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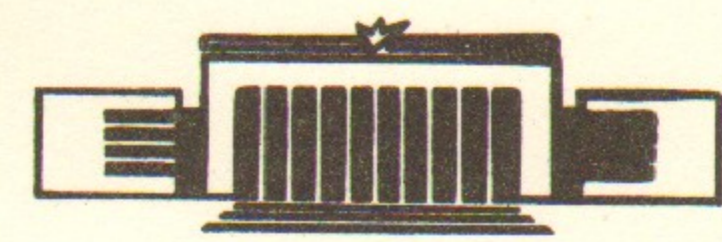
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THEORY AND PHENOMENOLOGY OF THE QCD VACUUM

3. QCD ON THE LATTICE

НОВОСИБИРСК

A B S T R A C T

This preprint contains discussion of the Euclidean formulation of gauge theories on the four-dimentional lattice. In order to make the presentation more transparant we start with the simplest example of quantum mechanical system, namely a particle in time-independent one-dimentional potential (see section 3.1) and explain how numerical methods currently used work in this case (see section 3.2). Only then we pass to gauge theories on the lattice and review some results obtained numerically, see sections 3.3 and 3.4. Finally we turn to complications caused by the introduction of the quark fields.

3. QCD ON THE LATTICE

The formulation and recent development of lattice gauge theories have demonstrated once more how fruitful some well known method may happen to be in some completely new framework. Most of the ideas used in the considerations of lattice gauge theories originate from the theory of critical phenomena in statistical mechanics, especially of the spin systems. As far as these questions are discussed in details in reviews [3.1—3.5], we consider them very briefly in section 3.1. We use the simplest example of the quantum system, namely one particle in time-independent one-dimensional potential, for which the statistical analog system in lattice approximation can also be formulated.

Returning to QCD, one may say that lattice formulation has revealed completely new aspects of this theory, connected with strong coupling expansion, Wilson confinement on the lattice etc. However, the real turning point was the discovery of «early scaling» phenomenon, due to which numerical simulations turn out to be very practical method of investigations of complicated quantum field theory problems. Many theorists put aside other methods and took places at the computer terminals.

Now few years have passed and, naturally, some primary enthusiasm has disappeared. It is evident that this method is very powerful indeed, for it allows to deal with quantum systems of various types with thousends degrees of freedom! However, it became clear that straightforward applications to QCD problems is not very practical at the moment due to the complications connected with the presence of light quarks. In order to make adequate account for virtual quarks in the QCD vacuum one needs new ideas (and, possibly, new computers).

However, there is a lot of other problems in physics, being much simpler than those connected with the QCD vacuum, which can be effectively solved by numerical simulations in similar («lattice») approximation. They include different cases in which we have quantum system with several nonseparable variables—traditional methods based on Schredinger equation are in this case very inadequate. With this in mind, we present some introductory consideration of the method in section 3.2 using the simplest quantum mechanical examples.

Only after demonstration of the method performance in the simplest cases we come to gauge theories in section 3.3. The original idea of lattice formulation is to regularize the theory in an

exactly gauge invariant way. The next section 3.4 contains some review of particular results obtained by numerical methods for pure gauge systems, while the last section 3.5 deals with quarks on the lattice. Recent attempts to make some spectroscopic calculations in QCD are postponed till section 5.1.

Finally, I would like to make few remarks concerning some psychological and organizational changes in our science connected with wide application of numerical methods. The theorists now not only have mastered the numerical methods developed previously, but also actively participate in their development. This activity is now recognized as a part of nowdays theoretical physics, as in last century the development of analytical methods for the solution of the «mathematical physics» equations. It seems that this process of «computerization» is irreversible.

In principle, it reflects real situation. Many complicated problems can now be solved starting from first principles, provided suitable numerical method is developed. Moreover, in many cases this is the only possible approach. However, doing such «numerical experiments» one should keep in mind that their final aim is not only the needed numbers, but also deeper understanding of the phenomenon. This is especially true for the QCD vacuum problem, for now we make only first steps toward its understanding and are not able to answer even simplest qualitative questions.

There appear also important organizational changes, say rather large collaborations, previously typical for experimental works. Even in this respect theorists are going to have some leading position: there are rumors about «grand unification» of these groups into some unprecedent world-wide collaboration. And also, one should pay attention to new progressive series of «Nuclear physics» called «Field theory and statistical systems».

3.1. Lattice approximation in quantum mechanics

In this section we discuss some interesting correspondence between the problems of quantum and statistical mechanics. In very general terms their connections reflect the undeterministic, statistical nature of quantities considered, being a subject for quantum or thermal fluctuations. Of course, there are also more formal relations between them, allowing for fruitful information exchange.

Evidently, these two disciplines consider completely different kind of problems. In quantum mechanics one is interested in time development of some dynamical process. The calculated quantity is the probability amplitude for transition between definite initial and final states. In statistical mechanics time independent (equilibrium) configurations are usually considered, and one looks for the probability of some definite state in the ensemble. However, it is possible in some cases to reformulate the problem from one language to another, with mutual benefit.

It is reasonable to start with the simplest example of quantum system, being one particle in time-independent one-dimensional potential $V(x)$. In section 2.1 we have already considered the first necessary step—namely, transition from real time t_0 to imaginary one τ_0. As a result, the evolution operator $\exp(iHt_0)$ is substituted by $\exp(-H\tau_0)$, similar to statistical Boltsmann factor. The second step, necessary for numerical calculations, is the «lattice» approximation, in which the imaginary time τ_0 is split into $N+1$ steps of length a, so $\tau_0 = (N+1) \cdot a$. Thus, the time development of our system (or the particle path) can be parametrized by N numbers $x_k = x(\tau_k)$, $\tau_k = ak$ (the end points are so far fixed). Now our problem can be reformulated as a statmechanical problem of some analog system, being in this case the one-dimensional lattice with «spins» x_k at the sites, being arbitrary real numbers. The Hamiltonian of this statistical system can be written as follows

$$H = \sum_{k=0}^{N} \left[\frac{m}{a} (x_{k+1} - x_k)^2 + V(x_k)a \right] \tag{3.1}$$

where we have approximated time derivatives by simple difference of neighbouring «spins». Of course, one may use another approximation for the derivative which is more precise, but in this case the Hamiltonian (3.1) is substituted by some other and more complicated expression. The temperature of our statistical analog system is put to unity (for the original quantum problem it is, in fact, the Plank constant).

It is quite evident by the construction that at small a limit all such statistical analogs should produce essentially the same result. However, if one just starts with these systems without reference to quantum mechanics, such universality is not at all evident. This is an example of the benefit for statmechanical problems, resulting from such analogies.

Now, let us demonstrate their utility in other direction. It is intuitively clear that for long enough lattice «spins» forget the boundary conditions and form some equilibrium distribution over x.

In particular, the energy density per unite time τ_0 tends to some limit. The correlations between «spins» decay exponentially with distance, so some well defined correlation length τ_{cor} can be defined. What all this means in terms of the quantum problem?

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Long time propagation amplitude really factorizes as follows

$$G(x_i, x_f, \tau_0) \xrightarrow[\tau_0 \to \infty]{} \Psi_0^*(x_i)\Psi_0(x_f) \exp(-E_0\tau_0) \qquad (3.2)$$

where Ψ_0 and E_0 are the wave function and energy of the ground state. The energy density per unite τ_0, as it becomes clear from (3.2) is just E_0. The correlation length is the «mass gap», or the distance between the ground and first excited state . This statement was demonstrated in section 2.1, also by straightforward expansion over the stationary states of the quantum problem.

The next comment is that not only «the long lattice» provides valuable information: the finite one is also of great interest. Let us choose periodic boundary conditions $x_i = x_f = x$ and integrate over x. Using the normalization condition of the wave function one finds

$$\int G(x, x, \tau_0)\, dx = \sum_n \exp(-E_n\tau_0) \qquad (3.3)$$

which is the statistical sum for the quantum system at temperature $T = \hbar/\tau_0$. Therefore, finite lattice with the periodic boundary condition provides an information concerning quantum system at nonzero temperature.

The last remark in this section deals with the limit $a \to 0$, in which lattice is substituted by continuum case of interest. Clear that in real calculation a should be small compared to typical time scale of the problem. However, in terms of statistical analog systems this natural condition looks rather nontrivial: we are interested in special case in which the correlation length is much larger than the lattice spacing a! In statistical systems such long-range fluctuations take place only near phase transitions (remember the classical example of critical opalescence). It is near phase transitions where system behaviour becomes independent on the details of the Hamiltonian at the «atomic» scale a. The last statement is highly nontrivial, its understanding was a great achievement of the theory (say, it was marked by Nobel prize for K.Wilson).

In the case of field theory in four-dimensional space-time the statistical analog systems are usually formulated on

four-dimensional lattices (although one may also use the so called Hamiltonian formulation [3.18] with continuous time) and study their phase transitions. As soon as they are found, the continuum limit may be understood as some quantum field theory.

3.2. Numerical simulation of configuration ensemble

Feynman formulation of quantum field theory based on functional integrals is very elegant and convenient for derivation of various general formulae and therefore it is widely used in field theory framework. However, only Gaussian functional integrals can be calculate analytically, which make their applications rather restricted.

Recently it was realised that it can also be a good framework for application of powerful numerical methods, which are in principle applicable for the integrals of general type. It is probably useful to start with the explanation why it was not noticed much earlier. The main question can well be posed in the framework of the simplest statistical analog system, discussed in the preceeding section. How many «spins» should be taken in order to have good description of the ground state wave function?

There are two main conditions. First the typical time scale of the problem (the correlation length) should be much larger than lattice spacing

$$\tau_{cor} \gg a$$

The second necessary condition is that the whole length of the lattice should be much larger than the correlation length

$$a \cdot (N+1) = \tau_0 \gg \tau_{cor}$$

with two inequalities of such type one can see that it is hardly possible to obtain good results with N less than, say, few hundreds. Now, the statistical sum is taken over all «spins» x_k and therefore we have to compute the integral over all of them. However, it can easily be checked that all «simple minded» methods of integration fail in the case of so many variables. For example, the ordinary Monte-Carlo method of «random argument» works up to $O(10)$ variables, but for $O(100)$ ones most of the points happen to be in the inessential integration region and therefore application of this simple method is hopeless. Thus, one

should select points not completely randomly, but making some importance sampling.

In the computer mathematics some suitable methods of such selective type are known, which make such problem solvable up to about million of variables at existing computers. Although they were suggested about 30 years ago (and, in fact, by physicists [3.42]), they were not used for evaluation of functional integrals untill recent pionear works by M. Creutz. I think the most obvious explanation for this fact is that they were simply unknown for most of the theorists occupied by quantum field theory problems. Now the situation in this respect is improved, but still it is probably reasonable to explain main ideas of these algorithms, for they are rather simple. And also, as it was mentioned in the introduction of this section, the method under consideration may have much wider field of applications.

The first point is that instead of direct evaluation of the integral we generate the ensemble of the points distributed according to some probability function $W(x)$. With such ensemble average values of some functions $f(x)$

$$\langle f \rangle \equiv \frac{\int f(x) W(x)\, dx}{\int W(x)\, dx} \qquad (3.4)$$

can be calculated as simple arithmetic average value over the ensemble. If $W(x)$ is much more complicated than $f(x)$, this approach is reasonable.

For one variable x this problem is solved by simple method suggested at early days of computer mathematics. Let us take a point on the plane (x,y) where x and $W(x)$ are defined with homogeneous weight. If $y < W(x)$, the point is below the curve, it is taken into ensemble, otherwise it is rejected. It is however evident that for many variables this method is inadequate due to the same reason as simple Monte-Carlo integration: too large fraction of points are rejected.

Now we come to the main idea of Metropolis et al. [3.42]: the new trial point is taken by some small shift of the previouse one, after which the comparison between some auxiliary random variable y and $W(x)$ is made as explained above. As a result, the point never goes too far from the essential region where $W(x)$ is large, therefore the method is very effective. It is clear that the optimal variation of x is such that the weight is changed by the order of unity.

8

For spin systems (remember the analog system of the preceeding section) the method «updates» each spin along the lattice in some order, one after another. The updating of all variables is called the «iteration» of the system. Calculations start with some arbitrary configuration, than there is «relaxation» process in which all averages tend toward their equilibrium values. When they are reasonably stable, the «measurements» are made, consisting in multiple updating and evaluation of the needed average values (or recording of configurations) from time to time. The process of ensemble generation is finished when the statistics needed is collected.

Another method used is the heat bath method, in which new value of some variable x_k is found not from the old one, but directly inverting the weight function at fixed other variables. It produces relaxation per smaller amount of iterations, but leads to more complicated calculations and in most cases Metropolis method is faster. Last years Langevin equation is also much discussed [3.50—3.52], which corresponds to Metropolis method in the limit of very small displacements of the point x, so that its stochastic trajectory in «computer time» t can be described by the differential equation

$$\frac{dx_k}{dt} = -\frac{\partial H}{\partial x_k} + \eta_k \qquad (3.5)$$

Here H is the Hamiltonian and η_k random variable with Gaussian distribution and the correlator

$$\langle \eta_k(t) \eta_{k'}(t') \rangle = 2\delta_{kk'}\delta(t-t') \qquad (3.6)$$

The standard transition to Fokker-Plank equation for the probability distribution leads to

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2 P(x,t)}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x} P(x,t)\right) \qquad (3.7)$$

It is easy to see that equilibrium distribution is indeed $\exp(-H(x))$. With smaller step the relaxation becomes longer, but it produces more accurate correlators, see discussion of this idea in section 4.4. Also analytic approach based on Langevin equation is now developing, the so called stochastic quantization [1.8].

Now we turn to some examples of how this method works. Field theory is postponed, and we start with simplest quantum mechanical problems. First examples of this type were linear and two-well nonlinear oscillators considered by Freedman and Creutz [3.44], and some set of model examples of increasing complexity

9

(up to two electron atoms and four nucleon nuclei) was discussed by Zhirov and myself in [3.45].

At Fig.1 we show equilibrium x distribution for the two-well oscillator [3.44] compared to $|\Psi_0(x)|^2$. In order to get such result one needs about 200 points and few hundreds of iterations: even for rather modest computers it takes less than a minute of CPU.

The second example deals with the operator average values, in particular, with the average energy. Somewhat unexpected difficulty is the fact that «naive» kinetic energy diverges at $a \to 0$. Really, coordinate displacement per time a is $(a/m)^{1/2}$ and therefore

$$\langle \dot{x}^2 \rangle \simeq \langle \Delta x^2 \rangle / a^2 \sim 1/am \to \infty \tag{3.8}$$

So one may either use «renormalized» definition [3.44]

$$\langle \dot{x}^2 \rangle = -\frac{\langle (x_{k+1} - x_k)(x_k - x_{k-1}) \rangle}{a^2} \tag{3.9}$$

or, which is much better, to use the virial theorem

$$E_0 = \lim_{\tau_0 \to \infty} \left\{ \frac{a}{\tau_0} \sum_k \left[\frac{x_k}{2} \frac{\partial V}{\partial x_k} + V(x_k) \right] \right\} \tag{3.10}$$

The ground state energy of the two-well oscillator, obtained in such way, is shown at Fig.2. The action has one parameter f

$$S = \int_0^{\tau_0} d\tau \left[\frac{\dot{x}^2}{4} + (x^2 - f^2)^2 \right] \tag{3.11}$$

and we plot E as a function of $(4/3) \cdot f^3$, the action of the instanton solution. We have found completely different dependence on the lattice spacing a in case of narrow and wide barriers, see Fig.3 where data are shown for $f = 1.4$ (transparent barrier) and $f = 1.6$ (wide barrier). Completely wrong results at large step a in case of wide barrier is connected with the specific phenomenon, «lattice instantons», known in more complicated framework as well. Due to discrete time it is possible to jump over the potential barrier per one step, so that no point is under it. Evidently, action is in this case evaluated incorrectly and tunneling is not suppressed. In order to overcome this phenomenon we have used the modified action in which $[V(x_k) + V(x_{k+1})]/2$ is changed to the integral over straight line

$$\int_0^a d\tau \, V\left(x = x_k + \frac{\tau}{a}(x_{k+1} - x_k) \right) \tag{3.12}$$

In between: this is sufficient to obtain much better results. Such simple tricks can also be very important for field theory applications, where it is not possible technically to get rid of such complication «by brut force», or very small a. The last example of numerical calculations in quantum mechanics is shown at Fig.4. They correspond to measured correlation function

$$K(\tau) = \langle x(\tau) x(0) \rangle \tag{3.13}$$

in the two-well oscillator [3.45]. The exponential behaviour at large enough time is seen, and from its slope the splitting of two first states is evaluated, see Fig.1 in chapter 2.

Finally, some comments on other possible applications. The ground state energies of simplest atoms and nuclei can easily be evaluated by this method [3.45]. We have also observed particle evaporation from these systems if τ_0 was not large enough (the temperature $T = 1/\tau_0$ sufficiently large), which is also of some interest for applications. Method is also adequate for different problems in which one is interested in tunneling through some multidimensional barrier—say in evaluation of the probability for chemical reaction of some complicated molecules. In this case it is also very helpful that nonzero temperature can be included in simple way, while in traditional approach one has to compute energy spectra and make explicit evaluation of statistical sums. Among difficulties of this approach we may mention account for identical Fermions: the amplitudes should be antisymmetrized, but statistical methods do not work for nonpositive weight functions.

3.3. Gauge fields on the lattice

In this section we consider formulation of the quantum gauge field theory on the lattice in its simplest form suggested by K. Wilson [3.4] (other possibilities are discussed in Refs [3.10—3.18]). It is defined on the oriented links of the cubic four dimensional lattice, which are usually indicated as (x, μ) where x is the initial point and $\mu = \pm 1,2,3,4$ gives the direction. Note that $(x_\mu + ae_\mu, -\mu)$ has the orientation opposite to (x, μ). We also will need elementary squares or plaquettes, numerated as (x, μ, ν).

Definition of the gauge field variables directly follows from the original idea of Yang and Mills, the principle of local gauge invariance. According to it, the direction of the axes in internal space at each lattice site is completely arbitrary. Derivatives in such theory enter only togather with vector «compensating fields» in covariant combinations

$$iD_\mu = i\partial_\mu + \frac{g}{2} A_\mu^a t^a \qquad (3.14)$$

which have simpler transformation law according to gauge tranformations. This consideration explains why gauge fields are defined on the links rather than sites, as «matter» fields.

It is also clear that the needed variables should describe relative orientation of axis in two neighbouring sites and be therefore transformed as

$$U'_{x,\mu} = \Omega^{-1}(x_\mu) U_{x,\mu} \Omega(x_\mu + ae_\mu) \qquad (3.15)$$

at gauge transformation $\Omega(x)$. For small a the relation to potential is simple

$$U_{x,\mu} = 1 + \frac{ig}{2} A_\mu^a t^a a + O(a^2) \qquad (3.16)$$

while for finite length it is the so called path-ordered exponent

$$U(c) = \mathrm{Pexp}\left(\frac{ig}{2} \int_c A_\mu^a t^a \, dx_\mu\right) = \lim_{N \to \infty} \prod_{i=1}^{N} \left(1 + \frac{ig}{2} A_\mu^a(x_i) t^a \Delta_i x\right) \qquad (3.17)$$

which depends on the path and has the transformation law (3.15). (By the way, interesting attempts to formulate gauge field theory in terms of such «contour variables» are discussed in [1.46]).

As it follows from (3.14), for the formulation of the gauge invariant action one needs closed contours. The minimal possibility is the elementary plaquette, so we come to Wilson action

$$S(U) = \frac{1}{g^2} \sum_\square \left[1 - \frac{1}{N_c} \mathrm{Re} \, \mathrm{Tr} \, (UUU^+U^+)\right] \qquad (3.18)$$

in which summation is made over the all plaquettes on the lattice. Respectively, statistical sum contains the integration over all $U_{x,\mu}$ at all links

$$Z = \int \prod_{x,\mu} dU_{x,\mu} \exp[-S(U)] \qquad (3.19)$$

over the gauge group volume. Theory of such invariant integration is known to mathematicians for a long time, but it may be reasonable to outline it for most important SU(2) and SU(3) gauge groups. In the former case one may use Euler angles

$$U = \exp(i\varphi' \vec{n}\vec{\sigma}/2)$$

$$\vec{n}^2 = 1, \quad n_z = \cos\theta, \quad n_x = \sin\theta\cos\varphi \qquad (3.20)$$

and the measure is just the element of the tree-dimensional sphere

$$dU = \frac{d^2\vec{n}}{4\pi} \frac{d\varphi'}{\pi} \sin^2\frac{\varphi'}{2} \qquad (3.21)$$

The SU(3) case also can be reduced to some spheres, in order to show this it is helpful to write the matrix as three complex vectors

$$U = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \qquad (3.22)$$

In such notations, unitarity means that

$$xx^* = yy^* = zz^* = 1$$
$$xy^* = xz^* = yz^* = 0 \qquad (3.23)$$

Now, the first condition means that 6 parameters of x lie on 5-dimensional sphere, y is orthogonal to x and lies on 3-dimensional sphere, and finally z is determined by x and y uniquely:

$$z_i = \varepsilon_{ikl} x_i^* y_l^* \qquad (3.24)$$

This method can be generalized in the obvious way.

The next step is the demonstration of the fact that the action (3.18) has correct continuous limit. Remind Stokes theorem for abelian field

$$\oint A_\mu \, dx_\mu = \int F_{\mu\nu} \, d\sigma_{\mu\nu} \qquad (3.25)$$

and apply it to elementary plaquette in some weakly varying field:

$$S(U) \simeq \frac{1}{g^2} \sum_\square \exp[F_{\mu\nu} a^2 + O(a^3)] \qquad (3.26)$$

so that at small a one has

$$S(U)\xrightarrow[a\to 0]{}\left(-\frac{1}{4g^2}\right)\int dx \sum_{\mu\nu}F^2_{\mu\nu} \tag{3.27}$$

For nonabelian field one does not have (3.25), but more lengthy calculation leads to the same result, which is essentially due to gauge invariance.

As for the correctness of the statistical sum (3.19) in this limit, we may only say that we do not precisely know what is the correct continuum limit, so the lattice theory may be considered as its constructive definition. Still one should define some particular limiting procedure, and also check whether the limit really exists.

The physical ground for this definition is the obvious consideration that physical quantities (like lambda parameter) should be stable in such limit

$$\Lambda = \frac{1}{a}\exp\left[-\frac{8\pi^2}{bg^2(a)}\right]\xrightarrow[a\to 0]{} \text{const} \tag{3.28}$$

which prescribe the particular behaviour of $g(a)$ at small a. As discussed in the section 3.1, existence of the limit can also be expressed as the following question: does the statistical analog system under consideration posess some second order phase transition at zero coupling? So far we have only numerical evidences that it is the case.

Finally at the end of this section we consider the confinement problem for the lattice theories, again following Wilson's work [3.6]. He has pointed out that at large g one may expand over $1/g$, which is analogous to the so-called high temperature expansion in statistical mechanics. The so-called Wilson loop average value is defined as follows:

$$W(C) = \frac{1}{N_c}\langle \text{Tr} \prod_c U_{x,\mu}\rangle = \frac{1}{N_c}\left\langle \text{Tr P} \exp\left[\frac{ig}{2}\int_c A^a_\mu t^a\, dx_\mu\right]\right\rangle \tag{3.29}$$

and for plain contour C it can easily be computed in this approximation. It is clear that average U is zero and in order to have nozero result one needs n-th order in $1/g^2$ (see Fig.5 for the explanation) where n is the area (in unites of a^2).

$$W(c)\sim \exp(-n\cdot k) \tag{3.30}$$

This result means confinement. In order to see this let us take contour $T\times L$ with T being oriented along the time axis. If $T\gg L\gg a$ one has

$$W(c)\sim \exp[-T\cdot E(L)] \tag{3.31}$$

where $E(L)$ is by definition the energy of two charges at distance L. Comparing to (3.29) one finds the famous linear potential

$$E(L) = k\cdot L \tag{3.32}$$

corresponding to a string with tension k.

In order to prove that this property is present in continuous theory one still has to show that it is not lost when one comes from large to small g. Recently this property at any finite g was proven for SU (2) theory by Tomboulis [3.37], and even in the limit of small g which is unfortunately somewhat weaker than that needed for transition to continuum limit.

3.4. Numerical studies of lattice gauge theories

The simplest gauge theory is that with abelian U (1) internal symmetry, so it was studied first. We remind the reader that Wilson proof of confinement in strong coupling regime outlined above corresponds to this case as well as for the nonabelian theories, but at small coupling existence of free photons is naturally expected. The obvious way out is some deconfinement phase transition at finite coupling. Its discovery in Ref. [3.54] (see also references therein) was the first important result obtained by numerical methods.

However, the turning point of this approach was the discovery made by Creutz [3.48] of the so called «early scaling» phenomenon in the simplest nonabelian SU (2) theory. It was very important psychologically, for it has demonstrated that some problems of quantum field theory are solvable at available computers.

In this work Wilson confinement criterium considered in the preceeding section was checked by evaluation of Wilson loop average value over the configuration ensemble generated numerically. Some helpful trick was suggested in this work: the following combination was measured

$$\chi(i,j) = \frac{W(i,j)\,W(i-1,j-1)}{W(i,i-1)\,W(i-1,j)} \tag{3.33}$$

where $W(i,j)$ stands for Wilson loop value for planar loop $i\times j$ (in lattice unites a^2). In the combination (3.33) two types of divergencies (at small a) are canceled. They are mass renormalization of the point charge (proportional to loop perimeter) as well as the logarithmic singularity due to rapid turn of the

charge at the loop corners.

The measured value of (3.33) directly provides the string tension k values, plotted as a function of g at Fig.6. The remarkable fact observed is some turn of this curve and reasonable agreement with weak coupling behaviour fixed by the asimptotic freedom

$$k \doteq \text{const} \cdot \Lambda_L^2$$

$$\Lambda_L = \frac{1}{a} (\beta_0 g^2)^{-\beta_1/2\beta_0} \exp\left(-\frac{1}{2\beta_0 g^2}\right)$$

$$\beta_0 = \frac{11}{3} \cdot \frac{N}{16\pi^2} \qquad \beta_1 = \frac{34}{3} \cdot \left(\frac{N}{16\pi^2}\right)^2 \qquad (3.34)$$

Note the following aspect of this relation: both the string tension k and Λ_L are physical quantities which can be measured experimentally. Therefore, connection between them (3.34) can be checked! The value of k can be taken from masses of large orbital momentum excitations, it is expressed in terms of Regge slope α'

$$k = \frac{1}{2\pi\alpha'} \simeq (420 \text{ MeV})^2 \qquad (3.35)$$

but the value of lambda is so far known with rather poor precision.

For pure gauge theories considered in this section one should introduce some convention, relating their parameters to physical unites. The most natural choice is to put lambda equal to that in real world, but in practice people consider fixed k (3.35) and use (3.34) for «lambda measurements». For SU(2) theory increase of the lattice size has not lead to significant violation of scaling (3.34), but for SU(3) case the situation is different, see Fig.7. Most likely, it is some trace of «lattice artifacts» connected with specific phase transitions, see original discussion in [3.13] and recent considerations in Ref.[3.67], from which the Fig.7 is taken.

This violation of scaling is rather large: the lambda values obtained by Creutz [3.58] and Pietarinen [3.59] for SU(3) group are as follows

$$\Lambda_L/\sqrt{k} = \begin{cases} (5\pm 1.5) \cdot 10^{-3} \\ (7\pm 2) \cdot 10^{-3} \end{cases} \qquad (3.36)$$

while at larger lattice sites number it is possible to study smaller couplings and obtain quite different value

$$\Lambda_L/\sqrt{k} = (10\div 12)10^{-3} \simeq 4.2\div 5 \text{ MeV} \qquad (3.37)$$

which is about a factor 2 larger! So, different authors use different scales and one should be careful at this point. In what follows we consider latest scale (3.37).

During last few years rather impressive work have been done, devoted to various tests of internal consistency of the calculations. In particular, the rotational symmetry is shown to be recovered [3.55], results do not depend on the particular lattice action [3.56], etc.

Moreover, some particular results were obtained, being of some interest for the phenomenology. Bhanot and Rebbi [3.61] have found the simplest perturbative phenomenon—the Coulomb force between static quarks at small distances. Rather impressive work was made for the calculation of glue-ball masses, see Table 1. Note, that the lightest state is scalar glue-ball, with mass of about 300 Λ_L. The «old» scale of lambda put it to 0.6—0.7 GeV (the place of the so-called «sigma meson» seen as some enhancement in pion-pion scattering), but the «new» one shift it to about 1.3 GeV. The last value corresponds to ε-meson, quite similar value is also suggested by various estimates by QCD sum rules, see section 5.8.

The value of gluonic condensate was evaluated by two different methods. The most straightforward one is the subtraction of the perturbative part from the «average plaquette» [3.76], but it is more covenient [3.74] to consider larger loops and connect their average values with OPE expression (to be derived in section 4.2). We present results of measurements by both methods for SU(3) group

$$\langle(gG)^2\rangle/\Lambda_L^4 = \begin{cases} (8\pm 5) \cdot 10^{10} & [3.76] \\ (8.96\pm .56) \cdot 10^9 & [3.78] \end{cases}$$

from which one can see that accuracy is rather poor and connected not only with statistical errors given.

With the scale (3.36) these results correspond to the condensate in pure gauge SU(3) theory to be about 30, 3 GeV^4, respectively. It is hardly reasonable to make averaging of these conflicting data, but in most cases results obtained seem to produce too large effect compared to phenomenological value 0.5 GeV^4. The inclusion of quarks makes the condensate smaller, so this tendency may contain some truth in it.

Interesting to note that for the SU(2) group results are essentially smaller, which can not be explained by naive large N counting $\langle (gG)^2 \rangle = O(N)$. For example, high statistics calculation. for SU(2) group was recently reported by Ishikawa et al. Using the former method they have found

$$\langle (gG)^2 \rangle / \Lambda_L^4 = (0.8 \pm 0.04) \cdot 10^9 \qquad [3.79]$$

The most dramatic part of the story is connected with «hunting» for the topologically nontrivial objects in gluonic vacuum. Some lattice monopoles and other artifacts are now rather well understood, but they are not present in continuous limit and are important only for the optimal choice of the action. Evidently, the most interesting type of such objects is the instantons, predicted by the semiclassical approximation for the continuous theory.

However, the topological arguments leading to their existence are not applicable on the lattice in strict sence, say «continuous gauge transformations» become meaningless. This fact makes definition of the topological charge on the lattice rather tricky, see discussion in Refs [3.80—3.84, 3.88]. Most of the work in this direction is concentrated around the evaluation of the so called to-pological susceptibility

$$A = \langle \int dx \, Q(x) \, Q(0) \rangle$$

$$Q = \frac{g^2}{32\pi^2} (G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a) \qquad (3.38)$$

In section 5.8 we obtain some estimate for it [5.58]

$$A = (3 \div 6) \cdot 10^{-4} \text{ GeV}^4 \qquad (3.39)$$

while in earlier works [5.60—5.62] similar ideas in somewhat different context have lead to

$$A = \frac{f_\pi^2}{3} (m_\eta^2 + m_{\eta'}^2 - 2m_k^2) \simeq 10^{-3} \text{ GeV}^4 \qquad (3.40)$$

In order to put it in simple terms we may mention that these numbers correspond approximately to one unite of the topological charge (positive or negative) per (fermi)^4 (in good agreement with the instanton liquid model).

Measurements of A on the lattice made in Refs [3.85—3.89] have produced positive results: nonzero A was found, with

reasonable scaling behaviour. Unfortunately, the numerical value was found to be about

$$A = (10^4 \div 10^5) \Lambda_L^4 \qquad (3.41)$$

which is about 1—2 orders of magnitude smaller than (3.39, 3.40)! (There is one exception: in Ref. [3.88] A was found of reasonable magnitude, in contradiction to Ref. [3.89] using similar algorithm). This observation is in evident contrast to the gluon condensate value considered above. In short, lattice configurations are very far from being «locally selfdual» (see more on this in sect. 5.8). Ishikawa et al. have made special studies of this interesting point [3.87], looking for the distribution over topological charge inside some small volume inside the lattice. Again, no trace of large amount of instantons suggested by (3.39, 3.40) and other phenomenological considerations!

One should not consider this fact to be very unexpected, because experience of simplests model examples (including $1 + 1$ dimensional sigma model [3.80], nonlinear oscillator [3.45] and pendilum [3.82]) clearly demonstrates that in order to have correct description of tunneling phenomena (instantons) one needs lattices with hundreds of points along the time axis. In other words, with lattice spacing a about 0.2—0.3 fermi it is hardly possible to describe correctly instantons of radius 0.3 fermi.

3.5. Quarks on the lattice

At first sight the lattice formulation for quark fields is much simpler than for gluonic ones, for they are not connected with derivatives and have rather simple gauge transformation law:

$$\Psi'(x) = \Omega(x) \Psi(x) \qquad (3.42)$$

so their natural place is sites of the lattice. The form of action suggested originally by Wilson [3.6] is rather evident:

$$S = -\frac{1}{2} \sum_{x,\mu} [\overline{\Psi}_x \gamma_\mu U_{x,\mu} \Psi_{x+\mu} - \overline{\Psi}_{x+\mu} \gamma_\mu U_{x,\mu}^+ \Psi_x] \qquad (3.43)$$

where the second term is needed in order to make it real. Note that this action is not only gauge invariant, but also posesses exact chiral symmetry

$$\Psi'_L = \Omega_1 \Psi_L, \qquad \Psi'_R = \Omega_2 \Psi_R,$$

$$\Psi_{L,R} = \frac{1 \pm \gamma_5}{2} \Psi, \qquad \Omega_1 \neq \Omega_2 \qquad (3.44)$$

Unfortunately, so natural action has serious defect, pointed out by Wilson in [3.9]. It is most transparently seen if one calculates the inverse propagator for a quark with momentum p

$$S^{-1}(p) = \frac{i}{a} \sum_{\mu=1}^{4} \gamma_\mu \sin(p_\mu a) \qquad (3.45)$$

In local limit of small a it seems to be reduced to correct continuum limit $S^{-1} = i p_\mu \gamma_\mu$, but in fact there are additional modes with $p \sim 1/a$ which are not decoupled in this limit. Note that propagating modes (in Minkowski space) are those which satisfy the equation

$$sh^2(p_0 a) = \sin^2(p_x a) \qquad (3.46)$$

if the propagation is along coordinate x. The substitution

$$p_x = \pi/2 - p_x \qquad (3.47)$$

does not violate (3.46) and therefore each solution is copied many times! Note also, that in (3.45) one has the first power of sin (pa), so half of the copies give the contribution of the opposite sign. The last comment is relevant for the observation, that although this action posesses exact chiral symmetry, the Adler-Bell-Jackiw anomaly [1.23] is absent: «normal» fermions are canceled by the «wrong sign copies»! Obviously such formulation should be improved, for it does not correspond to real world.

Another (improved) formulation for the lattice quarks was suggested by Wilson in [3.9], the action in this case is equal to

$$S = k \sum_{x,\mu} [\overline{\Psi}_x (1 - \gamma_\mu) U_{x,\mu} \Psi_{x+\mu} + \Psi_{x+\mu}(1 + \gamma_\mu) U_{x,\mu}^+ \Psi_x] - \sum_x \overline{\Psi}_x \Psi_x \qquad (3.48)$$

and in contrast to (3.43) it contains new «hopping» parameter K connected with quark mass. This can be seen in transition to continuous limit according to expressions

$$\Psi_x \to \left(\frac{a^3}{2k}\right)^{1/2} \Psi(x)$$

$$\Psi_{x+\mu} = \left(\frac{a^3}{2k}\right)^{1/2} \cdot \left[\Psi(x) + a\frac{\partial \Psi}{\partial x_\mu}\right] \qquad (3.49)$$

one finds the correct action

$$S = -\int dx \, (\overline{\Psi} \hat{\partial} \Psi + m \overline{\Psi} \Psi) \qquad (3.50)$$

where

$$m = (1 - 8K)/2Ka \qquad (3.51)$$

Therefore, massless quarks are obtained in some additional limit $K \to 1/8$ (the latter number is modified in the interacting theory and is substituted by some critical value K_c). Resrectively, chiral symmetry is recovered only in this limit.

The inverse propagator is for this action as follows:

$$S^{-1}(p) = 1 - K \sum_\mu [(1 + \gamma_\mu)e^{-ip_\mu a} + (1 - \gamma_\mu)e^{ip_\mu a}] \qquad (3.52)$$

and it has only one massless mode in the continuous limit, so the problem of unwanted copies is solved «by brute force».

Another possibility posessing exact chiral symmetry is «thinning of degrees of freedom» with direct use of extra copies, see comprehensive discussion in [3.5] (the original work is [3.91]).

With the theory formulated as statistical problem one should still develop practical computational algorithms. The obvious first step is formal integration over quark fields, so that Matew-Salam determinant appears:

$$\int D\overline{\Psi} D\Psi \, e^{-\overline{\Psi} M \Psi} = \det M \qquad (3.53)$$

At first sight the problem is reduced to the previous one for only gauge fields remain. However, in contrast to usual local actions this functional determinant is extremely nonlocal quantity. In particular, for the updating process one needs evaluation of its variation if only one $U_{x,\mu}$ is changed. As noted in [3.89, 3.90] the following formula is helpful here

$$\det(M + \delta M)/\det(M) = \det(1 + M^{-1}\delta M) \qquad (3.54)$$

but it contains the inverse matrix. Standard methods need about n^2 operations with n, the order of the matrix, is of the order of the number of lattice sites! Some modern development we discuss in section 4.4.

Another popular method [3.84] is based on the introduction of

some auxiliary «pseudofermionic» field and application of the relation

$$\int D\varphi \exp\left(-\varphi M\varphi\right) \sim \det{}^{-1}(M) \tag{3.55}$$

Note that in principle one need antiperiodic boundary conditions for pseudofermions.

One more method suggested in [3.85—3.87] is expansion in powers of «hopping parameter» K. Clear that L-th term is connected with contours of length $L \cdot a$, so this method is practical if quarks are rather heavy and long loops are not important. In practice such loops are taken randomly, but the efficiency problem is in this case rather severe.

Now the main question can be posed: is this determinant-induced interaction really important, or just makes some small correction to gauge field action? We consider it in somewhat more general framework in section 8.3, and now just present some examples from current literature.

First applications of the hopping parameter method have lead to optimistic conclusion [3.94, 3.95], but later results (such as obtained in [3.93] by the pseudofermion method) have demonstrated the quark importance. At Fig.8 we compare these results for the effective action dependence on hopping parameter K, where

$$S_{eff} = -\log \det(M)$$

These results show strong increase of the action with K, which clearly demonstrates that it can not be neglected for K values which in physical unites still correspond to rather heavy quarks, with mass about 800 MeV! Similar results (see sections 7.2, 7.3) were found for thermodynamical quantities. Also it is clear from these data that one can not use hopping parameter expansion in this case.

Unfortunately, all methods for the account of fermions are too slow: individual determinant is evaluated by CPU time of the order of one hour even at largest computers. Therefore light quarks can not be effectively taken into account with existing algorithms and available computers, while earlier hopes that their effect is rather small have definitely failed.

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T a b l e 1.

Some results on glueball masses in SU (3) gauge theory

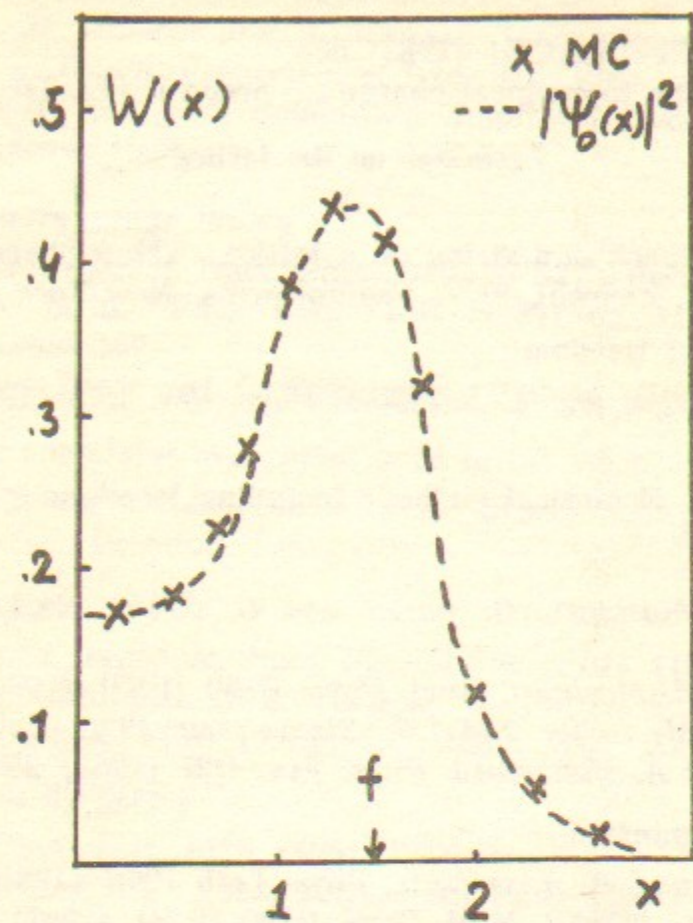| Quantity	Value	Reference
$M(0^{++})/\Lambda_L$	720 ± 100	[5.1]
	350 ± 50	[3.69]
	300 ± 16	[3.70]
$M(0^{--})/M(0^{++})$	$4.6 \pm^{1.7}_{0.6}$	[3.69]
$M(1^{+-})/M(0^{++})$	4.2 ± 0.3	[3.69]
$M(2^{++})/M(0^{++})$	2.93 ± 0.07	[3.69]
	2.27 ± 0.2	[3.79]
$M(2^{+-})/M(0^{++})$	> 4.9	[3.69]
$M(2^{-+})/M(0^{++})$	3.47 ± 0.17	[3.69]
$M(2^{--})/M(0^{++})$	$4.8 \pm 1.$	[3.69]
$M(3^{++})/M(0^{++})$	3.14 ± 0.13	[3.69]
$M(3^{+-})/M(0^{++})$	3.7 ± 0.3	[3.69]

Fig.1. Probability distribution over coordinate x for the two-well oscillator at $f = 1.4$. Points correspond to Monte-Carlo data [3.44] while the dashed curve is the ground state wave function squared.
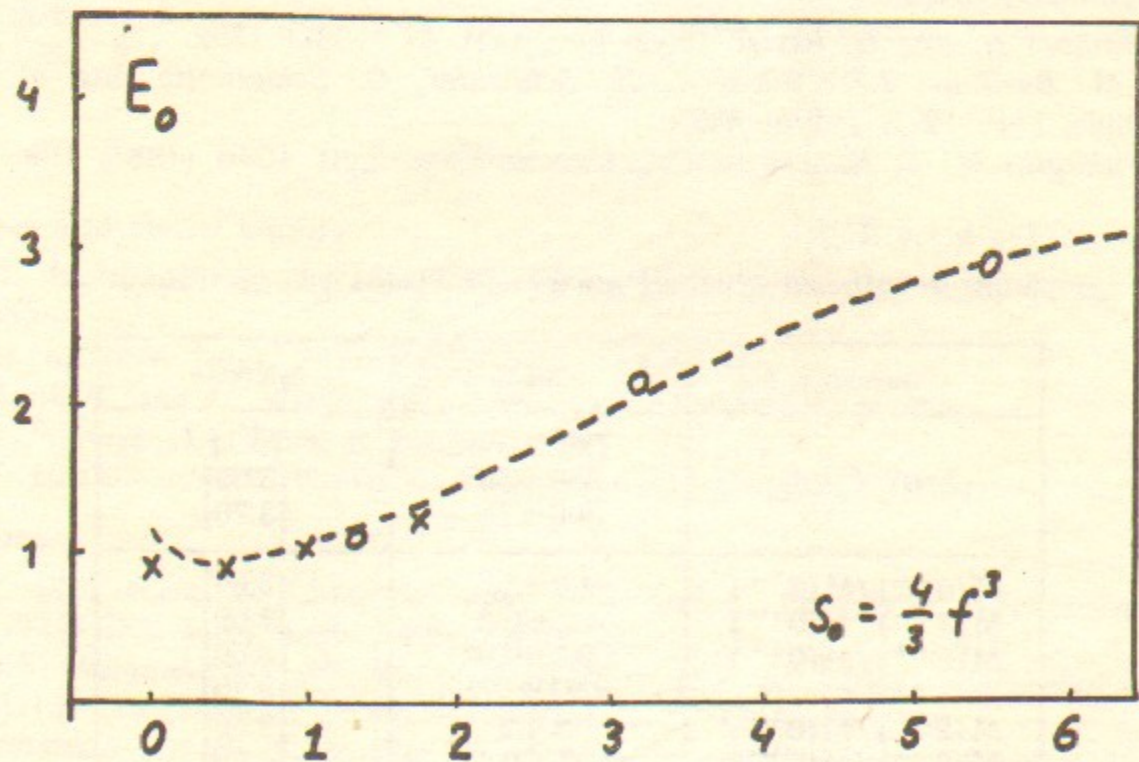

Fig.2. The ground state energy of the two-well oscillator versus S, the instanton action. Crosses and points correspond to [3.44] and [3.45].
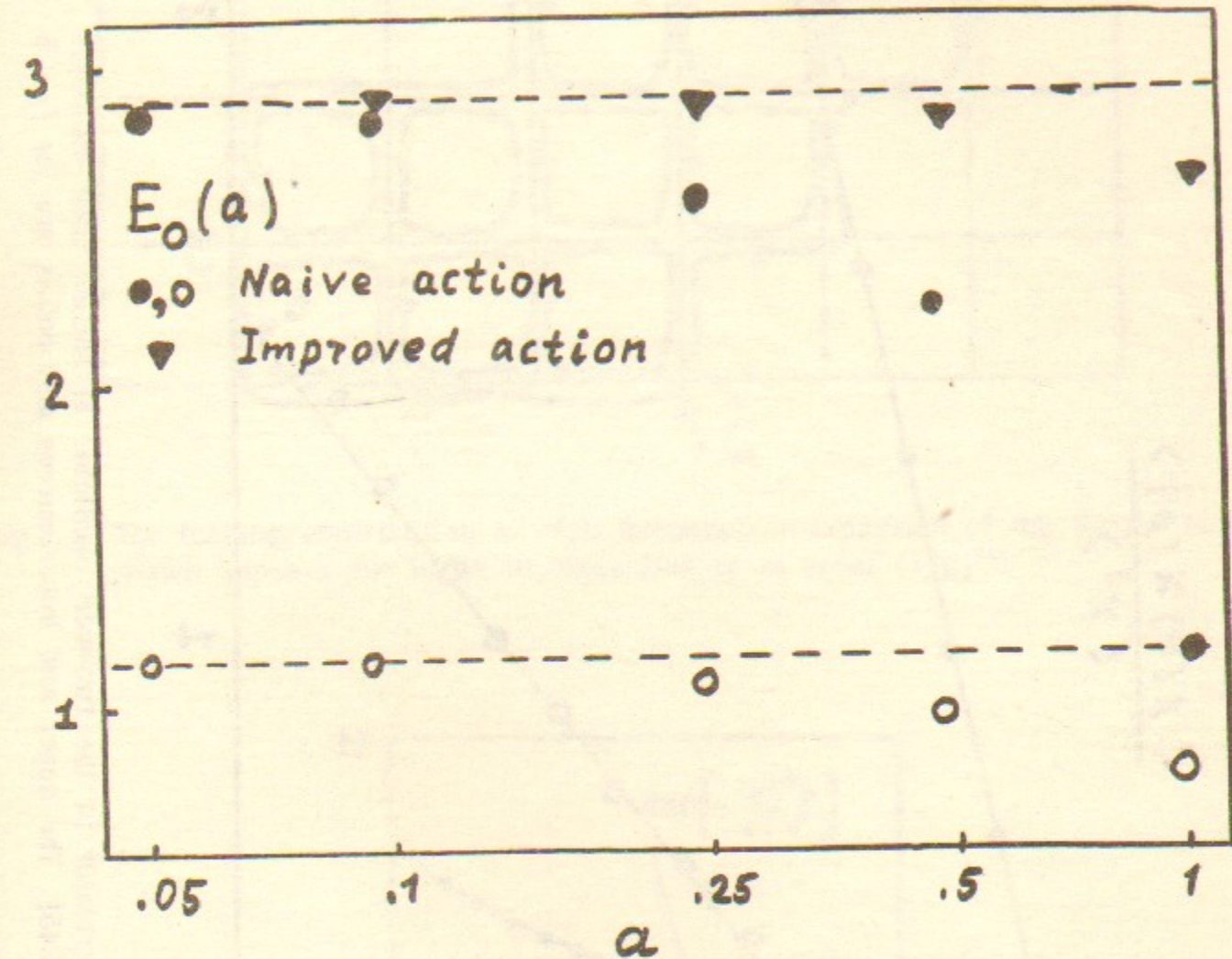

Fig.3. The ground state energy of the two-well oscillator versus a, the lattice spacing, at $f = 1.6$ and $f = 1.4$ (upper and lower part of the Figure, respectively) according to [3.45]. The triangles correspond to improved action, see text.
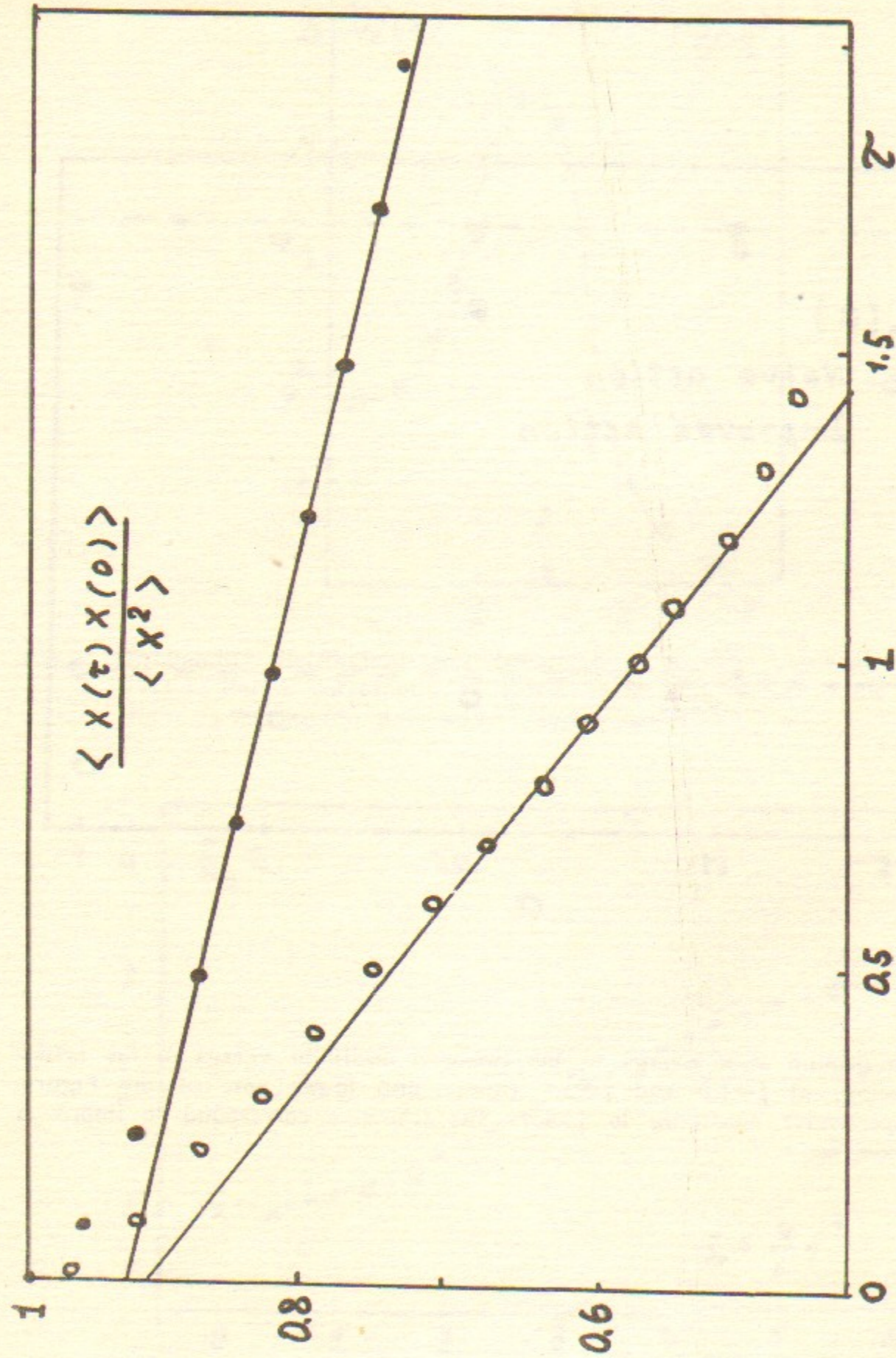
Fig.4. Coordinate correlator in the two-well oscillator at shifted time moments $\langle x(\tau)x(0)\rangle$ [3.45]. The upper and lower curves and points are for $f=1.6$ and $f=1.4$.
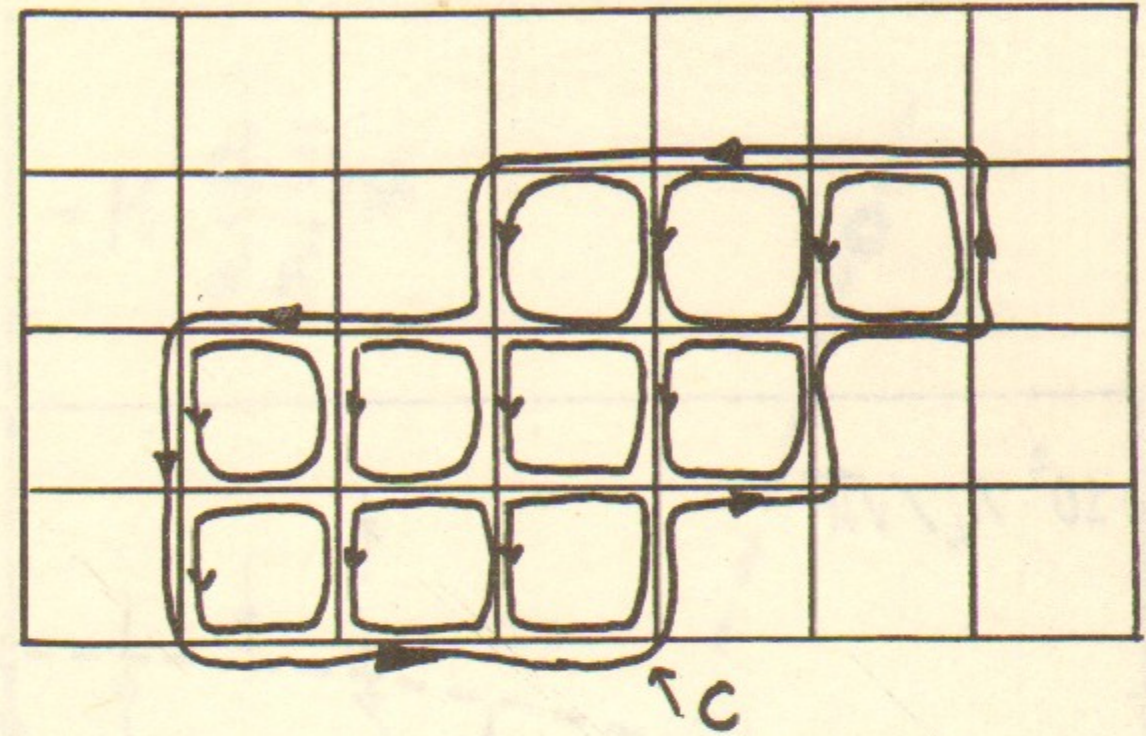
Fig.5. The leading contribution to high temperature expansion of the Wilson loop shown appears due to its 10 plaquettes, or in order $(1/g)^{20}$.
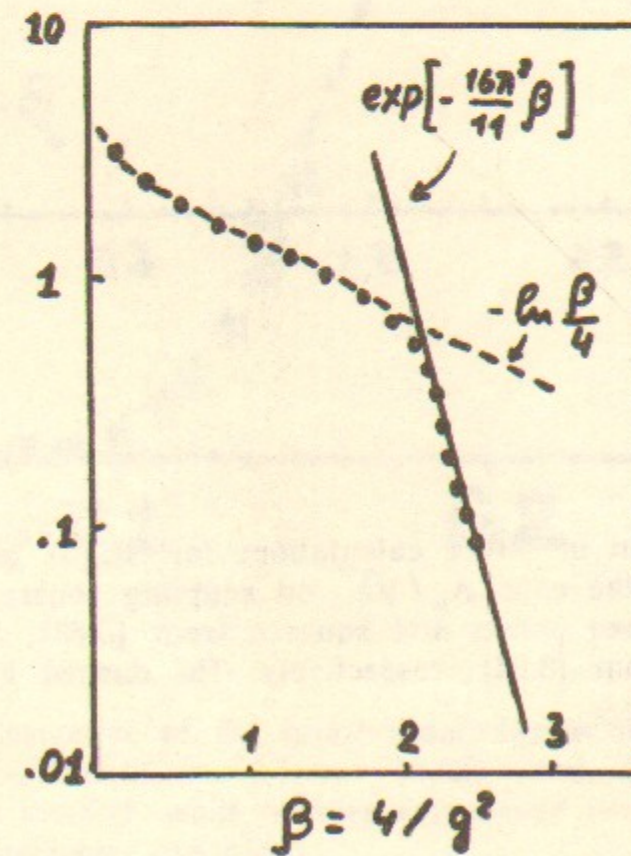

$$\exp\left[-\frac{16\pi^2}{11}\beta\right]$$

$$-\ln\frac{\beta}{4}$$

$$\beta = 4/g^2$$

Fig.6. Creutz results [3.58] for the string tension for SU(2) group versus bare coupling constant. Crossover from weak to strong coupling regimes is indicated.
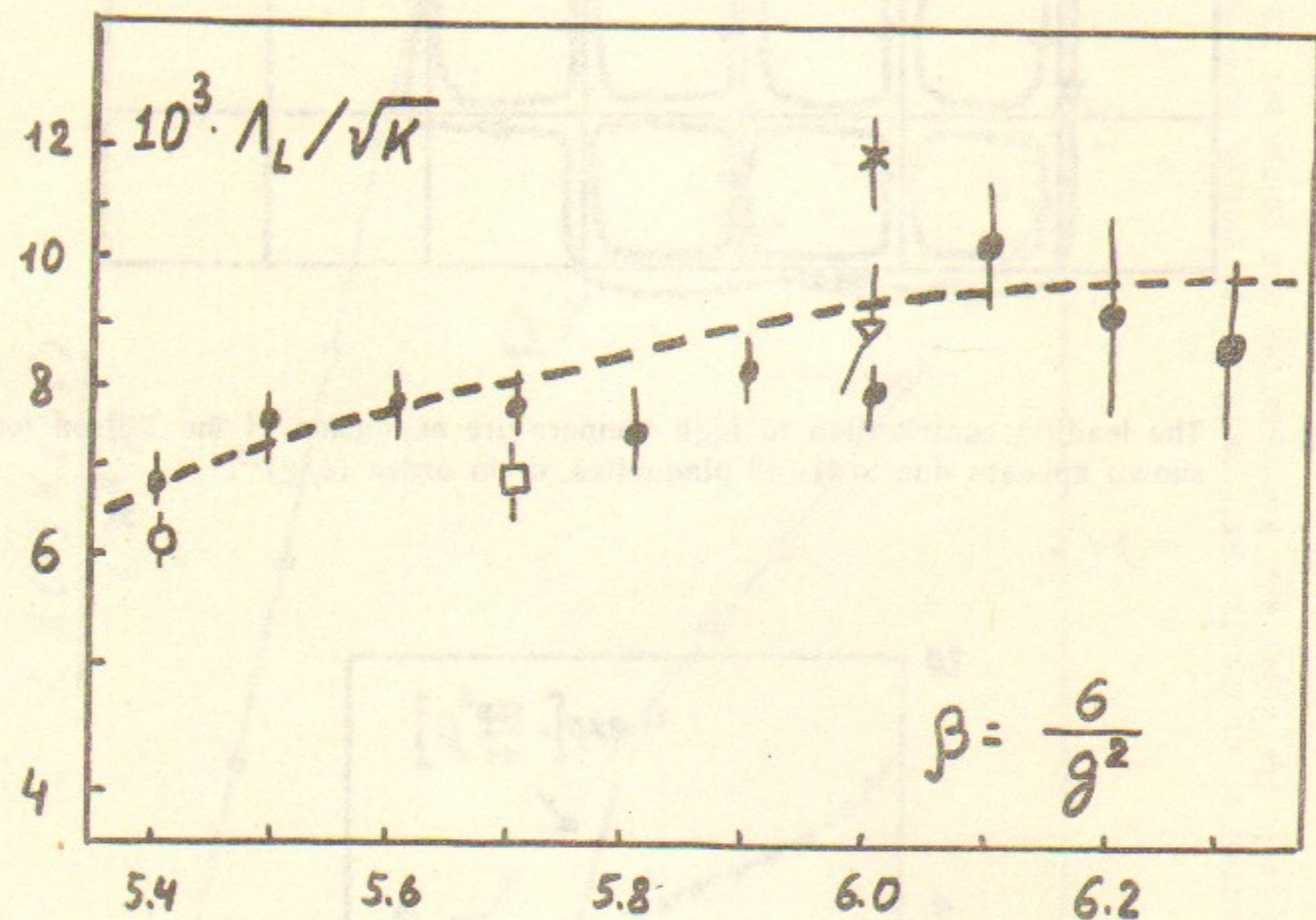
Fig.7. Scaling violation in lattice calculations for SU(3) group, shown as some dependence of the ratio $\Lambda_L / \sqrt{k}$ on coupling constant. Closed points are from [3.65], open points and squares from [3.63], triangles from [3.66] and crosses from [3.64], respectively. The dashed line is some result of Ref. [3.67].
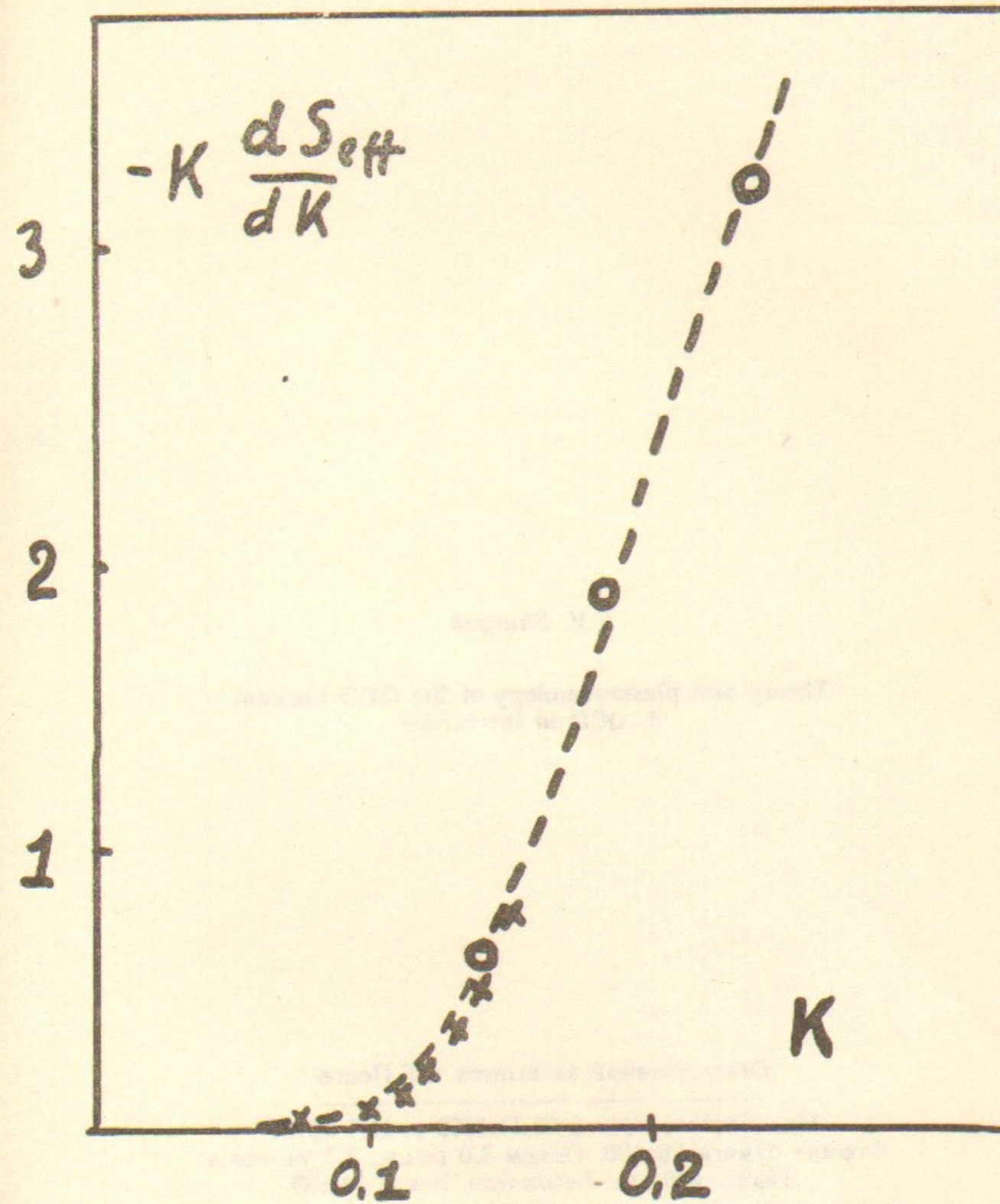

Fig.8. Logarithmic derivative of the quark-induced effective action over hopping parameter K. Points are from [3.93] and are obtained by the pseudofermion method, while crosses are earlier results [3.95] obtained by the hopping parameter expansion.

E.V. Shuryak

Theory and phenomenology of the QCD vacuum
***3.* QCD in the lattice**