in}_{\Delta_N}(x,\tilde{y},y,t)\tilde{P}(x,y,t) + \tag{2.17}$$
$$+(\rho^N(x,\tilde{y},t)-1)\partial_y^m\partial_{\tilde{y}}\overset{++}{\varphi}{}^{in}_{\Delta_{N-k}}(x,\tilde{y},y,t)P_\infty^k(P_\infty^T)^{N-1} - (\rho^N(x,y,t)-1)\partial_{\tilde{y}}^{m+1}\overset{++}{\varphi}_{\Delta_N}(x,\tilde{y},y)(P_\infty^T)^{N-1}P_\infty^k\Big]\Big\} = 0$$

where Δ_k denotes the projection of matrix φ onto k-th column: $(\varphi_{\Delta_k})_{mn} = \delta_{nk}\varphi_{mk}$.

III. Recursion operators

For further transformation of the equality (2.17) it is necessary to find the relations between the quantities $(\partial_y^n\varphi(x,\tilde{y},y,t))|_{\tilde{y}=y}$ and $(\partial_{\tilde{y}}^n\varphi(x,\tilde{y},y,t))|_{\tilde{y}=y}$ with different n , i.e. to calculate the recursion operators. We will calculate these operators using bilocal tensor product $\overset{++}{\varphi}(x,\tilde{y},y,t)$. We will proceed to the limit $\tilde{y}=y$ at the very end of the calculations. Our calculations are the generalization of the calculations of the recursion operators given in [18].

Starting equation for the calculation of the recursion operators is equation (2.14). Let us rewrite this equation is the form

$$[A,\rho^N(x,\tilde{y},t)(\partial_{\tilde{y}}\overset{++}{\varphi}(x,\tilde{y},y,t))] = -\partial_x\overset{++}{\varphi}(x,\tilde{y},y,t) - (\rho^N(x,\tilde{y},t)\partial_{\tilde{y}} + \partial_y\rho^N(x,y,t))\times$$
$$\times\overset{++}{\varphi}(x,\tilde{y},y,t)A - P(x,\tilde{y},t)\overset{++}{\varphi}(x,\tilde{y},y,t) + \overset{++}{\varphi}(x,\tilde{y},y,t)P(x,y,t) = 0. \tag{3.1}$$

Then we apply the projection operation Δ_k to equation (3.1). Taking into account (2.2) we obtain

$$A\rho^N(x,\tilde{y},t)\partial_{\tilde{y}}\overset{++}{\varphi}_{\Delta_k} - \delta_{k1}\rho^N(x,\tilde{y},t)\partial_{\tilde{y}}\overset{++}{\varphi}_{\Delta_N}\cdot A = -\partial_x\overset{++}{\varphi}_{\Delta_k} - (\rho^N(x,\tilde{y},t)\partial_{\tilde{y}} + \partial_y\rho^N(x,y,t))\times \tag{3.2}$$
$$\times\delta_{k1}\overset{++}{\varphi}_{\Delta_N}\cdot A - P(x,\tilde{y},t)\overset{++}{\varphi}_{\Delta_k} + (\overset{++}{\varphi}_{\Delta_N}\cdot\tilde{P}(x,y,t))_{\Delta_k} - (1-\delta_{k1})\overset{++}{\varphi}_{\Delta_{k-1}}P_\infty = 0 \quad (k=1,\ldots,N).$$

Solving the recurrent relations (3.2) with respect to $\overset{++}{\Phi}_{\Delta_k}$ we get

$$\overset{++}{\Phi}_{\Delta_k} = \sum_{m=0}^{N-k} \left(\mathcal{P}^{N-k-m} \overset{++}{\Phi}_{\Delta_N} \circ V_{N-m}\right)\left(A^T + P_\infty^T\right)^{N-k} \qquad (3.3)$$

where $(\mathcal{P} \circ V_m)_{ik} \overset{def}{=} \varphi_{ik} V_m$ and

$$\mathcal{P} \overset{def}{=} -\partial_x - P(x,\tilde{y},t) - A\rho^N(x,\tilde{y},t)\,\partial_{\tilde{y}}. \qquad (3.4)$$

Substitution of φ_{Δ_1} given by (3.3) into equation (3.2) for k = 1 gives the following relation which contains only φ_{Δ_N} :

$$\sum_{m=0}^{N} \left(\mathcal{P}^m \overset{++}{\Phi}_{\Delta_N} \circ V_m\right) = -\rho^N(x,\tilde{y},t)\,\partial_{\tilde{y}} \overset{++}{\Phi}_{\Delta_N} + \left(\rho^N(x,\tilde{y},t)\,\partial_{\tilde{y}} + \partial_y \rho^N(x,y,t)\right) \overset{++}{\Phi}_{\Delta_N} \qquad (3.5)$$

where $V_{N-1} = 0,\ V_N = 1$.

In virtue of (2.2) we have

$$\mathcal{P}^\ell = \tilde{\mathcal{P}}^\ell - \sum_{k_1+k_2=\ell-1} \tilde{\mathcal{P}}^{k_1} A\rho^N(x,\tilde{y},t)\, \tilde{\mathcal{P}}^{k_2}\,\partial_{\tilde{y}} = \tilde{\mathcal{P}}^\ell - r_\ell\,\partial_{\tilde{y}} \qquad (3.6)$$

where

$$\tilde{\mathcal{P}} \overset{def}{=} -\partial_x - P(x,\tilde{y},t)\ ,\ r_\ell \overset{def}{=} \sum_{k_1+k_2=\ell-1} \tilde{\mathcal{P}}^{k_1} A\rho^N(x,\tilde{y},t)\, \tilde{\mathcal{P}}^{k_2}. \qquad (3.7)$$

Let us introduce N-component columns

$$V \overset{def}{=} (V_0,\ldots,V_{N-2},0)^T,\ \chi \overset{def}{=} \left(\overset{++}{\Phi}_{1N},\ldots,\overset{++}{\Phi}_{NN}\right)^T = (\chi_1,\ldots,\chi_N)^T. \qquad (3.8)$$

Substituting (3.6) into (3.5) and using (3.8), one gets

$$\mathcal{G}\,\partial_{\tilde{y}}\,\chi(x,\tilde{y},y,t) = \mathcal{F}\chi(x,\tilde{y},y,t) \qquad (3.9)$$

where

$$\mathcal{G} \overset{def}{=} \sum_{\ell=0}^{N} r_\ell V_\ell - \rho^N(x,\tilde{y},t)\,1,\ \ \mathcal{F} \overset{def}{=} \sum_{m=0}^{N} \tilde{\mathcal{P}}^m V_m(x,y,t) - \rho^N(x,\tilde{y},t)\,\partial_{\tilde{y}} - \partial_y \rho^N(x,y,t). \qquad (3.10)$$

It follows from (3.10) that the operator $\mathcal{G}$ is a lower-triangular one: $\mathcal{G}_{ik} = 0$ for $k \geq i$ $(i,k = 1,\ldots,N)$. It follows also from (3.10) that the relation (3.9) allows us to express $\partial_{\tilde{y}}\chi(x,\tilde{y},y,t)$ through $\chi(x,\tilde{y},y,t)$. To do this we must take into account the degeneracy of the operator $\mathcal{G}$.

In virtue of this circumstance the first equation (3.9) is the constraint, i.e. the relation between the quantities $\chi_k(x,\tilde{y},y,t)$.

Using (3.10) and formulas

$$\begin{aligned}
&(\tilde{\mathcal{P}}^k)_{1\ell} = C_k^{\ell-1}(-\partial_x)^{k+1-\ell},\ \ell = 1,\ldots,k+1,\\
&(\tilde{\mathcal{P}}^k)_{1\ell} = 0,\ \ell = k+2,\ldots,N,\ (k = 1,\ldots,N-1); \qquad (3.11)\\
&(\tilde{\mathcal{P}}^N)_{1\ell} = C_N^{\ell-1}(-\partial_x)^{N+1-\ell} - V_{\ell-1}(x,\tilde{y},t),\ \ell = 1,\ldots,N,
\end{aligned}$$

we obtain this constraint in the form,

$$\sum_{k=1}^{N} \ell_k\,\chi_k(x,\tilde{y},y,t) = 0 \qquad (3.12)$$

where the operators ℓ_k act as follows

$$\begin{aligned}
\ell_k = {} & \delta_{k1}\left(\rho^N(x,\tilde{y},t)\,\partial_{\tilde{y}} + \partial_y \rho^N(x,y,t)\right) - V_{k-1}(x,y,t) + V_{k-1}(x,\tilde{y},t) - \qquad (3.13)\\
& - \sum_{n=1}^{N-k+1} C_{n+k-1}^{k-1}(-\partial_x)^n\, V_{n+k-1}(x,y,t).
\end{aligned}$$

Resolve the constraint (3.12) with respect to χ_N we obtain

$$\chi(x,\tilde{y},y,t) = M\,\chi_\Delta(x,\tilde{y},y,t) \qquad (3.14)$$

where $\chi_\Delta \overset{def}{=} (\chi_1,\ldots,\chi_{N-1},0)^T$

$$M_{ik} = \delta_{ik} - \delta_{iN}\,\ell_N^{-1}\ell_k\ \ (i,k = 1,\ldots,N) \qquad (3.15)$$

and $\ell_N = N\partial_x,\ \ell_N^{-1} = \frac{1}{N}\int_{+\infty}^{x}$. From (3.15) for the quantity $\partial_{\tilde{y}}^n M$ we have

$$\partial_{\tilde{y}}^n M = \sum_{p=0}^{n} O_{(n,p)}\,\partial_{\tilde{y}}^p \qquad (3.16)$$

where

$$\begin{aligned}
(O_{(n,p)})_{ik} = {} & M_{ik}\delta_{np} - \delta_{iN}\,\ell_N^{-1}\Big[(1-\delta_{p0})\,\delta_{k1}\,C_n^{p-1}\,\rho^N_{(n-p+1)} + \qquad (3.17)\\
& + (1-\delta_{pn})\,C_n^p\, V_{k-1\,(n-p)}\Big]
\end{aligned}$$

In general case $\omega_{kn} \neq 0$ ($k = 0,\dots,N-1$, $n = 0,1,2,\dots$) equations (4.5) at $\partial V_k/\partial y = 0$, $\partial \rho/\partial y = 0$ convert into nontrivial onedimensional equations, integrable by the problem (4.6)

There exist a subclass of equations (4.5) which have a special behaviour in the one-dimensional limit. This class of equations corresponds to $\omega_{0n} \neq 0$ and all the rest $\omega_{kn} = 0$. These equations are of the form

$$\frac{\partial V}{\partial t} - \hat{L}_1^{+} \frac{\partial \rho^N}{\partial t} u_{(0)} - \sum_{n=1}^{\infty} \omega_{0n}(t) \left[\mathcal{L}^{+}_{(0,n)} V - (\mathcal{M}^{+}_{(0,n)} + \mathcal{N}^{+}_{(0,n)})(\rho^N - 1) u_{(0)} \right] = 0. \tag{4.7}$$

Indeed, in virtue of $G_{(0)} = 0$, $\mathcal{F}_{(0)} = 1$ and $\rho^N_{\infty} = 0$, we have

$$\mathcal{L}^{+}_{(0,n)} = (-1)^n (1 - \delta_{n0}) \sum_{q=0}^{n-1} \sum_{r=0}^{q} C_n^q \hat{L}_r^{+} O^{+}_{(q,r)} \partial_y^{n-q}, \tag{4.8}$$

$$\mathcal{M}^{+}_{(0,n)} + \mathcal{N}^{+}_{(0,n)} = (-1)^n (1 - \delta_{n0}) \sum_{q=0}^{n-1} \sum_{r=0}^{q+1} C_n^q \hat{L}_r^{+} O^{+}_{(q+1,r)} \partial_y^{n-q}.$$

Therefore for $\frac{\partial V_k}{\partial y} = 0$, $\frac{\partial \rho}{\partial y} = 0$ the expression in the bracket in (4.7) is equal to zero.

Evolution law of the scattering matrix which corresponds to equations (4.7) is of the form

$$\frac{d \hat{S}(\mu,\lambda,t)}{d t} = (\Omega_0(\mu^N) - \Omega_0(\lambda^N)) \hat{S}(\mu,\lambda,t).$$

In the one dimensional limit $\partial V_k/\partial y = 0$, $\partial \rho/\partial y = 0$ equations (4.7) are

$$\frac{\partial V}{\partial t} - \hat{L}_1^{+} \frac{\partial \rho^N}{\partial t} u_{(0)} = 0. \tag{4.9}$$

Since in the onedimensional limit $\hat{S}(\mu,\lambda,t) = \delta_1(\mu - \lambda) S(\lambda,t)$ then evolution law of the scattering matrix for equations (4.9) is a trivial one: $\frac{d S(\lambda,t)}{d t} = 0$.

In the special case $\rho \equiv 1$ equations (4.7) are

$$\frac{\partial V}{\partial t} - \sum_{n=1}^{\infty} \omega_{0n}(t) \mathcal{L}^{+}_{(0,n)} V = 0 \tag{4.10}$$

which in the onedimensional limit $\frac{\partial V}{\partial y} = 0$ are reduced to the trivial equations $\frac{\partial V}{\partial t} = 0$.

So equations (4.10) are purely two-dimensional ones.

An example of equation (4.7) with $\omega_{01} \neq 0$, $\omega_{02} \neq 0$ and all the rest $\omega_{0n} = 0$ is

$$\frac{\partial V}{\partial t} - \hat{L}_1 \frac{\partial}{\partial t} (\rho^N - 1) u_{(0)} - \omega_{01} (-\partial_y V + \hat{L}_1^{+} \partial_y (\rho^N - 1) u_{(0)}) -$$
$$- \omega_{02} (\partial_y^2 V + 2 \hat{L}_1^{+} \partial_y V - \hat{L}_1^{+} \partial_y^2 (\rho^N - 1) u_{(0)} - 2 \hat{L}_2^{+} \partial_y (\rho^N - 1) u_{(0)}) = 0. \tag{4.11}$$

V. An example: N = 2

The system of equations (4.5) for $N = 2$ is of the form

$$\frac{\partial}{\partial t} \begin{pmatrix} V_0 \\ 0 \end{pmatrix} - \hat{L}_1^{+} \frac{\partial}{\partial t} \begin{pmatrix} \rho^2 \\ 0 \end{pmatrix} - \sum_{n=1}^{\infty} \omega_{0n}(t) \left[\mathcal{L}^{+}_{(0,n)} \begin{pmatrix} V_0 \\ 0 \end{pmatrix} - (\mathcal{M}^{+}_{(0,n)} + \mathcal{N}^{+}_{(0,n)}) \begin{pmatrix} \rho^2 - 1 \\ 0 \end{pmatrix} \right] -$$
$$- \sum_{n=0}^{\infty} \omega_{1n}(t) \left[\mathcal{L}^{+}_{(1,n)} \begin{pmatrix} V_0 \\ 0 \end{pmatrix} - \mathcal{M}^{+}_{(1,n)} \begin{pmatrix} \rho^2 - 1 \\ 0 \end{pmatrix} - \mathcal{N}^{+}_{(1,n)} \begin{pmatrix} 0 \\ \rho^2 - 1 \end{pmatrix} \right] = 0. \tag{5.1}$$

Second equation in (5.1) is an identity $0 \equiv 0$. Therefore system (5.1) is the one equation for two functions V_0 and ρ. Equations (5.1) is a two-dimensional generalization of the equations integrable by the onedimensional problem

$$\partial_x^2 \psi + V_0(x,t) \psi = \lambda^2 \rho^2(x,t) \psi \quad [3].$$

Here we present the quantities which we need for the calculations of the operators $\hat{L}_1^{+}$, $\mathcal{L}^{+}_{(k,n)}$, $\mathcal{M}^{+}_{(k,n)}$, $\mathcal{N}^{+}_{(k,n)}$. In virtue of (3.7) and (3.10) we get

$$\mathcal{G} = \begin{pmatrix} 0 & 0 \\ -\partial_x \rho^2(x,\tilde{y},t) - \rho^2(x,\tilde{y},t) \partial_x, & 0 \end{pmatrix} \tag{5.2}$$

$$\mathcal{F} = \begin{pmatrix} \partial_x^2 - \rho^2(x,\tilde{y},t) \partial_{\tilde{y}} - \partial_y \rho^2(x,y,t) + V_0(x,y,t), & -2\partial_x \\ \partial_x V_0(x,\tilde{y},t) + V_0(x,\tilde{y},t) \partial_x, & \partial_x^2 - \rho^2(x,\tilde{y},t) \partial_{\tilde{y}} - \partial_y \rho^2(x,y,t) + V_0(x,y,t) \end{pmatrix}. \tag{5.3}$$

For operator $\tilde{\mathcal{G}}$ such that $\tilde{\mathcal{G}}\psi = E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ we have $\tilde{\mathcal{G}}_{12} = -\frac{1}{2\rho(x,\tilde{y},t)}\partial_+^{-1}\frac{1}{\rho(x,\tilde{y},t)}$ and $\tilde{\mathcal{G}}_{11} = \tilde{\mathcal{G}}_{21} = \tilde{\mathcal{G}}_{22} = 0$. Operator $\tilde{\mathcal{G}}^+$ is of the form

$$\tilde{\mathcal{G}}^+ = \begin{pmatrix} 0 & 0 \\ \frac{1}{2\rho(x,\tilde{y},t)}\partial_-^{-1}\frac{1}{\rho(x,\tilde{y},t)} & 0 \end{pmatrix}. \tag{5.4}$$

The operators $l_k|_{\tilde{y}=y}$ $(k=1,2)$, due to (3.13), are

$$l_1 = -\partial_x^2 + \partial_y\rho^2, \quad l_2 = 2\partial_x, \quad l_2^{-1} = \frac{1}{2}\partial_+^{-1}, \quad \partial_\pm^{-1} \overset{def}{=} \int_{\pm\infty}^{x} \tag{5.5}$$

The operator $M|_{y=\tilde{y}}$ is (see (3.15))

$$M = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\partial_+^{-1}(\partial_x^2 - \partial_y\rho^2) & 0 \end{pmatrix} \tag{5.6}$$

Then from (3.17) and (3.20) follows that $O_{(0,0)} = M$, $\mathcal{F}_{(0,0)} = \mathcal{F}$. Formula (3.23) gives $\hat{L}_1 = \tilde{\mathcal{G}}\mathcal{F}_{(0,0)}O_{(0,0)}|_{\tilde{y}=y} = \tilde{\mathcal{G}}\mathcal{F}M|_{\tilde{y}=y}$. Substituting the expressions (5.4), (5.3), (5.6) for $\tilde{\mathcal{G}}, \mathcal{F}, M$ into this formula we obtain $\hat{L}_1$. For adjoint operator $\hat{L}_1^+$, we, as a result, have

$$\hat{L}_1^+ = \begin{pmatrix} -[\partial_x V_0 + V_0\partial_x + \frac{1}{2}(\partial_x^2+\rho^2\partial_y)\partial_-^{-1}(\partial_x^2+\rho^2\partial_y)\frac{1}{2\rho}\partial_-^{-1}\frac{1}{\rho}, & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.7}$$

Formulas (4.1) also give

$$G_{(0)} = 0, \quad \mathcal{F}_{(0)} = 1, \quad G_{(1)} = \begin{pmatrix} 0, & 0 \\ -\rho^2(x,\tilde{y},t), & 0 \end{pmatrix},$$
$$\mathcal{F}_{(1)} = \begin{pmatrix} -\partial_x & , -1 \\ -V_0(x,\tilde{y},t), & -\partial_x \end{pmatrix}. \tag{5.8}$$

Using the expressions (5.2)-(5.8) and recurrent relations (3.25) one can calculate by formulas (4.3), (4.4) all the operators $\mathcal{L}_{(k,n)}$, $\mathcal{M}^+_{(k,n)}$, and $\mathcal{N}^+_{(k,n)}$ and therefore one can calculate explicitly any concrete equation of the form (5.1).

Here we present some examples of equations (5.1).

Equation (4.11) is of the form

$$\frac{\partial}{\partial t}\begin{pmatrix} V_0 \\ 0 \end{pmatrix} - \hat{L}_1^+\frac{\partial}{\partial t}\begin{pmatrix} \rho^2 \\ 0 \end{pmatrix} - \omega_{01}\left[-\begin{pmatrix} V_{0y} \\ 0 \end{pmatrix} + \hat{L}_1^+\begin{pmatrix} (\rho^2)_y \\ 0 \end{pmatrix}\right] -$$
$$- \omega_{02}\left[\begin{pmatrix} V_{0yy} \\ 0 \end{pmatrix} + 2\hat{L}_1^+\begin{pmatrix} V_{0y} \\ 0 \end{pmatrix} - \hat{L}_1^+\begin{pmatrix} (\rho^2)_{yy} \\ 0 \end{pmatrix} - 2\hat{L}_2^+\begin{pmatrix} (\rho^2)_y \\ 0 \end{pmatrix}\right] = 0. \tag{5.9}$$

At $\rho \equiv 1$ equation (5.9) reduce to the equation

$$\frac{\partial V_0}{\partial t} + \omega_{01}\partial_y V_0 + \omega_{02}\left[\frac{1}{2}\partial_x^2\partial_y V_0 + \right.$$
$$\left. + 2V_0\partial_y V_0 + (\partial_x V_0)\partial_-^{-1}\partial_y V_0 + \frac{1}{2}\partial_-^{-1}\partial_-^{-1}\partial_y^3 V_0\right] = 0. \tag{5.10}$$

Equation (5.10) is the example of purely two-dimensional equation. Consider now another reduction of equation (5.9), namely $V_0 \equiv 0$. From (3.25) we have: $\hat{L}_2^+ = \mathcal{D}^+_{(1,0)} + \hat{L}_1^+\mathcal{D}^+_{(1,1)}$ where $\mathcal{D}^+_{(1,0)} = O^+_{(0,0)}\mathcal{F}^+_{(1,0)}\tilde{\mathcal{G}}^+|_{\tilde{y}=y} + O^+_{(1,0)}\mathcal{F}^+_{(1,1)}\tilde{\mathcal{G}}^+|_{\tilde{y}=y}$ and $\mathcal{D}^+_{(1,1)} = O^+_{(1,1)}\mathcal{F}^+_{(1,1)}\tilde{\mathcal{G}}^+|_{\tilde{y}=y}$. For $V_0 \equiv 0$ operator $\mathcal{D}^+_{(1,0)} = 0$ and for $\mathcal{D}^+_{(1,1)}$ one obtains

$$\mathcal{D}^+_{(1,1)} = \begin{pmatrix} -[\partial_x V_0 + V_0\partial_x + \partial_x(\rho^2)_y + (\rho^2)_y\partial_x + \frac{1}{2}(\partial_x^2+\rho^2\partial_y-(\rho^2)_y)\times & 0 \\ \times\partial_-^{-1}(\partial_x^2+\rho^2\partial_y-(\rho^2)_y)\frac{1}{2\rho}\partial_-^{-1}\frac{1}{\rho} & \\ 0 & 0 \end{pmatrix}. \tag{5.11}$$

Using (5.11) and the relation $\hat{L}_2^+|_{V_0\equiv 0} = \hat{L}_1^+\mathcal{D}^+_{(1,1)}|_{V_0\equiv 0}$ we obtain from equation (5.9) at $V_0 \equiv 0$ the following pure two-dimensional equation

$$\frac{\partial}{\partial t}\rho^2 + \omega_{01}(\rho^2)_y + \omega_{02}\left\{-(\rho^2)_{yy} + \left[\partial_x(\rho^2)_y + (\rho^2)_y\partial_x + \right.\right.$$
$$\left.\left. + \frac{1}{2}(\partial_x^2+\rho^2\partial_y-(\rho^2)_y)\partial_-^{-1}(\partial_x^2+\rho^2\partial_y-(\rho^2)_y)\right]\frac{1}{2\rho}\partial_-^{-1}\frac{(\rho^2)_y}{\rho}\right\} = 0. \tag{5.12}$$

The equations (5.1) for $\omega_{0n} = 0$, $\omega_{1n} \neq 0$ and $\rho \equiv 1$ have been considered in details in [16,18]. We will not describe they here. We only note that these equations contain Kadomtsev-Petviashvili (KP) equation and higher KP equations.

Let us consider now the equations (5.1) with $\omega_{0n} = 0$ and $\omega_{1n} \neq 0$. Under the reduction $V_0 \equiv 0$ they are of the form

$$\hat{L}_1^{+} \frac{\partial}{\partial t}\begin{pmatrix}\rho^2\\0\end{pmatrix} - \sum_{n=0}^{\infty} \omega_{1n}(t)\Big[\mathcal{M}^{+}_{(1,n)}\begin{pmatrix}\rho^2-1\\0\end{pmatrix} + \mathcal{N}^{+}_{(1,n)}\begin{pmatrix}0\\\rho^2-1\end{pmatrix}\Big] = 0 . \qquad (5.13)$$

All the operators in (5.13) are calculated at $V_0 \equiv 0$.

Using (3.17) and (3.20) one can show that at $V_0 \equiv 0$

$$\mathcal{M}^{+}_{(1,n)} = \hat{L}_1^{+}\widetilde{\mathcal{M}}^{+}_{(1,n)} , \quad \mathcal{N}^{+}_{(1,n)} = \hat{L}_1^{+}\widetilde{\mathcal{N}}^{+}_{(1,n)} . \qquad (5.14)$$

In virtue of (5.14) equation (5.13) is equivalent to the following one

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho^2\\0\end{pmatrix} - \sum_{n=0}^{\infty} \omega_{1n}(t)\Big[\widetilde{\mathcal{M}}^{+}_{(1,n)}\begin{pmatrix}\rho^2-1\\0\end{pmatrix} + \widetilde{\mathcal{N}}^{+}_{(1,n)}\begin{pmatrix}0\\\rho^2-1\end{pmatrix}\Big] = 0 . \qquad (5.15)$$

Equations (5.15) are the two-dimensional generalization of the Harry Dym's family of equations [3].

Consider the example of equation (5.15) for which $\omega_{10} \neq 0$, $\omega_{11} \neq 0$ and $\omega_{1n} = 0$ (n = 2,3,...). Using the expressions for the operators $\mathcal{M}^{+}_{(1,0)}$ and $\mathcal{N}^{+}_{(1,0)}$ we find $\widetilde{\mathcal{M}}^{+}_{(1,0)}\begin{pmatrix}\rho^2-1\\0\end{pmatrix} + \widetilde{\mathcal{N}}^{+}_{(1,0)}\begin{pmatrix}0\\\rho^2-1\end{pmatrix} = \begin{pmatrix}(\rho^2)_x\\0\end{pmatrix}$. Analogously, from (4.4), using (3.20), (3.25), (3.29) one can get

$$\widetilde{\mathcal{M}}^{+}_{(1,1)}\begin{pmatrix}\rho^2-1\\0\end{pmatrix} + \widetilde{\mathcal{N}}^{+}_{(1,1)}\begin{pmatrix}0\\\rho^2-1\end{pmatrix} = -\begin{pmatrix}\frac{1}{2}\partial_x^3\frac{1}{\rho} + \frac{3}{2}(\rho^2)_y\,\partial_-^{-1}\rho_y - \frac{3}{2}\rho^2\partial_-^{-1}\rho_{yy}\\0\end{pmatrix}.$$

Substituting the obtained expressions into (5.15) we obtain the following equation

$$\rho_t - \omega_{10}\rho_x + \omega_{11}\Big(\frac{1}{4\rho}\partial_x^3\frac{1}{\rho} - \frac{3}{4}\rho^4\Big(\frac{\partial_-^{-1}\rho_y}{\rho^2}\Big)_y\Big) = 0 . \qquad (5.16)$$

For variable $r \overset{def}{=} \frac{1}{\rho}$ equation (5.16) is of the form

$$r_t = \omega_{10}r_x + \frac{\omega_{11}}{4}\Big(r^3r_{xxx} + \frac{3}{r}\Big(r^2\partial_-^{-1}\Big(\frac{1}{r}\Big)_y\Big)_y\Big) . \qquad (5.17)$$

Then if one introduce the variable $\eta = r^{-2}$ equation (5.17) is rewritten as

$$\eta_t = \omega_{10}\eta_x + \frac{\omega_{11}}{4}\Big(-2(\eta^{-\frac{1}{2}})_{xxx} + 6\eta^2(\eta^{-1}\partial_-^{-1}(\sqrt{\eta})_y)_y\Big) . \qquad (5.18)$$

In the onedimensional limit $\eta_y = 0$ equation (5.18) (for $\omega_{10} = 0$, $\omega_{11} = 4$) is the well known Harry Dym equation [3] $\eta_t = -2(\eta^{-\frac{1}{2}})_{xxx}$. So equation (5.18) is the two-dimensional integrable generalization of Harry Dym equation.

VI. Examples: N = 3

The system of equations (4.5) for N = 3 has a form

$$\frac{\partial}{\partial t}\begin{pmatrix}V_0\\V_1\\0\end{pmatrix} - \hat{L}_1^{+}\frac{\partial}{\partial t}\begin{pmatrix}\rho^3\\0\\0\end{pmatrix} - \sum_{n=1}^{\infty}\omega_{0n}(t)\Big[\mathcal{L}^{+}_{(0,n)}\begin{pmatrix}V_0\\V_1\\0\end{pmatrix} - (\mathcal{M}^{+}_{(0,n)} + \mathcal{N}^{+}_{(0,n)})\begin{pmatrix}\rho^3-1\\0\\0\end{pmatrix}\Big] -$$

$$-\sum_{n=0}^{\infty}\omega_{1n}(t)\Big[\mathcal{L}^{+}_{(1,n)}\begin{pmatrix}V_0\\V_1\\0\end{pmatrix} - \mathcal{M}^{+}_{(1,n)}\begin{pmatrix}\rho^3-1\\0\\0\end{pmatrix} - \mathcal{N}^{+}_{(1,n)}\begin{pmatrix}0\\\rho^3-1\\0\end{pmatrix}\Big] - \qquad (6.1)$$

$$-\sum_{n=0}^{\infty}\omega_{2n}(t)\Big[\mathcal{L}^{+}_{(2,n)}\begin{pmatrix}V_0\\V_1\\0\end{pmatrix} - \mathcal{M}^{+}_{(2,n)}\begin{pmatrix}\rho^3-1\\0\\0\end{pmatrix} - \mathcal{N}^{+}_{(2,n)}\begin{pmatrix}0\\0\\\rho^3-1\end{pmatrix}\Big] = 0.$$

System (6.1) is the system of two equations for three functions V_0, V_1 and ρ.

The basic quantities which serve for calculation of the recursion operators are of the following form.

Operator $\mathcal{Y}$ is

$$\mathcal{Y} = \begin{pmatrix} 0 & 0 & 0\\ -2\partial_x\rho^3(x,\tilde{y},t) - \rho^3(x,\tilde{y},t)\partial_x & 0 & 0\\ \partial_x^2\rho^3(x,\tilde{y},t) + \partial_x\rho^3(x,\tilde{y},t)\partial_x + \rho^3(x,\tilde{y},t)\partial_x^2, & -2\rho^3(x,\tilde{y},t)\partial_x - \partial_x\rho^3(x,\tilde{y},t), & 0\end{pmatrix}. \qquad (6.2)$$

Operator $\widetilde{\mathcal{Y}}^{+}|_{\tilde{y}=y}$ is of the form

$$\tilde{\mathcal{G}}^{+}\Big|_{\tilde{y}=y}=\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3\rho}\partial_{-}^{-1}\frac{1}{\rho^{2}}\,, & -\frac{1}{9\rho}\partial_{-}^{-1}\frac{1}{\rho^{2}}(\partial_x^2\rho^3+\partial_x\rho^3\partial_x+\rho^3\partial_x^2)\frac{1}{\rho^2}\partial_{-}^{-1}\frac{1}{\rho}\,, & 0 \\ 0 & \frac{1}{3\rho^2}\partial_{-}^{-1}\frac{1}{\rho}\,, & 0 \end{pmatrix} \quad (6.3)$$

The operators $\ell_k|_{\tilde{y}=y}$ are

$$\ell_1=\partial_y\rho^3+\partial_x^3+\partial_x V_1\,,\quad \ell_2=-3\partial_x^2\,,\quad \ell_3=3\partial_x\,. \quad (6.4)$$

Using (3.15) for operator M^{+} one finds

$$M^{+}=\begin{pmatrix} 1\,, & 0\,, & -\frac{1}{3}(\partial_x^3+V_1\partial_x+\rho^3\partial_y)\partial_{-}^{-1} \\ 0 & 1\,, & -\partial_x \\ 0 & 0 & 0 \end{pmatrix}. \quad (6.5)$$

The operator $\mathcal{F}=\mathcal{F}_{(0,0)}$ has the following matrix elements

$$\mathcal{F}_{11}=-\partial_x^3-V_0(\tilde{y})+V_0(y)-\partial_x V_1(y)-\rho^3(\tilde{y})\partial_{\tilde{y}}-\partial_y\rho^3(y)\,,$$

$$\mathcal{F}_{12}=3\partial_x^2-V_1(\tilde{y})+V_1(y)\,,\quad \mathcal{F}_{13}=-3\partial_x\,,$$

$$\mathcal{F}_{21}=2\partial_x V_0(\tilde{y})+V_0(\tilde{y})\partial_x\,,\quad \mathcal{F}_{23}=3\partial_x^2-V_1(\tilde{y})+V_1(y)\,,$$

$$\mathcal{F}_{22}=-\partial_x^3-V_0(\tilde{y})+V_0(y)+2\partial_x V_1(\tilde{y})+V_1(\tilde{y})\partial_x-\partial_x V_1(y)-\rho^3(\tilde{y})\partial_{\tilde{y}}-\partial_y\rho^3(y)\,, \quad (6.6)$$

$$\mathcal{F}_{31}=-V_0(\tilde{y})\partial_x^2-\partial_x V_0(\tilde{y})\partial_x-\partial_x^2 V_0(\tilde{y})+V_1(\tilde{y})V_0(\tilde{y})-V_0(\tilde{y})V_1(y)\,,$$

$$\mathcal{F}_{32}=-V_1(\tilde{y})\partial_x^2-\partial_x V_1(\tilde{y})\partial_x-\partial_x^2 V_1(\tilde{y})+2V_0(\tilde{y})\partial_x+\partial_x V_0(\tilde{y})+V_1^2(\tilde{y})-V_1(\tilde{y})V_1(y)\,,$$

$$\mathcal{F}_{33}=-\partial_x^3+2V_1(\tilde{y})\partial_x+\partial_x V_1(\tilde{y})-\partial_x V_1(y)-V_0(\tilde{y})+V_0(y)-\rho^3(\tilde{y})\partial_{\tilde{y}}-\partial_y\rho^3(y)\,.$$

Using (6.5 and (6.6) one can show that nonzero elements of the operator $M^{+}\mathcal{F}^{+}|_{\tilde{y}=y}$ are:

$$(M^{+}\mathcal{F}^{+})_{12}=-\rho^3\partial_x\partial_y-\partial_x^4-V_1\partial_x^2-2V_0\partial_x-\partial_x V_0\,,$$

$$(M^{+}\mathcal{F}^{+})_{13}=-\frac{1}{3}\partial_x^2\rho^3\partial_y-\frac{1}{3}V_1\rho^3\partial_y-\frac{1}{3}\rho^3\partial_y\partial_x^2-\frac{1}{3}\rho^3\partial_{-}^{-1}\partial_y^2+$$
$$+\frac{2}{3}\rho^3\partial_y V_1-\frac{1}{3}\partial_x^5+\frac{2}{3}\partial_x^3 V_1-\frac{1}{3}V_1\partial_x^3+\frac{2}{3}V_1\partial_x V_1-V_0\partial_x^2-\partial_x V_0\partial_x-\partial_x^2 V_0\,, \quad (6.7)$$

$$(M^{+}\mathcal{F}^{+})_{22}=\rho^3\partial_y-2\partial_x^3-\partial_x V_1-V_1\partial_x\,,$$

$$(M^{+}\mathcal{F}^{+})_{23}=-\partial_x\rho^3\partial_y-\partial_x^4-2\partial_x V_0-V_0\partial_x-\partial_x^2 V_1-\partial_x V_1\partial_x-V_1\partial_x^2\,.$$

The operator $\mathcal{L}_1^{+}$ is

$$\mathcal{L}_1^{+}=M^{+}\mathcal{F}^{+}\tilde{\mathcal{G}}^{+} \quad (6.8)$$

and it can be calculated explicitly with the use of (6.3),(6.7).

In virtue of (4.1) the operators $G_{(k)}$ and $\mathcal{F}_{(k)}$ are of the form

$$G_{(0)}\equiv 0,\; G_{(1)}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\rho^3(\tilde{y}) & 0 & 0 \end{pmatrix},\; G_{(2)}=\begin{pmatrix} 0 & 0 & 0 \\ -\rho^3(\tilde{y}) & 0 & 0 \\ \rho^3(\tilde{y})\partial_x+\partial_x\rho^3(\tilde{y})\,, & -\rho^3(\tilde{y})\,, & 0 \end{pmatrix}, \quad (6.9)$$

$$\mathcal{F}_{(0)}\equiv 1,\; \mathcal{F}_{(1)}=\begin{pmatrix} -\partial_x\,, & 1\,, & 0 \\ 0\,, & -\partial_x\,, & 1 \\ -V_0(\tilde{y})\,, & -V_1(\tilde{y})\,, & -\partial_x \end{pmatrix},\; \mathcal{F}_{(2)}=\begin{pmatrix} \partial_x^2+V_1(y)\,, & -2\partial_x\,, & 1 \\ -V_0(y)\,, & \partial_x^2+V_1(y)-V_1(\tilde{y})\,, & -2\partial_x \\ V_0(\tilde{y})\partial_x+\partial_x V_0(\tilde{y})\,, & -V_0(\tilde{y})+V_1(\tilde{y})\partial_x+\partial_x V_1(\tilde{y})\,, & \partial_x^2-V_1(\tilde{y})+V_1(y) \end{pmatrix} \quad (6.10)$$

Using (6.2), (6.6), (6.9) and (6.10) and recurrent relations (3.25) one can calculate all the recursion operators $\hat{L}_n^+$, $\mathcal{L}^+_{(k,n)}$, $\mathcal{M}^+_{(k,n)}$, $N^+_{(k,n)}$ and therefore one can obtain the explicit form of equations (6.1).

The reduction problem for equation (6.1) is much more complicated problem in comparison with equation (5.1).

Here we present one example of the deep reduction $V_0 = V_1 = 0$. As we will see this reduction is possible for $\omega_{21} \neq 0$ and all other $\omega_{kn} = 0$.

Using (4.3), (4.4), (3.25), (3.26) and (6.2)-(6.10) we have at $V_0 = V_1 = 0,\ \omega_{21} \neq 0$:

$$\hat{L}_1^+ \frac{\partial}{\partial t}\begin{pmatrix}\rho^3\\0\\0\end{pmatrix} + \omega_{21}(t)\hat{L}_1^+\left\{O^+_{(1,1)}\mathcal{F}^+_{(1,1)}\tilde{\mathcal{Y}}^+\begin{pmatrix}(\rho^3)_{xx}\\2(\rho^3)_x\\0\end{pmatrix} + O^+_{(2,1)}\begin{pmatrix}(\rho^3)_{xxy}\\2(\rho^3)_{xy}\\(\rho^3)_y\end{pmatrix}\right\} = 0. \quad (6.11)$$

From formulas (3.20), (3.26) we get

$$O^+_{(1,1)}\Big|_{\substack{V=0\\ \tilde{y}=y}} = \begin{pmatrix} 1, & 0, & -\frac{1}{3}\partial_x^2 - \frac{1}{3}\rho^3\partial_y\partial_-^{-1} + \frac{1}{3}(\rho^3)_y\partial_-^{-1}\\ 0 & 1 & -\partial_x\\ 0 & 0 & 0\end{pmatrix}, \quad (6.12)$$

$$\mathcal{F}^+_{(1,1)}\Big|_{\substack{V=0\\ \tilde{y}=y}} = \begin{pmatrix} \partial_x^3 + \rho^3\partial_y - (\rho^3)_y, & -2(\rho^3)_y\partial_x - \partial_x\rho^3(y), & -\partial_x^2(\rho^3)_y - \partial_x(\rho^3)_y\partial_x - (\rho^3)_y\partial_x^2\\ 3\partial_x^2, & \partial_x^3 + \rho^3\partial_y - (\rho^3)_y, & -2\partial_x(\rho^3)_y - (\rho^3)_y\partial_x\\ 3\partial_x, & 3\partial_x^2, & \partial_x^3 + \rho^3\partial_y - (\rho^3)_y\end{pmatrix}. \quad (6.13)$$

The use of (6.3), (6.12), (6.13) gives

$$O^+_{(1,1)}\mathcal{F}^+_{(1,1)}\tilde{\mathcal{Y}}^+\begin{pmatrix}(\rho^3)_{xx}\\2(\rho^3)_x\\0\end{pmatrix} + O^+_{(2,1)}\begin{pmatrix}(\rho^3)_{xxy}\\2(\rho^3)_{xy}\\(\rho^3)_y\end{pmatrix} = \begin{pmatrix}w\\0\\0\end{pmatrix} \quad (6.14)$$

where

$$w = -\frac{1}{6}\left(\frac{1}{\rho^2}\right)_{xxxxx} - \frac{5}{3}\rho^6\left(\frac{\partial_-^{-1}\rho_y}{\rho^3}\right)_y - \frac{1}{6}\rho^3\left(\frac{1}{\rho}\right)_{xxy} + \frac{4}{3}\rho_{xxy} + \frac{2}{3}(\rho^3)_y\left(\frac{1}{\rho^2}\right)_{xx} + \frac{1}{2}(\rho^3)_{xy}\left(\frac{1}{\rho^2}\right)_x .$$

The formula (6.14) just leads to the conclusion that equation (6.1) admit the reduction $V_0 = V_1 = 0$ for $\omega_{21} \neq 0$ and other $\omega_{kn} = 0$.

Using (6.14) we obtain from (6.11) (at $\omega_{21} = -6$) the following equation for new variable $r \overset{def}{=} \rho^{-2}$:

$$r_t = r^{\frac{5}{2}}r_{xxxxx} + 10r^{-\frac{1}{2}}\left(r^{\frac{3}{2}}\partial_-^{-1}(r^{-\frac{1}{2}})_y\right)_y + 5rr_{xxy} - \frac{15}{2}r_{xy}r_x + \frac{15}{4}r_x^2r_yr^{-1}. \quad (6.15)$$

For variable $\eta \overset{def}{=} r^{-\frac{3}{2}}$ equation (6.15) is of the form

$$\eta_t = -\frac{3}{2}\left(\eta^{-\frac{2}{3}}\right)_{xxxxx} - 15\eta^2\left(\eta^{-1}\partial_-^{-1}(\eta^{1/3})_y\right)_y - \frac{15}{2}\left(\eta^{-\frac{2}{3}}\right)_{xxy} - \frac{15}{2}\eta_x\left(\eta^{-\frac{2}{3}}\right)_{xy} + \frac{5}{3}\eta^{-\frac{8}{3}}\eta_y(\eta_x)^2. \quad (6.16)$$

In the onedimensional limit equations (6.15) and (6.16) are

$$r_t = r^{5/2}r_{xxxxx}, \qquad \eta_t = -\frac{3}{2}\left(\eta^{-3/2}\right)_{xxxxx} \quad (6.17)$$

that are differ from the first higher equation from Harry Dym family.

In conclusion we give an example of pure twodimensional equation (4.11). It is of the form

$$\frac{\partial}{\partial t}\begin{pmatrix}V_0\\V_1\\0\end{pmatrix} - \hat{L}_1^+\frac{\partial}{\partial t}\begin{pmatrix}\rho^3\\0\\0\end{pmatrix} - \omega_{01}\left[-\partial_y\begin{pmatrix}V_0\\V_1\\0\end{pmatrix} + \hat{L}_1^+\begin{pmatrix}(\rho^3)_y\\0\\0\end{pmatrix}\right] - \quad (6.18)$$

$$-\omega_{02}\Big[\begin{pmatrix} V_{0yy} \\ V_{1yy} \\ 0 \end{pmatrix} + 2\hat{L}_1^+\begin{pmatrix} V_{0y} \\ V_{1y} \\ 0 \end{pmatrix} - \hat{L}_1^+\begin{pmatrix} (\rho^3)_{yy} \\ 0 \\ 0 \end{pmatrix} - 2\hat{L}_2^+\begin{pmatrix} (\rho^3)_y \\ 0 \\ 0 \end{pmatrix}\Big] = 0.$$

For $\rho^3 = 1$ the system (6.18) is reduced to

$$\frac{\partial}{\partial t}\begin{pmatrix} V_0 \\ V_1 \\ 0 \end{pmatrix} + \omega_{01}\begin{pmatrix} V_{0y} \\ V_{1y} \\ 0 \end{pmatrix} - \omega_{02}\Big[\begin{pmatrix} V_{0yy} \\ V_{1yy} \\ 0 \end{pmatrix} + 2\hat{L}_1^+\begin{pmatrix} V_{0y} \\ V_{1y} \\ 0 \end{pmatrix}\Big] = 0. \qquad (6.19)$$

Using the formula (6.8) one can easily obtain the explicit form of these equations.

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ОБЩАЯ СТРУКТУРА НЕЛИНЕЙНЫХ УРАВНЕНИЙ В I+2 ИЗМЕРЕНИЯХ, ИНТЕГРИРУЕМЫХ ОБОБЩЕННОЙ ДВУМЕРНОЙ ДИФФЕРЕНЦИАЛЬНОЙ СПЕКТРАЛЬНОЙ ЗАДАЧЕЙ

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